On Uniqueness and Dimensional Rigidity of Bar-and-Joint Frameworks

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What is a Framework?

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- An $r$-configuration is a mapping $p : V \rightarrow \mathbb{R}^r$, where $p^1, p^2, \ldots, p^n$ are not contained in a proper hyper-plane.
- A Bar-and-Joint Framework $G(p)$ in $\mathbb{R}^r$ is an $r$-configuration + a graph $G = (V, E)$ such that every two points corresponding to adjacent vertices of $G$ are constrained to stay the same distance apart.
Congruent versus Equivalent Frameworks

- $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be equivalent if $\|p^i - p^j\| = \|q^i - q^j\|$ for all $(i, j) \in E$. 
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![Diagram of congruent and equivalent frameworks](image-url)
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- $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be congruent if $\|p^i - p^j\| = \|q^i - q^j\|$ for all $i, j = 1, \ldots, n$. 
Rigidity and Global Rigidity

- $G(p)$ in $\mathbb{R}^r$ is said to be rigid if for some $\epsilon > 0$, there does not exist $G(q)$ in $\mathbb{R}^r$ which is equivalent to $G(p)$ such that $\|p^i - q^i\| < \epsilon$ for all $i = 1, \ldots, n$. 

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- $G(p)$ in $\mathbb{R}^r$ is said to be dimensionally rigid if there does not exist $G(q)$ in $\mathbb{R}^s$ for some $s \geq r + 1$ which is equivalent to $G(p)$. 
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In this talk we will present some new results on:
- global rigidity of frameworks.
- dimensional rigidity of frameworks.
- uniqueness of frameworks.
Generic Framework

- A framework $G(p)$ in $\mathbb{R}^r$ with $n$ vertices is said to be **generic** if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers.
A framework $G(p)$ in $\mathbb{R}^r$ with $n$ vertices is said to be generic if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, there does not exist a non-zero polynomial $f(x_1, \ldots, x_{nr})$ with integer coefficients such that $f(p^1_1, \ldots, p^n_1, \ldots, p^1_r, \ldots, p^n_r) = 0$. 

**Generic Framework**
Global rigidity

- Theorem (Hendrickson ’92): Let $G(p)$ be a generic framework in $\mathbb{R}^r$ with at least $r + 1$ vertices. If $G(p)$ is globally rigid, then graph $G = (V, E)$ is $r + 1$ vertex connected and $G(p)$ is redundantly rigid.
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- Graph $G$ is $k$ vertex connected if $G$ remains connected after deleting fewer than $k$ of its vertices.

- A framework $G(p)$ is said to be redundantly rigid if it is rigid, and it remains so even after deleting one edge of $G$. 
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Jackson and Jordán proved that Hendrickson’s conjecture is true for $r = 2$. 
Global rigidity

- Theorem (Jackson and Jordán ’05): Given a generic framework $G(p)$ in $\mathbb{R}^2$, then $G(p)$ is globally rigid in $\mathbb{R}^2$ if and only if $G$ is either a complete graph on at most 3 vertices or $G$ is 3-vertex connected and redundantly rigid.
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- Theorem (Connelly ’05): Given a generic framework $G(p)$ with $n$ vertices in $\mathbb{R}^r$, let $S$ be the stress matrix associated with an equilibrium stress $\omega$ for $G(p)$ such that $\text{rank } S = n - 1 - r$. Then $G(p)$ is globally rigid in $\mathbb{R}^r$. 
Stress Matrix $\mathcal{S}$

- For each $i, j = 1, \ldots, n, i \neq j$ let $\omega_{ij}$ be a scalar such that $\omega_{ij} = \omega_{ji}$ and $\omega_{ij} = 0$ if $(i, j) \notin E$. Then $\omega = (\ldots, \omega_{ij}, \ldots)$ is called a stress for $G(p)$. 
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\(\omega\) is an equilibrium stress for \(G(p)\) if:
\[
\sum_j \omega_{ij} (p_i - p_j) = 0 \text{ for all } i = 1, \ldots, n.
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Stress Matrix $S$

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- $\omega$ is an equilibrium stress for $G(p)$ if:
  $$\sum_j \omega_{ij} (p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n.$$

- The stress matrix $S = (s_{ij})$ is defined as:
  $$s_{ij} = -\omega_{ij} \text{ for } i \neq j,$$
  $$s_{ii} = \sum_j \omega_{ij} \text{ for } i = 1, \ldots, n.$$
Given $p^1, \ldots, p^n$ let $P^T = [p^1 p^2 \ldots p^n]$. Let $e = (1, 1, \ldots, 1)^T$. Let $\bar{r} = n - 1 - r$. 

\textbf{Gale Matrix $Z$}
Given $p^1, \ldots, p^n$ let $P^T = [p^1 \, p^2 \, \ldots \, p^n]$. Let $e = (1, 1, \ldots, 1)^T$. Let $\bar{r} = n - 1 - r$.

Let $\Lambda$ be the $n \times \bar{r}$ matrix whose columns for a basis for nullspace $\begin{bmatrix} P^T \\ e^T \end{bmatrix}$. 

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- Let $\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}$ and $Z = \Lambda \Lambda_1^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 \Lambda_1^{-1} \end{bmatrix}$. $Z$ is called the Gale matrix for $G(p)$. 
Lemma (Alfakih ’07): Given $G(p)$ in $\mathbb{R}^r$, let $Z$ be the Gale matrix for $G(p)$ and let $\bar{r} = n - 1 - r$. Further, let $S$ be the stress matrix for $G(p)$. Then

$$S = Z\Psi Z^T$$ for some $\bar{r} \times \bar{r}$ symmetric matrix $\Psi$.

Furthermore, if $\Psi'$ is any $\bar{r} \times \bar{r}$ symmetric matrix such that $z_i^T \Psi' z_j = 0$ for all $(i, j) \not\in E$, where $z_i$ is the $i$th row of $Z$. Then $Z\Psi' Z^T$ is a stress matrix for $G(p)$. 
Euclidean distance matrices (EDMs)

An $n \times n$ matrix $D = (d_{ij})$ is said to be an EDM iff $\exists \ p^1, p^2, \ldots, p^n \in \mathbb{R}^r$ such that

$$d_{ij} = \|p^i - p^j\|^2 \ \forall \ i, j = 1, \ldots, n.$$
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\]

- The dimension of the affine span of \( p^1, p^2, \ldots, p^n \) is called the embedding dim of \( D \).
Example

\[
D = \begin{bmatrix}
0 & 4 & 5 & 1 \\
4 & 0 & 1 & 5 \\
5 & 1 & 0 & 4 \\
1 & 5 & 4 & 0
\end{bmatrix}
\]
is an EDM.
Example

\[ D = \begin{bmatrix} 0 & 4 & 5 & 1 \\ 4 & 0 & 1 & 5 \\ 5 & 1 & 0 & 4 \\ 1 & 5 & 4 & 0 \end{bmatrix} \] is an EDM.

The points that generate \( D \) are:

\( p^1 \)

\( p^2 \)

\( p^3 \)

\( p^4 \)
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The points that generate $D$ are:

- $p^1$
- $p^2$
- $p^3$
- $p^4$

Embedding dim of $D$ is 2.
EDM Characterization

(Schoenberg ’35, Young and Householder ’38) An $n \times n$ symmetric $D$ with $\text{diag}(D) = 0$ is EDM iff $X := -\frac{1}{2}V^TDV \succeq 0$, and embedding dim of $D = \text{rank } X$, where $V$ is $n \times (n - 1)$ satisfying $V^Te = 0$, $V^TV = I_{n-1}$. 
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The points $p^1, \ldots, p^n$ that generate $D$ are given by the rows of $P$ where $V X V^T = P P^T$. 

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- The points $p^1, \ldots, p^n$ that generate $D$ are given by the rows of $P$ where $VXV^T = PP^T$.

- Since we don’t distinguish between congruent frameworks, $P$, $D$ and $X$ uniquely determine one another.
\( \Omega \), the set of all equivalent frameworks of \( G(p) \)

Given \( G(P_1) \) in \( \mathbb{R}^r \), let \( X_1 = V^T P_1 P_1^T V \) and for each \( (i, j) \notin E \), define \( M^{ij} = -\frac{1}{2} V^T E^{ij} V \).
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- Let
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\Omega := \{ y : \mathcal{X}(y) := X_1 + \sum_{(i,j) \notin E} y_{ij} M^{ij} \succeq 0 \}.
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- The set of all equivalent frameworks to \( G(P_1) \) in \( \mathbb{R}^r \) is:
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- The set of all equivalent frameworks to \( G(P_1) \) in all spaces is : \( \{ \mathcal{X}(y) : y \in \Omega \} \).
Properties of $\Omega$

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- $\Omega$ is bounded whenever graph $\mathcal{G}$ is connected.
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Ω, An Example

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Set $\Omega$ for $G(p)$.
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Characterizing Dimensional Rigidity

Theorem (Alfakih ’07): Let \( G(p) \) be a given framework in \( \mathbb{R}^r \) for some \( r \leq n - 2 \). If

\[
\exists \Psi > 0 : z^i^T \Psi z^j = 0 \quad \forall (i, j) \notin E, \quad (*)
\]

holds, then \( G(p) \) is dimensionally rigid. Otherwise, if \((*)\) does not hold, then \( G(p) \) is dimensionally flexible iff \( \exists y \neq 0 \) such that \( Z^T \mathcal{E}(y) Z \) is nonzero PSD and

\[
\text{nullspace } Z^T \mathcal{E}(y) Z \subseteq \text{nullspace } P^T \mathcal{E}(y) Z,
\]

where \( \mathcal{E}(y) = \sum_{(i,j) \notin E} y_{ij} E^{ij} \).
Some Corollaries

- Checking the validity of Condition (*) is a Semi-Definite Programming problem which can be solved efficiently.
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- Corollary (Alfakih ’07): Let $G(p)$ be a given framework in $\mathbb{R}^{n-2}$. Then Condition (*) is necessary and sufficient for $G(p)$ to be dimensionally rigid.
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- A framework $G(p)$ in $\mathbb{R}^r$ is in general position if no $r + 1$ of the points $p^i, \ldots, p^n$ are affinely dependent.
More Corollaries

- Corollary (Alfakih ’07): Let $G(p)$ be a given framework in $\mathbb{R}^{n-2}$. Assume that $G = (V, E)$ is not complete graph and $G(p)$ is in general position then $G(p)$ is dimensionally flexible.
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- Corollary (Alfakih ’07): Let $G(p)$ be a given framework in $\mathbb{R}^{r}$ for some $r \leq n - 2$. Assume that $G(p)$ is in general position. If $\delta(G) \leq r$ then $G(p)$ is dimensionally flexible.
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- These Corollaries are false if $G(p)$ is not in general position.
Examples

\[ \text{Dim rigid } G(p) \text{ where } \delta(G) = r = 2 \]
Examples

- Dim rigid $G(p)$ where $\delta(G) = r = 2$

- Dim rigid $G(p)$ where $r = n - 2 = 2$
Characterizing Uniqueness

- Theorem (Alfakih ’07): Let $G(p)$ be a given framework in $\mathbb{R}^r$ for some $r \leq n - 2$. If $G(p)$ is both rigid and dimensionally rigid, then $G(p)$ is unique.
Characterizing Uniqueness

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Theorem (Alfakih ’07): Let $G(p)$ be a given generic framework in $\mathbb{R}^r$ for some $r \leq n - 2$. If $G(p)$ is dimensionally rigid, then $G(p)$ is unique.
Finally

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- (Connelly ’05) If $\exists$ a nonsingular symmetric matrix $\Psi$ such that: $z^i \Psi z^j = 0 \ \forall (i, j) \notin E$, then $G(p)$ is globally rigid.

- (Alfakih ’07) If $\exists$ a positive definite symmetric matrix $\Psi$ such that: $z^i \Psi z^j = 0 \ \forall (i, j) \notin E$, then $G(p)$ is unique.