Putnam Competition December 6, 2008.

# 1 Part A

A1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that
$f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers $x, y, z$.
Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

A2. Alan and Barbara play a game in which they take turns filling entries of an initially empty $2008 \times 2008$ array. Alan plays first. At each turn a player chooses a real number and places it in a vacant entry. The game ends when all entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

A3. Start with a finite sequence of positive integers $a_1, a_2, \ldots, a_n$. If possible, choose two indices $j < k$ such that $a_j$ does not divide $a_k$ and replace $a_j$ and $a_k$ by $gcd(a_j, a_k)$ and by $lcm(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop, and that the final sequence does not depend on the choices made.

A4. Define $f : \mathbb{R} \to \mathbb{R}$ by
$f(x) = \begin{cases} x & \text{if} \quad x \leq e \\ xf(\ln x) & \text{if} \quad x > e. \end{cases}$
Does $\sum_{n=1}^{\infty} \dfrac{1}{f(n)}$ converge?

A5. Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular $n$-gon in counterclockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n - 1$.

A6. Prove that there exists a constant $c > 0$ such that in every nontrivial finite group $G$ there exists a sequence of length at most $c \ln |G|$ with the property that each element of $G$ equals the product of some subsequence. (The elements of $G$ in the sequence are not required to be distinct. A subsequence of the sequence is obtained by selecting some of the terms, not necessarily consecutive, without reordering them; for example, 4,4,2 is a subsequence of 2,4,6,4,2, but 2,2,4 is not.)

B1. What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose center is not a rational point? (A rational point is a point both of whose coordinates are rational numbers.)

B2. Let $F_n(x) = \ln x$. For $n \geq 0$, and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t)dt$. Evaluate $\lim_{n\to\infty} \dfrac{n!F_n(1)}{\ln n}$.

B3. What is the largest possible radius of a circle contained in a 4 dimensional hypercube of side length 1?

B4. Let $p$ be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \ldots, h(p^2-1)$ are distinct modulo $p^2$. Show that $h(0), h(1), \ldots, h(p^3-1)$ are distinct modulo $p^3$.

B5. Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number f(q) is rational and has the same denominator as $q$. (The denominator of a rational number $q$ is the unique positive integer $b$ such that $q = a/b$ for some integer $a$ with $gcd(a, b) = 1$.)

B6. Let $n$ and $k$ be positive integers. Say that a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ is $k$-limited if $|\sigma(i) - i| \leq k$ for all $i$. Prove that the number of $k$-limited permutations of $\{1, 2, \ldots, n\}$ is odd iff $n \equiv 0$ or $1 \bmod 2k + 1$.