Family Name ______________________ Given Name ______________________
ID. No. ______________________

DEPARTMENT OF MATHEMATICS AND STATISTICS
Theory of Interest 62-392 Test 1 M. Hlynka
Wednesday, February 9, 2011. Time allowed: 75 minutes.

Directions: You must show your work! You will be graded on your completeness as well as your correctness. Choose the closest answer. Calculators encouraged.

1. (Kozubowski) An investment of $100 accumulates to $107 after 6 months. If a period is one year, find (a) $i^{(2)}$ (b) i (c) $d^{(3)}$ (c) δ
SOLUTION: $1+i = 1.07^2 = 1.1449$ so $i = .1449$
$1+i = (1+\frac{i^{(2)}}{2})^2$ so $i^{(2)} = .14$
$\delta = ln(1+i) = ln(1.1449) = .1353.$
$(1-\frac{d^{(3)}}{3})^{-3} = (1-d)^{-1} = 1+i = 1.1449$ so $d^{(3)} = 3(1-(1+i)^{-1/3}) = .1323$

2. (Michigan1.17) Jennifer deposits 1000 into an account. Interest is credited at a nominal annual rate of i convertible seminannually for the first 7 years and at rate $2i$ convertible quarterly for the all years thereafter. The accumulated amount after 5 years is X. The accumulated amount at the end of 10.5 years is 1980. Calculate X.
(A) 1200 (B) 1225 (C) 1250 (D) 1275 (E) 1300
SOLUTION: $1000 = (1+\frac{i}{2})^{14}(1+\frac{2i}{4})^{14} = 1000(1+\frac{i}{2})^{28} = 1980.$
$X = 1000(1+\frac{i}{2})^{10} = 1000(1+\frac{i}{2})^{28*10/28} = 1000*1.980^{10/28} = 1276.3$ D

3. (SOA, May 2003, #33) At an annual effective interest rate of i, $i > 0$, both of the following annuities have a present value of X:
1. a 20-year annuity-immediate with annual payments of 55
2. a 30-year annuity-immediate with annual payments that pays 30 per year for the first 10 years, 60 per year for the second 10 years, and 90 per year for the final 10 years.
Calculate X.
(A) 575 (B) 585 (C) 595 (D) 605 (E) 615
SOLUTION:
$55a_{\overline{20}|} = 30a_{\overline{10}|} + 60a_{\overline{10}|}v^{10} + 90a_{\overline{10}|}v^{20}$ so
$55(1-v^{20}) = 55(1-v^{10})(1+v^{10}) = 30(1-v^{10}) + 60(1-v^{10})v^{10} + 90(1-v^{10})v^{20}$. But $v^{10} \neq 1$ so $55(1+v^{10}) = 30 + 60v^{10} + 90v^{20}$ so $11(1+x) = 6 + 12x + 18x^2$ where $x = v^{10}$. Thus $18x^2 + x - 5 = 0$. So $v^{10} = x = \frac{-1 \pm \sqrt{1+4(18)(5)}}{36} = 18/36 = .5$ so $i = 2^{.1} - 1 = .072.$
Thus $PV = 55a_{\overline{20}|} = 55\frac{1-v^{20}}{i} = 55\frac{1-.5^2}{.072} = 574.725$ A.

4. (SpecimenExam30-3-72N24) For a given positive integer n, a rate of interest i can be found for which $4a_{\overline{2n}|i} = 5a_{\overline{n}|i}$. Express in terms of n how long it will take for money to doubole at this rate of interest.
(A) $\sqrt{n}$ (B) $n/2$ (C) $5n/8$ (D) $n/\sqrt{2}$ (E) $2n$
SOLUTION : $4\frac{1-v^{2n}}{i} = 5\frac{1-v^n}{i}$ so $4\frac{(1-v^n)(1+v^n)}{i} = 5\frac{1-v^n}{i}$. If $v^n = 1$ then $v = 1$ so $i = 1$ which cannot give $4a_{\overline{2n}|i} = 5a_{\overline{n}|i}$. Thus $4(1+v^n) = 5$ so $v^n = .25$ so $(1+i)^n = 4$ so $(1+i)^{n/2} = 2$ at which point money is doubled. Thus $n/2$ is the answer.

5. (Michigan2.4) Fund A accumulates at rate 12% convertible monthly. Fund B accumulates with force of interest $\delta_t = t/6$, in years. At time zero, 1 is deposited into each fund. Let T be the time when the two funds are equal. Determine T, in years.

(A) $12ln(1.01)$ (B) $12ln(1.12) - ln(1.01)$ (C)$12ln(1.12)$ (D) $144ln(1.01)$ (E) $144ln(1.12)$
SOLUTION: Fund A has value $(1.01)^m$ after m months.
Fund B has value $e^{\int_0^t r/6dr}$ after t years or $e^{\int_0^{m/12} r/6dr}$ after m months. Thus

$$1.01^m = e^{r^2/12|_0^{m/12}} = e^{m^2/12^3} \text{ so}$$

$m\, ln(1.01) = m^2/12^3$ so $m = 12^3\, ln(1.01)$ months. Thus $T = m/12 = 12^2\, ln(1.01)$.

6. (SOA1972SpecimenExam23) Find the first derivative with respect to i of $f(i) = \frac{\delta}{d}$.
(A) 0 (B) $\frac{i-\delta}{i^2}$ (C) $\frac{d^2}{i^2}(1-\frac{i}{\delta})$ (D) $\frac{\delta - i}{i^2}$ (E) $\frac{d^2}{i^2}(\frac{i}{\delta}-1)$
SOLUTION: $\delta = ln(1+i)$ and $d = i/(1+i) = 1 - 1/(1+i)$. Thus $\frac{df}{di} = \frac{d}{di}\frac{(1+i)ln(1+i)}{i} = \frac{d}{di}(1 + 1/i)ln(1+i) = \frac{-1}{i^2}ln(1+i) + (1+\frac{1}{i})\frac{1}{1+i}) = \frac{-ln(1+i)}{i^2} + \frac{1}{i} = \frac{-\delta}{i^2} + \frac{1}{i} = \frac{i-\delta}{i^2}$.

7. (Michigan6.5) An annuity immediate pays an initial benefit of 1 per year, increasing by 10.25% every four years. The annuity is payable for 40 years. Using an annual effective rate of 5%, find the present value of this annuity.
SOLUTION: Find accumulated amount of groups of 4 payments at years $4, 8, 12, 16, \ldots, 40$. Call them $K_1, K_2, \ldots, K_{10}$. Then find PV.
$\{K_1, \ldots, K_{10}\} = \{s_{\overline{4}|.05}, 1.1025 s_{\overline{4}|.05}, \ldots, 1.1025^9 s_{\overline{4}|.05}\}$. Note that $1.05^2 = 1.1025$. Thus
$PV = K_1 1.05^{-4} + K_2 1.05^{-8} + \cdots = K_1 1.1025^{-2} + K_2 1.1025^{-4} + \ldots$
$= s_{\overline{4}|.05}(1.1025^{-2} + 1.1025^{-3} + \cdots + 1.1025^{-11}) = s_{\overline{4}|.05} 1.1025^{-1}(1.1025^{-1} + \cdots + 1.1025^{-10})$
$= s_{\overline{4}|.05} 1.1025^{-1} a_{\overline{10}|.1025} = 23.76580$

8. (SpecimenExam30-3-72N27) An annual perpetuity immediate is divided among four charities W,X,Y,Z. W is to receive the first n annual payments, X the next n, Y the third n, and Z the rest. Denote W's share by w, X's share by x, Y's share by y, Z's share by z. If $w - 2x = z - y$, evaluate w/z.
(A)1/2 (B) 1 (C) 2 (D) 4 (E) 8
SOLUTION:
$w = a_{\overline{n}|}$; $x = a_{\overline{n}|}v^n$; $y = a_{\overline{n}|}v^{2n}$. $z = v^{3n+1} + v^{3n+2} + \cdots = \frac{v^{3n+1}}{1-v}$.
Now $w - 2x = a_{\overline{n}|}(1 - 2v^n) = z - y = \frac{v^{3n+1}}{1-v} - a_{\overline{n}|}v^{2n}$. Solve for $a_{\overline{n}|}$ to get $a_{\overline{n}|}(1 - 2v^n + v^{2n}) = \frac{v^{3n+1}}{1-v}$ so $\frac{(1-v^n)^3}{i} = \frac{v^{3n+1}}{i/(1+i)} = \frac{v^{3n}}{i}$. Thus $(1-v^n)^3 = v^{3n}$ so $1 - v^n = v^n$ so $v^n = .5$. Finally $w/z = \frac{a_{\overline{n}|}}{v^{3n+1}/(1-v)} = \frac{1-v^n}{v^{3n}} = \frac{1-.5}{.5^3} = 4$.