DEPARTMENT OF MATHEMATICS AND STATISTICS
Theory of Interest 62-392 Test 1 M. Hlynka
Wednesday, February 13, 2008

Family Name ____________________ Given Name ____________________
Id. No. ____________________

Directions: For questions 1-7, you will be graded on your completeness as well as your correctness. You must show your work!
On Questions 8 and 9, only the final answer will be graded. For questions 8,9 only, no work need be shown and your work will not be looked at.

1. (MH question) A loan of \$2000 is to be repaid at the end of each of the next 5 years at an annual interest rate of .06. Find the annual payment.
SOLUTION:
$\$2000 = Ka_{\overline{5}|.06}$ so $K = 2000/a_{\overline{5}|.06} = \dfrac{2000(.06)}{1-1.06^{-5}} = 474.79.$

2. (MH question) How many years will it take to double your money if you invest an amount A now at a nominal rate of 4% compounded semiannually?
SOLUTION: $2 = (1+\dfrac{i^{(2)}}{2})^n = (1.02)^{2n}$ so $2n = ln(2)/ln(1.02) = 35.0028$ so $n = 17.50$.

3. (from Assignment 1, modified) Money accumulates in a fund at an effective annual interest rate of i during the first three years and at an effective annual interest rate of 3i thereafter. A deposit of 100 is made into a fund at time 0. It accumulates to 178.66 at the end of 10 years and to 368.21 at the end of 20 years. What is the value of the fund at the end of 8 years? SOLUTION:

$$368.21 = 178.66(1+3i)^{10} \text{ so } 1+3i = (368.21/178.66)^{.1} = 1.074996$$

Thus $i = .074996/3 = .024999$.
Value after 8 years is $100(1+i)^3(1+3i)^5 = 100(1.024999)^3)(1.074996)^5 = 154.60$.

4. (from Assignment 3, modified) Two annuities have the same present value. The first annuity is a 12-year annuity-immediate paying \$K per year. The second annuity is an 4-year annuity-immediate paying \$2K per year. Both annuities are based on an annual effective interest rate of i, $i > 0$. Determine i.
SOLUTION:
$Ka_{\overline{12}|} = 2Ka_{\overline{4}|}$ so $a_{\overline{12}|} = 2a_{\overline{4}|}$ and
$\dfrac{1-(1+i)^{-12}}{i} = 2\dfrac{1-(1+i)^{-4}}{i}$. Let $x = (1+i)^{-4}$. Then
$1-x^3 = 2(1-x)$ so $x^3 - 2x + 1 = 0$. One root is clearly $x = 1$ so $(x-1)$ is a factor.
Thus $(x-1)(x^2+x-1) = 0$. By the quadratic formula, $x = \dfrac{-1 \pm \sqrt{1+4}}{2}$. But

$x = (1+i)^{-4}$ lies between 0 and 1. So $(1+i)^{-4} = x = \dfrac{-1+\sqrt{5}}{2} = .618034$.
Thus $i = (.618034)^{-1/4} - 1 = .12784 = 12.784\%$.

5. In Fund X money accumulates at force of interest $\delta_t = .01t + .10$, for $0 < t < 20$. In Fund Y money accumulates at annual effective rate i. An amount of \$1 is invested in each fund, and the accumulated values are the same at the end of 20 years. Find the value in Fund Y at the end of 1.5 years.
SOLUTION: $A_X(t) = 1e^{\int_0^t \delta_r dr} = e^{.005t^2 + .1t}$. $A_Y(t) = (1+i)^t$
Thus $e^{.005(20)^2 + .1(20)} = e^4 = set = (1+i)^{20}$. We want to find
$(1+i)^{1.5} = ((1+i)^{20})^{1.5/20} = (e^4)^{1.5/20} = e^{.3} = 1.34986$

6. (May 2003, #12) Eric deposits X into a savings account at time 0, which pays interest at a nominal rate of i, compounded semiannually. Mike deposits $2X$ into a different savings account at time 0, which pays simple interest at an annual rate of i. Eric and Mike earn the same amount of interest during the last 6 months of the 8th year. Calculate i.
SOLUTION: $X(1+i/2)^{16} - X(1+i/2)^{15} = 2X(1+8i) - 2X(1+7.5i)$ so $(1+i/2)^{15} i/2 = 2(.5)i$ so $(1+i/2)^{15} = 2$ so $i/2 = 2^{1/15} - 1 = .04729$ so $i = .09458$.

7. (question from Exercise 2.1.8, modified) If $s_{\overline{n}|} = 60$, $s_{\overline{2n}|} = 240$, find $s_{\overline{3n}|}$.
SOLUTION: $s_{\overline{2n}|} = s_{\overline{n}|} + (1+i)^n s_{\overline{n}|}$ so $240 = 60 + 60(1+i)^n$.
Thus $(1+i)^n = 3$. Hence

$$s_{\overline{3n}|} = s_{\overline{n}|} + (1+i)^n s_{\overline{2n}|} = 60 + 3(240) = 780.$$

8. (May 2003,#22) A perpetuity costs 77.1 and makes annual payments at the end of the year. The perpetuity pays 1 at the end of year 2, 2 at the end of year $3, \ldots, n$ at the end of year $(n+1)$. After year $(n+1)$, the payments remain constant at n. The annual effective interest rate is 10.5%. Calculate n.
(A) 17 (B) 18 (C) 19 (D) 20 (E) 21
SOLUTION:

$$\begin{aligned}
77.1 &= 1v^2 + 2v^3 + 3v^4 + \cdots + nv^{n+1} + n(v^{n+2} + \ldots) = v(Ia)_{\overline{n}|} + nv^{n+1} a_{\overline{\infty}|} \\
&= v\frac{\ddot{a}_{\overline{n}|} - nv^n}{i} + nv^{n+1}\frac{1}{i} \\
&= \frac{a_{\overline{n}|} - nv^{n+1}}{i} + \frac{nv^{n+1}}{i} = \frac{a_{\overline{n}|}i}{i} \\
&= \frac{1 - v^n}{i^2}
\end{aligned}$$

Thus $v^n = 1 - 77.1(i^2)$ so $n = \dfrac{ln(1 - 77.1i^2)}{ln(v)} = \dfrac{-1.8973}{-.099845} = 19$

9. Simplify the expression $\left(\frac{d}{dv}\delta\right)\left(\frac{d}{di}d\right)$.

(A) $-v$ (B) $-v^3$ (C) 1 (D) v (E) v^3

SOLUTION:

$\frac{d}{dv}\delta = \frac{d}{dv}ln(1+i) = \frac{d}{dv}ln(v^{-1}) = \frac{d}{dv}(-ln(v)) = \frac{-1}{v}$.

$\frac{d}{di}d = \frac{d}{di}\frac{i}{1+i} = \frac{1}{(1+i)^2}$

Hence $\left(\frac{d}{dv}\delta\right)\left(\frac{d}{di}d\right) = \frac{-1}{v}\frac{1}{(1+i)^2} = \frac{-1}{1+i} = -v$.