On the Set of Realizations of Edge-Weighted Graphs in Euclidean Spaces

Abdo Y. Alfakih *

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario N9B 3P4
Canada

February 22, 2005

AMS classification: 51K05, 90C22, 52A20, 05C50, 15A57
Keywords: graph realizations, Euclidean distance matrices, semidefinite programming, low rank solutions, Gale transform, convex sets.

Abstract

Let $G = (V, E, \omega)$ be an edge-weighted graph. A realization of G in $\Re^r$ is a mapping of the vertices $1, 2, \ldots, n$ of G into points $p^1, p^2, \ldots, p^n$ in $\Re^r$ such that $\|p^i - p^j\|^2 = \omega_{ij}$ for every edge $(i, j) \in E$. In this paper, we study the geometry of the set of all realizations of G, and we present a simple randomized algorithm for obtaining realizations of G in low dimensional Euclidean spaces. Some numerical results on randomly generated problems are also presented.

1 Introduction

Let $G = (V, E, \omega)$ be a given simple connected undirected graph, where $V = \{1, 2, \ldots, n\}$ is the set of vertices, E is the set of edges and $\omega : E \to \Re^+$ is a function which assigns to every edge $(i, j) \in E$ a positive number (weight) ω_{ij}. A realization of G in $\Re^r$ is a mapping of the vertices of G into points $p^1, p^2, \ldots, p^n$ in $\Re^r$ such that $\|p^i - p^j\|^2 = \omega_{ij}$ for every edge $(i, j) \in E$, where $\| \ \|$ denotes the Euclidean norm. The problem of determining whether or not a given edge-weighted

*E-mail: alfakih@uwindsor.ca

graph G has a realization in $\Re^r$ for some r, is known as the *graph realization problem (GRP)* or the *Euclidean distance matrix completion problem (EDMCP)*.

Recently, the GRP has received a great deal of attention due to its many applications in molecular conformations, protein folding, multidimensional scaling and wireless sensor network localization [9, 11, 19, 20, 27]. In most of these applications, obtaining a realization in the smallest possible dimension is desired. Unfortunately, it was shown by Saxe [24] that given a graph $G = (V, E, \omega)$ and an integer k, the problem of determining whether or not G has a realization in $\Re^k$ is NP-hard. In [7, 23, 14] it was shown that if G has a realization in $\Re^r$ for some r, then it has a realization in Euclidean space of dimension $\leq \lfloor \frac{\sqrt{8|E|+1}-1}{2} \rfloor$, and a method for obtaining such a realization was devised in [4]. This bound was later sharpened in [8] to $\lfloor \frac{\sqrt{8(|E|-1)+1}-1}{2} \rfloor$, however, devising a constructive proof of this latter result remains an open problem. Recently, a characterization of graphs that have realizations in $\Re^k$ for $k \leq 3$ was obtained in [10].

In this paper, using Euclidean distance matrices (EDMs), we study the set of all realizations of G, and we present a simple randomized algorithm for obtaining realizations of G in low dimensional Euclidean spaces. We also present some numerical results on randomly generated problems.

We denote by $\mathcal{S}_n$ the space of $n \times n$ real symmetric matrices. Positive semidefiniteness (positive definiteness) of a symmetric matrix A is denoted by $A \succeq 0$ ($A \succ 0$). For a matrix A in $\mathcal{S}_n$, diag(A) denotes the n-vector formed from the diagonal entries of A. We denote by e_n the vector of all ones in $\Re^n$; and we denote by E^{ij} the $n \times n$ symmetric matrix with ones in the (i, j)th and (j, i)th entries and zeros elsewhere. The identity matrix of order n will be denoted by I_n. Finally, $\backslash$ denotes the set theoretic difference.

2 Preliminaries

Let $G = (V, E, \omega)$ be a given edge-weighted connected simple graph with n nodes and m edges. Let $p^1, p^2, \ldots, p^n$ be given points in $\Re^r$ and assume that $p^1, p^2, \ldots, p^n$ are not contained in a proper hyperplane of $\Re^r$. Then the $n \times r$ matrix

$$P = \begin{bmatrix} {p^1}^T \\ {p^2}^T \\ \vdots \\ {p^n}^T \end{bmatrix}$$

is called a *realization* of G in $\Re^r$, or simply an r-*realization* of G, if

$$\|p^i - p^j\|^2 = \omega_{ij} \text{ for all } (i,j) \in E.$$

Let P be a realization of G in $\Re^r$. Then the $n \times n$ matrix $D_P = (d_{ij}) = \|p^i - p^j\|^2$ is called the *Euclidean distance matrix (EDM)* defined by P. Two r-realizations P and P' of G are said to be *congruent* if $D_P = D_{P'}$. Note that D_P is invariant under translations and rotations in $\Re^r$. Thus, all realizations P' obtained from P by a rigid motion are congruent to P. In order to avoid trivialities, no distinction between congruent realizations will be made. Consequently, we can assume without loss of generality that in all realizations P of G, the centroid of the points $p^1, p^2, \ldots, p^n$ coincides with the origin; i.e., $P^T e_n = 0$.

2.1 Equivalent Representations of a Realization P

Let P be an r-realization of G. In this subsection we use Euclidean distance matrices (EDMs) to present two equivalent representations of P.

An $n \times n$ matrix $D = (d_{ij})$ is said to be a *Euclidean distance matrix (EDM)* if and only if there exist points $p^1, p^2, \ldots, p^n$ in some Euclidean space such that $d_{ij} = \|p^i - p^j\|^2$ for all $i, j = 1, \ldots, n$. The dimension of the affine subspace spanned by $p^1, \ldots, p^n$ is called the *embedding dimension* of D.

It is well known [12, 13, 17, 26] that a symmetric $n \times n$ matrix D with zero diagonal is EDM if and only if D is negative semidefinite on the subspace

$$M := e_n^\perp = \{x \in \Re^n : e_n^T x = 0\}.$$

Let V be the $n \times (n-1)$ matrix whose columns form an orthonormal basis of M; that is, V satisfies:

$$V^T e_n = 0\,, \quad V^T V = I_{n-1}\,. \tag{1}$$

Then the orthogonal projection on M, denoted by J, is given by $J := VV^T = I_n - e_n e_n^T/n$. Hence, it immediately follows that if D is a symmetric matrix with zero diagonal, then

$$D \text{ is an EDM iff } \quad \mathcal{T}(D) := -\frac{1}{2}\, JDJ \succeq 0. \tag{2}$$

Furthermore, it is also well known that the embedding dimension of D is given by the rank of $\mathcal{T}(D)$.

Next we show that a given r-realization P of G can be equivalently represented by D_P, the EDM matrix defined by P. Recall that $D_P = (d_{ij}) = \|p^i - p^j\|^2$, which is equivalent to

$$D_P = \mathcal{K}(PP^T) := \operatorname{diag}(PP^T)\, e_n^T + e_n\, (\operatorname{diag}(PP^T))^T - 2PP^T. \tag{3}$$

Let $\mathcal{S}_H$ and $\mathcal{S}_C$ denote the subspaces of $\mathcal{S}_n$ defined by

$$\begin{aligned} \mathcal{S}_H :=& \quad \{B \in \mathcal{S}_n : \operatorname{diag}(B) = 0\}, \\ \mathcal{S}_C :=& \quad \{B \in \mathcal{S}_n : Be_n = 0\}. \end{aligned}$$

Then it can be easily verified that the linear operators $\mathcal{T} : \mathcal{S}_H \to \mathcal{S}_C$ and $\mathcal{K} : \mathcal{S}_C \to \mathcal{S}_H$ are mutually inverse [12]. Therefore, given an EDM D_P with embedding dimension r, the realization P that defines D_P can be recovered as follows. Let $B = \mathcal{T}(D_P)$. Then $B \succeq 0$, $B\, e_n = 0$, and rank $B = r$. Thus, P is obtained by factorizing B as $B = PP^T$. Note that $P^T e_n = 0$ since $B\, e_n = 0$. Also, note that the factorization of B into PP^T is not unique. However, if $B = PP^T = P'P'^T$, then the two r-realizations P and P' are congruent. Thus, P and D_P uniquely determine each other.

Next, we present a third equivalent representation of an r-realization P, which happens to be the most convenient for our purposes. Recall that $\mathcal{S}_{n-1}$ denotes the space of symmetric matrices of order $n-1$; and consider the two linear operators $\mathcal{K}_V : \mathcal{S}_{n-1} \to \mathcal{S}_H$ and $\mathcal{T}_V : \mathcal{S}_H \to \mathcal{S}_{n-1}$ such that

$$\mathcal{K}_V(X) := \mathcal{K}(VXV^T), \tag{4}$$

and

$$\mathcal{T}_V(D) := V^T \mathcal{T}(D) V = -\frac{1}{2} V^T D V, \tag{5}$$

where V is the $n \times (n-1)$ matrix defined in (1). Then we have the following lemma.

Lemma 2.1 ([3]) *The operators $\mathcal{T}_V$ and $\mathcal{K}_V$ are mutually inverse; and D in $\mathcal{S}_H$ is a Euclidean distance matrix of embedding dimension r if and only if $\mathcal{T}_V(D) \succeq 0$ and rank $\mathcal{T}_V(D) = r$.*

Therefore, let D_P be the EDM matrix defined by the r-realization P and let $X = \mathcal{T}_V(D)$. Then, D_P and X uniquely determine each other. Furthermore, we have the following relations.

$$\begin{array}{lclcl} X & = & -\frac{1}{2} V^T D_P V & = & V^T P P^T V, \\ D_P & = & \mathcal{K}_V(X) & = & \mathcal{K}(PP^T), \\ PP^T & = & VXV^T & = & \mathcal{T}(D_P). \end{array} \tag{6}$$

Hence, P, D_P and X uniquely determine one another. In a slight abuse of notation, we will use the term r-realization to refer to X as well as to P.

2.2 EDMs and Gale Transform

Let P be a given realization of G in $\Re^r$. Consider the $(r+1) \times n$ matrix

$$\begin{bmatrix} P^T \\ e_n^T \end{bmatrix} = \begin{bmatrix} p^1 & p^2 & \dots & p^n \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Since $p^1, \dots, p^n$ are not contained in a proper hyperplane in $\Re^r$; that is, the affine subspace spanned by $p^1, \dots, p^n$ has dimension r, it follows that $r \leq n-1$, and that $\begin{bmatrix} P^T \\ e_n^T \end{bmatrix}$ has full row rank. Let $\bar{r} = n - 1 - r$. For $\bar{r} \geq 1$, let Λ be the $n \times \bar{r}$ matrix, whose columns form a basis for null space $\begin{bmatrix} P^T \\ e_n^T \end{bmatrix}$. Λ is called a *Gale matrix* corresponding to P; and the ith row of Λ, considered as a vector in $\Re^{\bar{r}}$, is called a *Gale transform* of p^i [15]. Note that Λ is not unique. In fact, for any nonsingular $\bar{r} \times \bar{r}$ matrix Q, the columns of ΛQ span null space $\begin{bmatrix} P^T \\ e_n^T \end{bmatrix}$. Hence, ΛQ is also a Gale matrix. We will exploit this property to define a special Gale matrix Z which is more sparse than Λ and more convenient for our purposes.

Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is $\bar{r} \times \bar{r}$ and Λ_2 is $(r+1) \times \bar{r}$. Since Λ has full column rank, we can assume without loss of generality that Λ_1 is nonsingular. Then Z is defined as

$$Z := \Lambda \Lambda_1^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 \Lambda_1^{-1} \end{bmatrix}. \tag{7}$$

Let P and P' be two r-realizations of an EDM D. Then $B = \mathcal{T}(D) = PP^T = P'P'^T$. Therefore, null space P^T = null space P'^T. Hence, Z is uniquely determined by D. The following lemma establishes a connection between an EDM D and the Gale matrix Z corresponding to it.

Lemma 2.2 **([2])** *Let P be a given r-realization of edge-weighted graph G, and let Z be the Gale matrix corresponding to P. Further, let U and W be the matrices whose columns form orthonormal bases of the null space and the range space of $X = V^T PP^T V$ respectively, where V is defined in (1). Then*

1. *$VU = ZQ$ for some $\bar{r} \times \bar{r}$ nonsingular matrix Q, i.e., VU is a Gale matrix corresponding to P.*

2. *$VW = PQ'$ for some $r \times r$ nonsingular matrix Q'.*

Proof. Statement 1 holds since $XU = 0$ iff $P^TVU = 0$ and since $e_n^TVU = 0$. On the other hand, since $Z^TVW = Q^{-T}U^TV^TVW = 0$ and since $e_n^TVW = 0$, statement 2 also holds.

□

3 The Set of Realizations of $G = (V, E, \omega)$

Given an r_0-realization P_0 of $G = (V, E, \omega)$, where $|V| = n$ and $|E| = m$, let $\bar{G} = (V, \bar{E})$ denote the complement graph of G and let $\bar{m}$ be the cardinality of $\bar{E}$; i.e., $\bar{m} = n(n-1)/2 - m$. To avoid trivialities, assume that G is not a complete graph, thus $\bar{m} \geq 1$. For each edge of the complement graph $\bar{G}$ define the matrix

$$M^{ij} := \mathcal{T}_V(E^{ij}) = -\frac{1}{2}V^TE^{ij}V, \tag{8}$$

where E^{ij} is the $n \times n$ matrix with ones in the (i,j)th and (j,i)th entries and zeros elsewhere. It is easy to see that $\{M^{ij} : (i,j) \in \bar{E}\}$ is linearly independent. Let $X_0 = V^TP_0\,P_0^TV$, and define the linear map $\mathcal{X} : \Re^{\bar{m}} \to \mathcal{S}_{n-1}$ by

$$\mathcal{X}(y) := X_0 + \sum_{(i,j)\in\bar{E}} y_{ij}M^{ij}. \tag{9}$$

Let

$$\Omega = \{y \in \Re^{\bar{m}} : \mathcal{X}(y) \succeq 0\}. \tag{10}$$

Then it was shown in [1] that the set of all realizations of G in $\Re^r$ is given by

$$\{\mathcal{X}(y) : y \in \Omega \text{ and rank } \mathcal{X}(y) = r\}; \tag{11}$$

and that the set of all realizations of G in $\Re^s$, for some s, $1 \leq s \leq n-1$, is given by

$$\{\mathcal{X}(y) : y \in \Omega\}. \tag{12}$$

Note that Ω is a closed convex and generally nonpolyhedral set. Furthermore, Ω contains the origin since $\mathcal{X}(0) = X_0 \succeq 0$, and it is bounded whenever G is connected.

Next we present an equivalent definition of the set Ω which is convenient for computational purposes. In particular, it simplifies the task of describing Ω in terms of a finite set of nonlinear inequalities; and it leads to sparse optimization problems involving Ω, a property we will exploit in section 5.

The columns of matrix $\begin{bmatrix} -e_{n-1}^T \\ I_{n-1} \end{bmatrix}$ form a basis, albeit non-orthonormal, of the subspace M, the orthogonal complement of e_n. (The matrix $\begin{bmatrix} -e_{n-1}^T \\ I_{n-1} \end{bmatrix}$ was implicitly used in Schoenberg's original [26] characterization of EDMs). Therefore, there exists a nonsingular $(n-1) \times (n-1)$ matrix Q such that $VQ = \begin{bmatrix} -e_{n-1}^T \\ I_{n-1} \end{bmatrix}$. Recall that $\mathcal{X}(y) = -\frac{1}{2} V^T(D_{P_0} + \sum_{(i,j)\in\bar{E}} y_{ij}E^{ij}) V$. Thus

$$\Omega = \{y \in \Re^{\bar{m}} : \frac{1}{2} [e_{n-1} \quad -I_{n-1}] \, (D_{P_0} + \sum_{(i,j)\in\bar{E}} y_{ij}E^{ij}) \begin{bmatrix} -e_{n-1}^T \\ I_{n-1} \end{bmatrix} \succeq 0\}. \qquad (13)$$

Example 3.1 *Consider the weighted cycle graph C_4 with weights $\omega_{12} = \omega_{34} = 2$ and $\omega_{14} = \omega_{23} = 1$. It is easy to see that*

$$P_0 = \begin{bmatrix} -1 & 1/2 \\ 1 & 1/2 \\ 1 & -1/2 \\ -1 & -1/2 \end{bmatrix}, \quad \textit{and } D_{P_0} = \begin{bmatrix} 0 & 4 & 5 & 1 \\ 4 & 0 & 1 & 5 \\ 5 & 1 & 0 & 4 \\ 1 & 5 & 4 & 0 \end{bmatrix},$$

are a realization of this weighted graph in $\Re^2$ and its corresponding EDM respectively. From (13) and the fact that a symmetric matrix A is positive semidefinite iff every principle minor of A is nonnegative, it follows that Ω (see Figure 1) can be described by the following inequalities

$$\begin{aligned} -4 \leq \quad & y_{13} && \leq 4, \\ -4 \leq \quad & y_{24} && \leq 4, \\ & y_{13} + y_{24} && \leq 0, \\ 5y_{13} + 5y_{24} + & y_{13}y_{24} && \geq -16. \end{aligned}$$

Obviously, in this case Ω is non-polyhedral. Furthermore, the interior points, the smooth boundary points, and the singular boundary points of Ω correspond, respectively, to realizations of this graph in $\Re^3$, $\Re^2$ and $\Re$. Realizations corresponding to the origin (a smooth boundary point) and the two singular boundary points y^1, y^2 are also given in Figure 1. Definitions of smooth and singular boundary points of a convex set are given in Section 4.

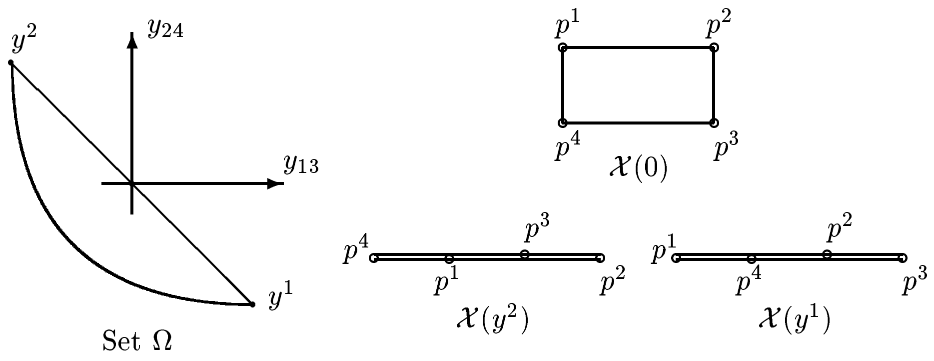

Figure 1: The set Ω corresponding to the cycle graph C_4 with weights $\omega_{12} = \omega_{34} = 2$ and $\omega_{14} = \omega_{23} = 1$. The realizations $\mathcal{X}(0)$, $\mathcal{X}(y^1)$ and $\mathcal{X}(y^2)$ are given to the right.

Suppose X_0 does not have the highest rank among all realizations $\{\mathcal{X}(y) : y \in \Omega\}$ of G. Then a realization of G with the highest rank among all realizations $\{\mathcal{X}(y) : y \in \Omega\}$ can be obtained, for example, by solving the semidefinite programming (SDP) formulation of the Euclidean distance matrix completion problem presented in [3]. This follows since various path-following interior-point algorithms for SDP compute the max-rank solution [16, 18, 21]. Therefore, without loss of generality we can make the following assumption.

Assumption 3.1 *The given realization $X_0 = \mathcal{X}(0)$ of G has the highest rank among all realizations $\{\mathcal{X}(y) : y \in \Omega\}$ of G.*

Now suppose that rank $X_0 = \mathcal{X}(0) < n-1$, i.e., G has no realization in $\Re^{n-1}$. Then Ω has an empty interior and is contained in a proper subspace $\mathcal{L}$ of $\Re^{\bar{m}}$. In such case, we will find it convenient to express Ω as the intersection of $\mathcal{L}$ with a convex set $\bar{\Omega}$ whose interior is nonempty.

Let W_0 and U_0 be the matrices whose columns for an orthonormal bases for the range space and the null space of X_0 respectively, where $X_0 = V^T P_0 P_0^T V$. Let $\mathcal{L}$ be the subspace of $\Re^{\bar{m}}$ defined by

$$\begin{array}{ll} \mathcal{L} := \Re^{\bar{m}} & \text{if } r_0 = n-1, \\ \mathcal{L} := \{ y \in \Re^{\bar{m}} : \sum_{(i,j)\in \bar{E}} y_{ij} M^{ij} U_0 = 0 \ \} & \text{if } r_0 \leq n-2, \end{array} \tag{14}$$

where $r_0 = $ rank X_0. Also define the set $\bar{\Omega}$ as

$$\bar{\Omega} := \{ y \in \Re^{\bar{m}} : W_0^T \mathcal{X}(y)\, W_0 \succeq 0 \}. \tag{15}$$

Note that $\bar{\Omega}$ has nonempty interior. Then we have.

Lemma 3.1 *Given an r_0-realization P_0 of edge-weighted graph G, then in light of Assumption 3.1 we have*

$$\Omega = \bar{\Omega} \cap \mathcal{L},$$

where Ω, $\bar{\Omega}$, and $\mathcal{L}$ are defined in (10),(15), and (14) respectively.

Proof. Let $Q = [W_0 \; U_0]$ and let $\mathcal{M}(y) = \sum_{(i,j)\in \bar{E}} y_{ij} M^{ij}$. Then $\mathcal{X}(y) \succeq 0$ iff $Q^T \, \mathcal{X}(y) \, Q \succeq 0$. But

$$Q^T \, \mathcal{X}(y) \, Q = \begin{bmatrix} W_0^T X_0 W_0 + W_0^T \mathcal{M}(y) \, W_0 & W_0^T \mathcal{M}(y) \, U_0 \\ U_0^T \mathcal{M}(y) W_0 & U_0^T \mathcal{M}(y) \, U_0 \end{bmatrix}.$$

It is obvious that $\bar{\Omega} \cap \mathcal{L} \subseteq \Omega$. On the other hand let $y \in \Omega$ then $y \in \bar{\Omega}$, $U_0^T \mathcal{M}(y) \, U_0 \succeq 0$ and null space $U_0^T \mathcal{M}(y) \, U_0 \subseteq$ null space $W_0^T \mathcal{M}(y) \, U_0$. Now by Assumption 3.1, $U_0^T \mathcal{M}(y) \, U_0 = 0$. Thus $W_0^T \mathcal{M}(y) \, U_0 = 0$ and $y \in \mathcal{L}$ and the result follows.

□

Using Lemma 2.2, one can easily show that

$$\bar{\Omega} = \{ y \in \Re^{\bar{m}} : (P_0^T P_0)^2 - \frac{1}{2} \sum_{(i,j)\in \bar{E}} y_{ij} \, P_0^T E^{ij} P_0 \succeq 0 \},$$

and that for $r_0 \leq n-2$,

$$\mathcal{L} := \{ y \in \Re^{\bar{m}} : \sum_{(i,j)\in \bar{E}} y_{ij} \, V^T E^{ij} Z_0 = \sum_{(i,j)\in \bar{E}} y_{ij} \, E^{ij} Z_0 = [\alpha_1 e_n \;\; \alpha_2 e_n \;\; \cdots \;\; \alpha_{\bar{r}_0} e_n] \, \},$$

for some scalars $\alpha_1, \ldots, \alpha_{\bar{r}_0}$, where Z_0 is the Gale matrix corresponding to P_0 and $\bar{r}_0 = n - 1 - r_0$.

4 The Facial Structure of Ω

A subset F of a closed convex set $K \subset \Re^n$ is called a *face* of K if for all $x \in F$, $y, z \in K$, and $0 < \alpha < 1$, $x = \alpha y + (1-\alpha) z$ implies that $y, z \in F$. Note that $F = \emptyset$ and $F = K$ are faces of K. A nonempty face $F \neq K$ is called a *proper* face of K, and a face of dimension 0 is called an *extreme point* of K. The boundary of K, denoted by ∂K, is the union of all proper faces of K.

For a point $y \in K$, let $F(y)$ denote the smallest face of K containing y. Let $\mathcal{P}_n$ denote the cone of positive semidefinite matrices of order n. The following result is well known [5].

Lemma 4.1 *Let A, $B \in \mathcal{P}_n$. Then*

$$B \in F(A) \text{ iff } \textit{ null space } A \subseteq \textit{ null space } B.$$

Furthermore, every face of $\mathcal{P}_n$ is of the form

$$F_L = \{ B \succeq 0 : L \subseteq \textit{ null space } B \},$$

for some subspace L of $\Re^n$.

Also well known (see for example [28, page 116]) is the fact that if K_1, K_2 are two closed convex sets, and if F_1, F_2 are two faces of K_1 and K_2 respectively, then $F_1 \cap F_2$ is a face of $K_1 \cap K_2$. Furthermore, every face of $K_1 \cap K_2$ is of this form. Hence, the following result follows immediately.

Lemma 4.2 *Let $\bar{y}$, $y \in \Omega$, and let $F(\bar{y})$ denote the smallest face of Ω containing $\bar{y}$. Then*

$$y \in F(\bar{y}) \text{ iff } \text{ null space } \mathcal{X}(\bar{y}) \subseteq \text{ null space } \mathcal{X}(y).$$

Furthermore, every face of Ω is of the form

$$F_L = \{y \in \Omega : L \subseteq \text{ null space } \mathcal{X}(y)\},$$

for some subspace L of $\Re^{n-1}$, where $\mathcal{X}(.)$ is defined in (9).

Lemma 4.3 *Let $y^1, y^2 \in \Omega$. Then $F(y^1) = F(y^2)$ if and only if the realizations $\mathcal{X}(y^1)$ and $\mathcal{X}(y^2)$ have the same Gale matrix.*

Proof. This follows from Lemma 2.2 and the fact that $F(y^1) = F(y^2)$ if and only if null space $\mathcal{X}(y^1) =$ null space $\mathcal{X}(y^2)$.

□

Let $\hat{F}$ denote the relative interior of a face F of K. Then the family of sets $\mathcal{F}_K = \{\hat{F}_i : F_i \text{ is a face of } K\}$ is a partition of K; i.e., $\hat{F}_i \cap \hat{F}_j = \emptyset$ for every $i \neq j$ and $K = \cup_i \hat{F}_i$ (see [28, page 116]). It is well known [6] that if F is a face of $\mathcal{P}_n$, then $F = F(A)$ if and only if $A \in \hat{F}$. Therefore, if we denote the interior of Ω by int Ω, then Lemma 4.3 shows that there is a one-to-one correspondence between $\mathcal{F}_\Omega \backslash \{\text{int } \Omega\}$ and $\{Z : Z$ is the Gale matrix corresponding to realization $\mathcal{X}(y)$ for all $y \in \Omega\}$.

Example 4.1 *Consider the set Ω given in Figure 1. $\hat{F}(0)$ is given by the open line segment $]y^1, y^2[$. Figure 2 depicts the realizations corresponding to several points in $\hat{F}(0)$. Obviously, in all these realizations we have $p^1 + p^3 = p^2 + p^4$. Thus all realizations have the same Gale matrix $Z = [1 \ , \ -1 \ , \ 1 \ , \ -1]^T$.*

Given $x_0 \in K \subset \Re^n$, the normal cone to K at x_0, denoted by $N_K(x_0)$, is defined by

$$N_K(x_0) = \{c \in \Re^n : c^T x_0 \geq c^T x \text{ for all } x \in K\}.$$

Thus, $c \in \Re^n$ is in $N_K(x_0)$ if the hyperplane $c^T x = c^T x_0$ is a supporting hyperplane of K at x_0. Note that $N_K(x_0)$ is a closed convex cone. Also note that $0 \in N_K(x)$ for all $x \in K$ and $N_K(x) = \{0\}$ if and only if x is in the interior of K.

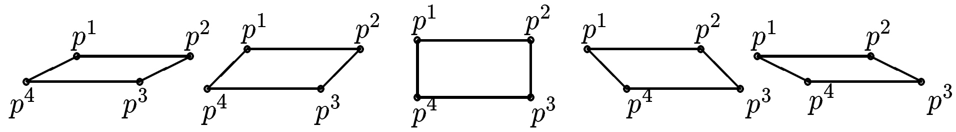

Figure 2: Realizations corresponding to several points in $\hat{F}(0) =]y^1, y^2[$ of the set Ω of Figure 1. Note that all these realizations have the same Gale matrix Z.

Normal cones are helpful in classifying the boundary points of a closed convex set $K \subset \Re^n$. Let x_0 be a boundary point of K, then x_0 is said to be a *vertex of order* α if the dimension of $N_K(x_0) = n - \alpha$. A vertex of order $n-1$ is called a *regular* or *smooth* point of K since K has a unique supporting hyperplane at such a vertex. On the other hand, vertices of order α for $0 \leq \alpha \leq n-2$ are called *singular* points of K.

The following lemma is well known [25, page 72].

Lemma 4.4 *Let $y^1, y^2 \in \partial K$. Then $N_K(y^1) = N_K(y^2)$ if and only if $F(y^1) = F(y^2)$.*

Lemma 4.5 *Let $\bar{y} \in \Omega$ and let $\bar{U}$ be the matrix whose columns form an orthonormal basis for null space $W_0^T \mathcal{X}(\bar{y})\, W_0$. Then*

$$N_\Omega(\bar{y}) = N_{\bar{\Omega}}(\bar{y}) + \mathcal{L}^\perp,$$

where $\bar{\Omega}$ is defined in (15), $\mathcal{L}^\perp$ is the orthogonal complement of $\mathcal{L}$ defined in (14), and

$$N_{\bar{\Omega}}(\bar{y}) = \{c \in \Re^{\bar{m}} : c_{ij} = -trace\,(\bar{U}^T W_0^T M^{ij}\, W_0 \bar{U} \Psi) \text{ for some } \Psi \succeq 0\},$$

where M^{ij} is defined in (8).

Proof. The fact that $N_\Omega(\bar{y}) = N_{\bar{\Omega}}(\bar{y}) + \mathcal{L}^\perp$ follows from Lemma 3.1 (see [25, page 71]). Now let $c \in \Re^{\bar{m}}$ where $c_{ij} = -\text{trace}\,(\bar{U}^T W_0^T M^{ij}\, W_0 \bar{U} \Psi)$ for some $\Psi \succeq 0$. Then for all $y \in \bar{\Omega}$ we have $c^T(\bar{y} - y) = \sum_{(i,j)\in \bar{E}} (y_{ij} - \bar{y}_{ij})$ trace $(W_0^T M^{ij}\, W_0 \bar{U} \Psi \bar{U}^T)$ $=$ trace $(W_0^T(\mathcal{X}(y) - \mathcal{X}(\bar{y}))\, W_0\, \bar{U}\Psi\bar{U}^T) =$ trace $(W_0^T \mathcal{X}(y)\, W_0\, \bar{U}\Psi\bar{U}^T) \geq 0$ since the matrices $W_0^T \mathcal{X}(y) W_0$ and $\bar{U}\Psi\bar{U}^T$ are positive semidefinite. Hence, $c \in N_{\bar{\Omega}}(\bar{y})$.

On the other hand let $c \in N_{\bar{\Omega}}(\bar{y})$ and consider the following pair of dual semidefinite programming problems

$$\begin{array}{lll} & p^* = \max & c^T y \\ \text{(Primal)} & \text{subject to :} & \\ & & W_0^T \mathcal{X}(y) W_0 \succeq 0, \end{array}$$

and

$$
\begin{array}{lll}
 & d^* = \min & \text{trace } (W_0^T X_0 W_0 Y) \\
\text{(Dual)} & \text{subject to :} & \\
 & & \text{trace } (W_0^T M^{ij} W_0 Y) = -c_{ij} \quad \text{for all } (i,j) \in \bar{E} \\
 & & Y \succeq 0.
\end{array}
$$

Now it follows from the definition of $N_{\bar{\Omega}}(\bar{y})$ that $p^* = c^T \bar{y}$. Since the Slater constraint qualification condition holds for the primal problem ($W_0^T X_0 W_0 \succ 0$), it follows from the strong duality of semidefinite programming that there exists $Y^* \succeq 0$ such that trace $(W_0^T M^{ij} W_0 Y^*) = -c_{ij}$, and that $d^* =$ trace $(W_0^T X_0 W_0 Y^*)$ $= p^* = c^T \bar{y}$. This implies that trace $(W_0^T \mathcal{X}(\bar{y}) W_0 Y^*) = 0$. Hence, $Y^* = \bar{U} \Psi \bar{U}^T$ for some $\Psi \succeq 0$. Therefore, $c_{ij} = -$trace $(\bar{U}^T W_0^T M^{ij} W_0 \bar{U} \Psi)$ and the the result follows.

□

The following are two immediate corollaries.

Corollary 4.1 *The supporting hyperplanes to $\bar{\Omega}$ at $\bar{y}$ are given by*

$$\sum_{(i,j)\in \bar{E}} (y_{ij} - \bar{y}_{ij}) \; \textit{trace} \; (\bar{U}^T W_0^T M^{ij} W_0 \bar{U} \Psi) = \textit{trace} \; (\bar{U}^T W_0^T \mathcal{X}(y) W_0 \bar{U} \Psi) = 0,$$

for some $\Psi \succeq 0$ where $\bar{U}$ is as defined in Lemma 4.5. In particular if $\mathcal{L} = \Re^{\bar{m}}$, i.e., if $\Omega = \bar{\Omega}$, then these hyperplanes are given by

$$\sum_{(i,j)\in \bar{E}} (y_{ij} - \bar{y}_{ij}) \; \textit{trace} \; (\bar{Z}^T E^{ij} \bar{Z} \Psi) = 0,$$

for some $\Psi \succeq 0$, where $\bar{Z}$ is the Gale matrix corresponding to $\mathcal{X}(\bar{y})$.

Corollary 4.2 *Assume $\Omega = \bar{\Omega}$ and let $y \in \partial \Omega$. If dim $N_\Omega(y) \geq \frac{s(s+1)}{2}$ for some positive integer s, then rank $\mathcal{X}(y) \leq n - 1 - s$.*

Corollary 4.2 is the motivation for our randomized algorithm for obtaining a realization of G in low dimensional Euclidean spaces presented in Section 5. The following result is well known and proofs of equivalent statements are given in [7, 14, 23].

Lemma 4.6 *Let $\hat{y}$ be an extreme point of $\Omega \subset \Re^{\bar{m}}$. Then*

$$\textit{rank} \; \mathcal{X}(\hat{y}) \leq \lfloor \frac{\sqrt{8|E| + 1} - 1}{2} \rfloor.$$

Proof. Let rank $\mathcal{X}(\hat{y}) = r$ and let W and U be the matrices whose columns form an orthonormal bases for the range space and the null space of $\mathcal{X}(\hat{y})$ respectively. Then the minimal face defined by $\hat{y}$ is given by $F(\hat{y}) = \{y \in \Omega : \mathcal{X}(y)U = 0\}$. Thus $y \in F(\hat{y})$ iff y is a solution of the following system of equations

$$\sum_{(i,j)\in\bar{E}} (y_{ij} - \hat{y}_{ij})M^{ij} = W\Phi W^T, \tag{16}$$

where Φ is a symmetric $r \times r$ matrix. (16) is a homogeneous system of $n(n-1)/2$ equations in the $\bar{m}+r(r+1)/2$ variables $(y_{ij} - \hat{y}_{ij})$ and Φ_{ij}. Thus if $r^2+r-2|E| > 0$, i.e., if $r > \lfloor \frac{\sqrt{8|E|+1}-1}{2} \rfloor$ then system (16) has a solution other than $\hat{y}$, hence $\hat{y}$ is not an extreme point.

$\square$

The bound in Lemma 4.6 was later sharpened by Barvinok [8] in the following lemma

Lemma 4.7 *Let $\Omega \subset \Re^{\bar{m}}$ be bounded, then there exists a point $\hat{y} \in \Omega$ such that*

$$\text{rank } \mathcal{X}(\hat{y}) \leq \lfloor \frac{\sqrt{8(|E|-1)+1}-1}{2} \rfloor.$$

5 Obtaining Realizations in Low Dimensions

As was mentioned in the introduction, obtaining a realization of a weighted graph G in the Euclidean space of the lowest possible dimension is a problem of great interest in many practical applications of the graph realization problem. Since this problem is NP-hard, an efficient algorithm for finding a point $\bar{y} \in \Omega$ such that $\bar{y} = \arg\min \{$ rank $\mathcal{X}(y) : y \in \Omega\}$ is unlikely to exist. Therefore, instead of trying to obtain $\bar{y}$, we will concentrate on obtaining a $y \in \Omega$ such that $\mathcal{X}(y)$ is of low rank.

A simple constructive proof of Lemma 4.6 was given in [4]. This proof consists of a method, which we present next, for constructing a realization of G in $\Re^k$ for some $k \leq \lfloor \frac{\sqrt{8|E|+1}-1}{2} \rfloor$ by obtaining an extreme point of Ω. Let y^0 be any point in Ω. If y^0 is not an extreme point of Ω, then system (16) has a solution $y \neq y^0$. Let $d = y - y^0$ then $d \neq 0$. By moving from y^0 along direction d until we reach the boundary of the face $F(y^0)$ we obtain a point $y^1 \in \Omega$ such that rank $\mathcal{X}(y^1) <$ rank $\mathcal{X}(y^0)$. This process can be repeated until an extreme point of Ω is obtained. On the other hand, devising a constructive proof of the sharpened bound in Lemma 4.7 is still an open problem.

The simple method discussed above differentiates only between extreme points and non-extreme points of Ω. But as we saw in Example 3.1, different extreme points

y's of Ω can have $\mathcal{X}(y)$'s of different ranks. Normal cones are helpful in classifying extreme points of Ω. Corollary 4.2 suggests that an extreme point $y \in \Omega$ with a high dimensional normal cone tends to have a low rank realization $\mathcal{X}(y)$. Thus we can exploit this fact to present the following simple randomized algorithm for obtaining a point y of Ω with a low rank realization $\mathcal{X}(y)$.

Algorithm 5.1 *Input : realization P_0 of $G = (V, E, \omega)$, positive integer t.*

1. *For $k = 1, \dots, t$ do*
   (a) *Let c^k be a random vector selected uniformly from the unit sphere in $\Re^{\bar{m}}$.*
   (b) *Solve the SDP problem: max $\{{c^k}^T y : y \in \Omega\}$ and let y^{*k} be the corresponding optimal solution.*
2. *Pick $y^*_{\min}$ = arg min $\{\mathcal{X}(y^{*k}) : k = 1, \dots, t\}$.*

Two remarks are in order here. First, $c \in \Re^{\bar{m}}$ in (a) is generated using the following simple method. Let c'_{ij} be $\bar{m}$ numbers randomly drawn from a standard normal distribution (mean 0 and variance 1). Then it can be easily shown [22] that the vector $c = \dfrac{1}{\|c'\|} c'$ is uniform over the unit sphere in $\Re^{\bar{m}}$, where $c' = (c'_{ij})$. Second, the characterization of Ω given in (13) is used in (b) since this characterization generates a sparse SDP problem. This SDP problem is then solved using the well-known solver SeDuMi by Jos Sturm [29].

Next we present numerical results on randomly generated connected weighted graphs $G = (V, E, \omega)$. We assume that we are given a realization X of G in $\Re^r$, where r = Barvinok bound in Lemma 4.7, i.e., $r = \lfloor \frac{\sqrt{8(|E|-1)+1}-1}{2} \rfloor$. Such X is generated as follows. First we generate an $(n-1) \times r$ matrix Y with entries uniformly distributed in the interval $[-5, 5]$, then we set $X = YY^T$. We used SeDuMi 1.05R5 to optimize $c^T y$ over set Ω and the numerical experiments were run on a 3.06 GHz Pentium-4 PC with 496 MB of RAM .

Tables 1 and 2 show the occurrences of rank $\mathcal{X}(y^*)$ for $n = 20$ and $n = 30$ respectively. The CPU time for optimizing $c^T y$ over set Ω was in the order of 1 sec for $n = 20$ and 15 sec for $n = 30$. As Tables 1 and 2 show, a rank which is much smaller than the Barvinok bound in Lemma 4.7 was consistently obtained.

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Table 1: Number of occurrences of rank $\mathcal{X}(y^*)$ in 100 randomly generated problems with $t = 10$ and $n = 20$.

	$m = 30$	40	50	60	70	80	90	100
Barvinok bound =	7	8	9	10	11	12	12	13
rank $\mathcal{X}(y^*)$								
11								13
10							45	497
9					3	152	671	481
8				3	305	750	283	9
7			30	460	649	98	1	
6		43	584	523	43			
5	18	674	380	14				
4	640	283	6					
3	342							

Table 2: Number of occurrences of rank $\mathcal{X}(y^*)$ in 100 randomly generated problems with $t = 5$ and $n = 30$.

	$m = 100$	140	180	220	260	300	340	380
Barvinok bound =	13	16	18	20	22	23	25	27
rank $\mathcal{X}(y^*)$								
24								5
23								296
22								194
21							128	5
20						2	348	
19						80	24	
18						358		
17					119	60		
16				2	335			
15				122	46			
14			2	332				
13			148	44				
12		1	317					
11		135	33					
10		331						
9	61	33						
8	346							
7	93							