Family Name ________________________ Given Name ____________________
ID. No. ____________________

DEPARTMENT OF MATHEMATICS AND STATISTICS
Theory of Interest 62-392 Test 1 SOLUTIONS M. Hlynka
Tuesday, February 9, 2010. Time allowed: 75 minutes.

Directions: You must show your work! You will be graded on your completeness as well as your correctness. Choose the closest answer. Calculators encouraged.

1. (Michigan1.11) On January 1, 2010, Amil has the following options for repaying a loan.
(i) Sixty monthly payments of 100 beginning February 1, 2010.
(ii) A single payment of 6000 at the end of K months.
Interest is at a nominal rate of 12% compounded monthly. The two options have the same present value. Determine K.
(A)29 (B)29.5 (C) 30 (D) 30.5 (E) 31
SOLUTION:
$100a_{\overline{60}|.01} = 6000v^K$ so
$100\dfrac{1-1.01^{-60}}{.01} = 6000(1.01)^{-K}$ so
$K = -(\ln(10/6) + \ln(1-1.01^{-60}))/\ln(1.01) = 29.01227$.

2. (Michigan1.6) You are given that $a(t) = Kt^2 + Lt + M$, for $0 \le t \le 2$, and that $a(0) = 100$, $a(1) = 110$, and $a(2) = 136$. Determine the force of interest at time $t = 1/2$.
(A) .030 (B) .049 (C) .061 (D) .095 (E) .097
SOLUTION: $100 = a(0) = M$. $110 = a(1) = K + L + M$ so $K + L = 10$. $136 = a(2) = 4K + 2L + M$ so $18 = 2K + L$ so $K = 8$ and $L = 2$. Thus $a(t) = 8t^2 + 2t + 100$. So $\delta(t) = \dfrac{a'(t)}{a(t)} = \dfrac{16t+2}{8t^2+2t+100}$ so $\delta(.5) = \dfrac{10}{103} = .097$.

3. (Michigan1.13) The present value of 200 paid at the end of n years, plus the present value of 100 paid at the end of $2n$ years is 200. Determine the annual effective rate of interest.
(A) $\left(\dfrac{\sqrt{3}+1}{2}\right)^{1/n} - 1$ (B) $1 - \left(\dfrac{\sqrt{3}-1}{2}\right)^{1/n}$ (C) $\left(\dfrac{\sqrt{3}-1}{2}\right)^{1/n} - 1$
(D) $\left(\dfrac{\sqrt{3}+1}{2}\right)^{1/2n} - 1$ (E) $1 - \left(\dfrac{\sqrt{3}-1}{2}\right)^{1/2n}$
SOLUTION:
$200v^n + 100v^{2n} = 200$ so $0 = x^2 + 2x - 2$ so $v^n = x = \dfrac{-2 \pm \sqrt{4+8}}{2} = -1 + \sqrt{3}$. But $v = 1/(1+i)$ so $i = (1/v) - 1 = \left(\dfrac{1}{\sqrt{3}-1}\right)^{1/n} - 1 = \left(\dfrac{\sqrt{3}+1}{3-1}\right)^{1/n} - 1$.

4. (Michigan1.10) A person deposits 100 at the beginning of each year for 20 years. Simple interest at a rate of i per year grows the account to 2840 at the end of 20 years. If compound interest at the same rate i had been used, what would be the accumulated value in the account after 20 years?
(A) 2890 (B) 3100 (C) 3200 (D)3310 (E) 3470
SOLUTION:
$2840 = 100(1+20i) + 100(1+19i) + \cdots + 100(1+i) = 2000 + i * 100(21)(20)/2$ so $i = 16.80/420 = .04$.
Using compound interest, accumulated amount is

$$100\ddot{s}_{\overline{20}|.04} = 100\frac{1.04^{21} - 1.04}{.04} = 3096.92$$

5. (Michigan1.17) Jennifer deposits 1000 into an account. Interest is credited at a nominal annual rate of i convertible seminannually for the first 7 years and at rate $2i$ convertible quarterly for the all years thereafter. The accumulated amount after 5 years is X. The accumulated amount at the end of 10.5 years is 1980. Calculate X.
(A) 1200 (B) 1225 (C) 1250 (D) 1275 (E) 1300
SOLUTION:
$1000(1+i/2)^{10} = X.(1)$
$1980 = 1000(1+i/2)^{14}(1+i/2)^{14} = 1000(1+i/2)^{28}$ (2)
From (2), $1.98 = (1+i/2)^{28}$ so $(1+i/2)^{10} = 1.98^{10/28}$
Thus $X = 1000(1+i/2)^{10} = 1000 * 1.98^{10/28} = 1276.30$.

6. (Michigan2.7) A perpetuity pays 1 at the end of every year plus an additional 1 at the end of every second year. The present value of the perpetuity is K for $i > 0$. Determine K.
(A)$\frac{i+3}{i(i+2)}$ (B) $\frac{i+2}{i(i+1)}$ (C) $\frac{i+1}{i^2}$ (D) $\frac{3}{2i}$ (E) $\frac{i+1}{i(i+2)}$
SOLUTION: $K = v + v^2 + \cdots + v^2 + v^4 + v^6 + \cdots = \frac{v}{1-v} + \frac{v^2}{1-v^2} = \frac{(1+v)v}{1-v^2} + \frac{v^2}{1-v^2} = \frac{v+2v^2}{1-v^2}$
$=$ (mult. num and den by $(1+i)^2$) $= \frac{1+i+2}{(1+i)^2-1} = \frac{3+i}{i(2+i)}$

7. (Michigan2.17) A deposit of 100 is made into a fund at time $t = 0$. The fund pays interest at a nominal annual rate of discount d compounded quarterly for the first two years. Beginning at time $t = 2$, interest is credited at a force of interest $\delta_t = \frac{1}{t+1}$. At time $t = 5$, the accumulated value of the fund is 260. Calculate d.
(A) 12.7% (B) 12.9% (c) 13.1% (D) 13.3% (E)13.5%
SOLUTION:
$260 = \frac{100}{(1-d/4)^8} e^{\int_2^5 1/(t+1)dt} = \frac{100}{(1-d/4)^8} e^{ln(t+1)|_2^5} = \frac{100}{(1-d/4)^8} e^{ln(6)-ln(3)} = \frac{100}{(1-d/4)^8} e^{ln(6/3)} = \frac{100}{(1-d/4)^8} 2.$
So $1 - d/4 = (200/260)^{.125}$ so $d = 4 * (1 - (200/260)^{.125}) = 0.129$.

8. (Michigan1.3)
Deposits are to be made to a fund each January 1 and July 1 for the years 1995 through 2005. The deposit made on each July 1 will be 10.25% greater than the one made on the immediately preceding January 1. The deposit made on each January 1 (except for January 1, 1995) will be the same amount as the deposit made on the immediately preceding July 1. The fund will be credited with interest at a nominal annual rate of 10%, compounded semi-annually. On December 31, 2005, the fund will have a balance of 11000. Determine the initial deposit to the fund.
(A) 160 (B) 165 (C) 175 (D) 195 (E) 200
SOLUTION:

$$\begin{gathered} January \qquad July \\ K(1.05)^{22} + K(1.1025)(1.05)^{21} \\ +K(1.1025)(1.05)^{20} + K(1.1025)^2(1.05)^{19} \\ +\ldots+\ldots \\ +K(1.1025)^9(1.05)^2 + K(1.1025)^{10}(1.05)^1 \\ = 11K(1.1025)^{11} + 11K(1.1025)^{11}(1.05) \\ \\ = 65.96463K = 11000 \end{gathered}$$

Thus $K = 11000/65.96463 = 166.7560$.