University of Windsor Undergraduate Mathematics Contest
October 14^{th}

Please explain your answers completely!

(1) Explain how to solve the following system of equations in your head:

$$\begin{aligned} 6751x + 3249y &= 26751 \\ 3249x + 6751y &= 23249. \end{aligned}$$

(2) ΔABC is such that $AB = 5$, $BC = 7$, and $AC = 9$. Point D is located between A and C with $BD = 5$. Use the Pythagorean Theorem and some algebra to find AD/DC. (Note that ΔABC is not a right angle triangle–how can you make right angle triangles so you can use the Pythagorean Theorem?)

(3) In this question, we explore a technique called *telescoping* where one takes advantage of cancellation to evaluate sums or products. Here is an example of telescoping: Evaluate $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{k-1} + \frac{1}{k-1}\right) + \left(-\frac{1}{k} + \frac{1}{k}\right) - \frac{1}{k+1} \\ &= 1 - \frac{1}{k+1}. \end{aligned}$$

(a) Evaluate the sum

$$\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{995}+\sqrt{997}} + \frac{1}{\sqrt{997}+\sqrt{999}}.$$

(b) Prove that $\displaystyle\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{997}+\sqrt{999}} < 24.$

(c) Compute the sum

$$\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \cdots + \sqrt{1 + \frac{1}{2007^2} + \frac{1}{2008^2}}.$$

(4) Find the sum of all real numbers x that satisfy

$$(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3.$$

(5) Show that the product of four consecutive positive integers can never be a perfect square.

				k			
					k		
						k	
							k

(6) Four knights are placed on a chess board as shown. Cut the chess board into four congruent pieces, each containing a knight. (Colouring doesn't matter and the knights may be on different locations on the congruent pieces.)