On Global Rigidity and Universal Rigidity of Generic Bar Frameworks with Few Vertices

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Abstract

Gale transform was successfully used to classify polytopes with few vertices; that is, r-dimensional polytopes with $r + k$ vertices, where k is small. In this paper, we use Gale transform to classify r-dimensional generic bar frameworks with $r + k$ vertices, where $k \leq 4$; as universally rigid, globally rigid or non-globally rigid.

1 Introduction

A *configuration* p in $\mathbb{R}^r$ is a finite set of points $p^1, \ldots, p^n$ whose affine hull is $\mathbb{R}^r$. Gale transform is a well-known technique in polytope theory for representing a configuration p in $\mathbb{R}^r$ by another configuration z in $\mathbb{R}^{n-r-1}$ [8]. Such a transform, which encodes the combinatorial structure of p, is particularly powerful when $n-r-1$ is small. Accordingly, Gale transform was successfully used in the classification of

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polytopes with few vertices, i.e., r-dimensional polytopes with $r+k$ vertices where k is small [10, 14].

A *bar-and-joint framework* (or a framework for short), denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph G and a configuration p in $\mathbb{R}^r$, such that for every $i = 1, \ldots, n$, vertex i of G is located at p^i. We will refer to the nodes and edges of graph G as the nodes and edges of the framework $G(p)$. Recently, a characterization of the global rigidity of generic frameworks was obtained by Connelly [5] and Gortler *et al* [9]. This characterization was given in terms of the stress matrix associated with the framework. On the other hand, Alfakih [2] obtained a sufficient condition for the universal rigidity of a generic framework in terms of the Gale transform of the configuration of the framework. Furthermore, by relating the stress matrix associated with a framework $G(p)$ to the Gale transform of its configuration, he showed that the Connelly-Gortler *et al* necessary and sufficient condition for generic global rigidity can be, equivalently, expressed in terms of the Gale transform of the configuration p.

In this paper, we use Gale transform to classify r-dimensional generic frameworks with $r+k$ vertices, where $k \leq 4$; as universally rigid, globally rigid or non-globally rigid.

Global rigidity and universal rigidity of bar frameworks have many applications in molecular conformation [6], multidimensional scaling [12] and wireless sensor networks [13, 7].

2 Preliminaries

Two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be *congruent* if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \ldots, n$, where $\|.\|$ denotes the Euclidean norm. On the other hand, two frameworks $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be *equivalent* if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i, j) \in E$, where E is the set of edges of graph G.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be *rigid* if there exists an $\epsilon > 0$ such that, if any other framework $G(q)$ in the same space $\mathbb{R}^r$ is equivalent to $G(p)$ and $\|q^i - p^i\| \leq \epsilon$ for all $i = 1, \ldots, n$, then $G(q)$ is actually congruent to $G(p)$. A framework $G(p)$ in $\mathbb{R}^r$ is said to be *globally rigid* if any other framework $G(q)$ in the same space $\mathbb{R}^r$ is equivalent to $G(p)$, then $G(q)$ is actually congruent to $G(p)$. A framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if any other framework $G(q)$ in any space $\mathbb{R}^s$ is equivalent to $G(p)$, then $G(q)$ is actually congruent to $G(p)$. Obviously, universal rigidity implies global rigidity, which in turn implies rigidity. However, the reverse implications do not hold.

A framework $G(p)$ is said to be *generic* if all the coordinates of the points $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, $G(p)$ is generic if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, p^2, \dots, p^n) = 0$. The notion of general position for points is weaker than that of genericness. Framework $G(p)$ in $\mathbb{R}^r$ is said to be in *general position* if no $r+1$ of the points $p^1, p^2, \dots, p^n$ are affinely dependent. For example $G(p)$ in the plane is in general position if no three of the points $p^1, \dots, p^n$ lie on a straight line. Gale transform of a configuration p encodes the affine dependencies among the points $p^1, \dots, p^n$. Consequently, it is particularly suited when the configuration p is in general position or generic.

2.1 Gale Transform

Let $G(p)$ be a given framework with n vertices in $\mathbb{R}^r$. Consider the $(r+1) \times n$ matrix

$$A := \begin{bmatrix} p^1 & p^2 & \dots & p^n \\ 1 & 1 & \dots & 1 \end{bmatrix}. \tag{1}$$

Recall that the affine hull of the points $p^1, \dots, p^n$ has dimension r, i.e., $p^1, \dots, p^n$ are not contained in a proper hyper-plane in $\mathbb{R}^r$. Then $r \leq n-1$, and matrix A has full row rank. Let $\bar{r}$ be the dimension of the null space of A; i.e., $\bar{r} = n - 1 - r$. For $\bar{r} \geq 1$, let Λ be the $n \times \bar{r}$ matrix whose columns form a basis for the null space of A. Λ is called a *Gale matrix* of the framework $G(p)$; and the ith row of Λ, considered as a vector in $\mathbb{R}^{\bar{r}}$, is called a *Gale transform* of p^i [8]. We take advantage of the fact that Λ is not unique to define a unique sparse Gale matrix Z as follows.

Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is $\bar{r} \times \bar{r}$ and Λ_2 is $(r+1) \times \bar{r}$. We can assume without loss of generality that Λ_1 is non-singular since Λ has full column rank. Then Z is defined as

$$Z := \Lambda {\Lambda_1}^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 {\Lambda_1}^{-1} \end{bmatrix}, \tag{2}$$

where $I_{\bar{r}}$ is the identity matrix of order $\bar{r}$. Two remarks are in order here. First, let ${z^i}^T$ denote the ith row of Z then obviously $z^1, z^2, \dots, z^{\bar{r}}$, the Gale transforms of $p^1, p^2, \dots, p^{\bar{r}}$ respectively, are simply the standard unit vectors in $\mathbb{R}^{\bar{r}}$. Second, if we apply the usual sign function to a column of Z component wise, we get a signed n-vector. In oriented matroid theory, this signed vector is called a *circuit* of the oriented matroid of the configuration p [3].

The Gale matrix Z has additional properties when the framework $G(p)$ is in general position or generic.

Lemma 2.1 ([1]) *Let $G(p)$ be a framework in general position with n vertices in $\mathbb{R}^r$ and let $z^1, \ldots, z^n$ be the Gale transforms of $p^1, \ldots, p^n$ respectively. Recall that $\bar{r} = n - 1 - r$. Then any subset of $z^1, \ldots, z^n$ of cardinality $\bar{r}$ is linearly independent.*

Lemma 2.2 *Let $G(p)$ be a generic framework with n vertices in $\mathbb{R}^r$ and let $z^1, \ldots, z^n$ be the Gale transforms of $p^1, \ldots, p^n$ respectively. Then all the coordinates of $z^{\bar{r}+1}$, $z^{\bar{r}+2}, \ldots, z^n$ are algebraically independent over the integers.*

Proof. Assume that $G(p)$ is generic and write the matrix A in block form as

$$A = \begin{bmatrix} p^1 & p^2 & \ldots & p^n \\ 1 & 1 & \ldots & 1 \end{bmatrix} = [P_1 \;\; P_2],$$

where P_1 is $(r+1) \times \bar{r}$ and P_2 is $(r+1) \times (r+1)$ and nonsingular. Then the Gale matrix $Z = \begin{bmatrix} I_{\bar{r}} \\ -{P_2}^{-1} P_1 \end{bmatrix}$. Note that the determinant function is a polynomial with integer coefficients. Hence by Cramer's rule, each coordinate of $z^{\bar{r}+1}, z^{\bar{r}+2}, \ldots, z^n$ is of the form

$$\frac{h(p^1, \ldots, p^n)}{\det(P_2)},$$

where h is a polynomial with integer coefficients.

Now assume that g is a non-zero polynomial of degree s with integer coefficients, such that $g(z^{\bar{r}+1}, z^{\bar{r}+2}, \ldots, z^n) = 0$. Then $(\det(P_2))^s \; g(z^{\bar{r}+1}, z^{\bar{r}+2}, \ldots, z^n) = f(p^1, \ldots, p^n) = 0$. This contradicts our assumption that $G(p)$ is generic since f is a non-zero polynomial with integer coefficients.

□

2.2 Stress Matrices

Let $G(p)$ be a framework in $\mathbb{R}^r$ where $G = (V, E)$ has n vertices and m edges. Associate with each edge (i, j) in E a scalar ω_{ij}. Any vector $\omega = (\omega_{ij})$ in $\mathbb{R}^m$ such that

$$\sum_{j:(i,j) \in E} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n, \tag{3}$$

is called an *equilibrium stress* for $G(p)$. Given an equilibrium stress ω, let $S = (s_{ij})$ be the $n \times n$ symmetric matrix defined by:

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E, \\ 0 & \text{if } (i,j) \notin E, \\ \displaystyle\sum_{k:\,(i,k) \in E} \omega_{ik} & \text{if } i = j. \end{cases}$$

S is called the *stress matrix* associated with ω. The following lemma shows that the Gale matrix Z of framework $G(p)$ and the stress matrix S associated with an equilibrium stress ω of $G(p)$ are closely related.

Lemma 2.3 (Alfakih [2]) *Given a framework $G(p)$ with n vertices in $\mathbb{R}^r$, let Z be the Gale matrix of $G(p)$ and recall that $\bar{r} = n - 1 - r$. Further, let S be the stress matrix associated with an equilibrium stress ω for $G(p)$. Then*

$$S = Z\Psi Z^T \textit{ for some } \bar{r} \times \bar{r} \textit{ symmetric matrix } \Psi. \tag{4}$$

Furthermore, let z^i be the Gale transform of p^i. If Ψ' is any $\bar{r} \times \bar{r}$ symmetric matrix such that ${z^i}^T \Psi' z^j = 0$ for all $(i, j) \notin E$, then $Z\Psi' Z^T$ is a stress matrix associated with an equilibrium stress ω for $G(p)$.

3 Generic Global Rigidity and Generic Universal Rigidity

The following two theorems are critical for this paper. The first theorem was originally stated in [5, 9] in terms of the stress matrix associated with a framework $G(p)$. Using Lemma 2.3, we equivalently state it in terms of the Gale transform of configuration p.

Theorem 3.1 (Connelly [5], Gortler et al [9]) *Let $G(p)$ be a generic framework with n vertices in $\mathbb{R}^r$ and let $z^1, \ldots, z^n$ be, respectively, the Gale transforms of $p^1, \ldots, p^n$. Recall that $\bar{r} = n - 1 - r$. Then $G(p)$ is globally rigid if and only if*

$$\exists \; \bar{r} \times \bar{r} \textit{ nonsingular symmetric matrix } \Psi : {z^i}^T \Psi z^j = 0, \forall \; (i, j) \notin E, \tag{5}$$

where E is the set of edges of graph G.

Theorem 3.2 (Alfakih [2]) *Let $G(p)$ be a generic framework with n vertices in $\mathbb{R}^r$ and let $z^1, \ldots, z^n$ be, respectively, the Gale transforms of $p^1, \ldots, p^n$. Further, let E be the set of edges of graph G and recall that $\bar{r} = n - 1 - r$. If*

$$\exists \; \bar{r} \times \bar{r} \textit{ symmetric positive definite matrix } \Psi : {z^i}^T \Psi z^j = 0, \forall \; (i, j) \notin E, \tag{6}$$

then $G(p)$ is universally rigid.

Since Conditions (5) and (6) in the previous two theorems are given in terms of the missing edges of graph G, we will find it convenient, in some cases, to use the complement graph $\bar{G}$ instead of G.

4 Classification of Generic Frameworks with Few Vertices

In this section, we classify generic frameworks $G(p)$ in $\mathbb{R}^r$ with $r+k$ vertices ($k \leq 4$); as universally rigid, globally rigid or non-globally rigid.

We begin with some notation. Let P_n (C_n) denote the graph consisting of a path (cycle) on n vertices. Let K_n denote the complete graph on n vertices. Thus, $K_3 \cup (n-3)K_1$ denotes the disconnected graph consisting of a triangle and $n-3$ isolated nodes. Let $K_{1,n}$ denote the star graph on $n+1$ vertices. A *paw* is the graph obtained from K_3 by adding one node adjacent to exactly one of the nodes of K_3. A *chair* is the graph obtained from P_4 by adding one node adjacent to exactly one of its internal nodes. Finally, subscripts are used to denote components of vectors, while superscripts are used to index a particular vector in a set of vectors. For example, z^1_3 denotes the 3rd component of vector z^1 while z^3_1 denotes the 1st component of vector z^3.

The following simple lemma follows from one of Hendrickson's necessary conditions for generic global rigidity [11], namely the condition concerning vertex connectivity. The proof is added to illustrate the main idea behind most of the proofs in this paper.

Lemma 4.1 *Let $G(p)$ be a generic framework with n vertices in $\mathbb{R}^r$ and let $\bar{G}$ denote the complement graph of G. If there exists a node in $\bar{G}$ with degree $\geq \bar{r} = n-1-r$, then $G(p)$ is not globally rigid.*

Proof. Without loss of generality, assume that node 1 has degree $\geq \bar{r}$, and nodes $2, 3, \ldots, \bar{r}+1$ are adjacent to node 1 in $\bar{G}$. Let $\Psi = (\psi_{ij})$ be any $\bar{r} \times \bar{r}$ symmetric matrix such that ${z^1}^T \Psi z^i = 0$ for all $i = 2, \ldots, \bar{r}+1$. Then $\psi_{12} = \psi_{13} = \cdots = \psi_{1\bar{r}} = 0$ since $z^1, \ldots, z^{\bar{r}}$ are the standard unit vectors in $\mathbb{R}^{\bar{r}}$. Furthermore, ${z^1}^T \Psi z^{\bar{r}+1} = 0$ implies that $\psi_{11} z_1^{\bar{r}+1} = 0$, which in turn implies that $\psi_{11} = 0$ since $z_1^{\bar{r}+1} \neq 0$. Thus Ψ is singular and hence, by Theorem 3.1, $G(p)$ is not globally rigid.

□

4.1 Frameworks in $\mathbb{R}^r$ with $r+1$ or $r+2$ Vertices

The case of a framework $G(p)$ with $r+1$ vertices in $\mathbb{R}^r$ is trivial. In this case $\bar{r} = 0$, and all the vertices of $G(p)$ are affinely independent. If G is not the complete graph K_{r+1}, then $G(p)$ is not rigid whether $G(p)$ is generic or not. Consequently, $G(p)$ is neither globally rigid nor universally rigid.

Next we consider the case where $G(p)$ has $r+2$ vertices.

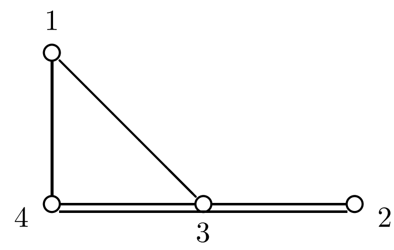

Figure 1: An example of a non-generic framework with 4 vertices in $\mathbb{R}^2$ that is universally rigid. (1,2) is the only missing edge.

Theorem 4.1 (Alfakih [1]) *Let $G(p)$ be a generic framework with $r+2$ vertices in $\mathbb{R}^r$. Assume that G is not the complete graph, then $G(p)$ is not globally rigid.*

Proof. Since $G=(V,E)$ is not the complete graph, assume, without loss of generality, that $(1,2) \notin E$. Since in this case $\bar{r}=1$, the genericness of $G(p)$ implies that z^1 and z^2 are nonzero scalars. Now let Ψ be any scalar such that $z^1 \Psi z^2 = 0$, thus Ψ must be zero. Hence it follows from Theorem 3.1 that $G(p)$ is not globally rigid.

□

The assumption in the previous theorem that $G(p)$ is generic can not be dropped. Figure 1 depicts a non-generic framework with 4 vertices in $\mathbb{R}^2$ that is universally rigid.

4.2 Frameworks in $\mathbb{R}^r$ with $r+3$ Vertices

The problem of determining the global rigidity or the universal rigidity of a generic framework $G(p)$ in $\mathbb{R}^r$ becomes interesting when $G(p)$ has at least $r+3$ vertices.

Theorem 4.2 *Let $G(p)$ be a generic framework with $r+3$ vertices in $\mathbb{R}^r$. Assume that G is not the complete graph and let $\bar{G}$ denote the complement graph of G. Then:*

1. *If $\bar{G} = K_2 \cup (r+1)K_1$, then $G(p)$ is universally rigid.*
2. *If $\bar{G} = 2K_2 \cup (r-1)K_1$, then $G(p)$ is globally rigid.*
3. *If $\bar{G} = P_3 \cup rK_1$, then $G(p)$ is not globally rigid.*
4. *If G has three or more missing edges, then $G(p)$ is not globally rigid.*

Proof. In this case $\bar{r}=2$.

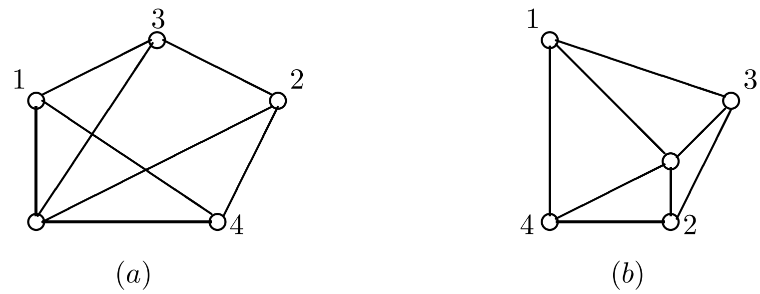

Figure 2: Two frameworks with 5 vertices in $\mathbb{R}^2$, and with two missing non-adjacent edges. Framework (a) is universally rigid while framework (b) is globally rigid but not universally rigid.

1. Without loss of generality, assume that the edge of K_2 is $(1,2)$ then $z^1 = (1\ 0)^T$ and $z^2 = (0\ 1)^T$. Clearly $\Psi = I_2$, the identity matrix of order 2, satisfies Condition (6). Thus $G(p)$ is universally rigid.

2. Wlog, assume that the edges of the two K_2 are $(1,2)$ and $(3,4)$. Then $z^1 = (1\ 0)^T$, $z^2 = (0\ 1)^T$, $z^3 = (z_1^3\ z_2^3)^T$ and $z^4 = (z_1^4\ z_2^4)^T$. Let $\Psi = (\psi_{ij})$ be a symmetric 2×2 matrix satisfying ${z^1}^T \Psi z^2 = {z^3}^T \Psi z^4 = 0$. Then $\psi_{12} = \psi_{21} = 0$ and $\psi_{11} z_1^3 z_1^4 + \psi_{22} z_2^3 z_2^4 = 0$. Note that $z_1^3 z_1^4 \neq 0$ and $z_2^3 z_2^4 \neq 0$ since $G(p)$ is generic. Let $\psi_{11} = 1$ and $\psi_{22} = -(z_1^3 z_1^4)/(z_2^3 z_2^4)$. Thus Ψ satisfies Condition (5) and $G(p)$ is globally rigid.

3. Wlog, assume that the edges of P_3 are $(2,1)$ and $(1,3)$. Then degree $(1) = 2 = \bar{r}$. Hence, by Lemma 4.1, $G(p)$ is not globally rigid.

4. If any two edges of $\bar{G}$ are adjacent, then the result follows from Statement 3. Therefore, assume that $\bar{G}$ has the edges $(1,2)$, $(3,4)$ and $(5,6)$. Let $\Psi = (\psi_{ij})$ be a symmetric 2×2 matrix satisfying ${z^1}^T \Psi z^2 = {z^3}^T \Psi z^4 = {z^5}^T \Psi z^6 = 0$. Then $\psi_{12} = \psi_{21} = 0$ and $\psi_{11} z_1^3 z_1^4 + \psi_{22} z_2^3 z_2^4 = \psi_{11} z_1^5 z_1^6 + \psi_{22} z_2^5 z_2^6 = 0$. Thus $\psi_{11} = \psi_{22} = 0$ since $G(p)$ is generic. Hence, Ψ is the zero matrix and the result follows.

$\square$

Two remarks concerning the previous theorem are in order here. First, in statement 2, $G(p)$ may or may not be universally rigid depending on whether ψ_{22} is positive or negative (see Figure 2). Second, the assumption that $G(p)$ is generic can not be dropped. Framework (a) in Figure 3, which is non-generic, is not globally rigid even though it has one missing edge. On the other hand, framework (b) in Figure 3, which is non-generic, is universally rigid even though it has two adjacent missing edge.

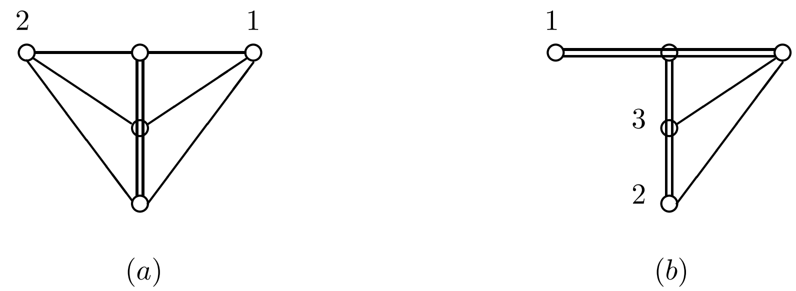

Figure 3: Two non-generic frameworks with 5 vertices in $\mathbb{R}^2$. Framework (a) is not globally rigid even though it has only one missing edge, namely (1,2). On the other hand, framework (b) is universally rigid even though it has two adjacent missing edge, namely (1,2) and (1,3).

4.3 Frameworks in $\mathbb{R}^r$ with $r+4$ Vertices

The following theorem classifies frameworks with one, two or three missing edges.

Theorem 4.3 *Let $G(p)$ be a generic framework with $r+4$ vertices in $\mathbb{R}^r$. Assume that G is not the complete graph and let $\bar{G}$ denote the complement graph of G. Then*

1. *If $\bar{G} = K_2 \cup (r+2)K_1$, or $\bar{G} = 2K_2 \cup rK_1$, or $\bar{G} = P_3 \cup (r+1)K_1$, then $G(p)$ is universally rigid.*
2. *If $\bar{G} = K_3 \cup (r+1)K_1$, or $\bar{G} = P_4 \cup rK_1$, then $G(p)$ is universally rigid.*
3. *If $\bar{G} = P_3 \cup K_2 \cup (r-1)K_1$, or $\bar{G} = 3K_2 \cup (r-2)K_1$, then $G(p)$ is globally rigid.*
4. *If $\bar{G} = K_{1,3} \cup rK_1$, then $G(p)$ is not globally rigid.*

Proof. In this case $\bar{r} = 3$. Recall that $z^1 = (1\ 0\ 0)^T$, $z^2 = (0\ 1\ 0)^T$ and $z^3 = (0\ 0\ 1)^T$.

1. If $\bar{G}$ has one edge say $(1,2)$, or two adjacent edges say $(1,2)$ and $(2,3)$, then $\Psi = I_3$ satisfies Condition (6) and the result follows. On the other hand, if $\bar{G}$ has two non-adjacent edges say $(1,2)$ and $(3,4)$, then $\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \psi_{22} & \psi_{23} \\ 0 & \psi_{32} & 1 \end{bmatrix}$, where $\psi_{23} = \psi_{32} = -z_3^4/z_2^4$ and $\psi_{22} > \psi_{23}^2$ satisfies Condition (6). Note that $z_2^4 \neq 0$ since $G(p)$ is generic. Thus $G(p)$ is universally rigid.
2. Assume that $\bar{G}$ consists of a K_3 and $r+1$ isolated nodes, and let the edges of K_3 be $(1,2)$, $(2,3)$ and $(3,1)$. Then $\Psi = I_3$ satisfies Condition (6). Hence, $G(p)$ is universally rigid.

Now Assume that $\bar{G}$ consists of a P_4 and r isolated nodes, and let the edges of P_4 be $(1,2)$, $(2,3)$ and $(3,4)$. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & \psi_{13} \\ 0 & 1 & 0 \\ \psi_{31} & 0 & 1 \end{bmatrix}$, where $\psi_{13} = \psi_{31} = -z_3^4/z_1^4$ and $\psi_{11} > \psi_{13}^2$ satisfies Condition (6). Hence, $G(p)$ is universally rigid.

3. Assume that $\bar{G}$ consists of a P_3 , a K_2 and $r+1$ isolated nodes, and let the edges of P_3 be $(1,2)$, $(2,3)$ and the edge K_2 be $(4,5)$. Then $\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \psi_{33} \end{bmatrix}$, where $\psi_{33} = -(z_1^4 z_1^5 + z_2^4 z_2^5)/(z_3^4 z_3^5)$ satisfies Condition (5). Note that $\psi_{33} \neq 0$ since $G(p)$ is generic. Hence, $G(p)$ is globally rigid.

   Now assume that $\bar{G}$ consists of 3 non-adjacent edges and $r-2$ isolated nodes, and let these edges be $(1,2)$, $(3,4)$ and $(5,6)$. Then $\Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \psi_{22} & \psi_{23} \\ 0 & \psi_{32} & 1 \end{bmatrix}$, where $\psi_{23} = \psi_{32} = -z_3^4/z_2^4$ and

$$\psi_{22} = -\frac{z_1^5 z_1^6 + z_3^5 z_3^6 + \psi_{23}(z_2^5 z_3^6 + z_3^5 z_2^6)}{z_2^5 z_2^6}$$

   satisfies Condition (5). Note that $\psi_{22} - \psi_{23}^2 \neq 0$ since $G(p)$ is generic. Hence, $G(p)$ is globally rigid.

4. Let the edges of the star $K_{1,3}$ be $(1,2)$, $(1,3)$ and $(1,4)$. Then deg $(1) = 3 = \bar{r}$ and the result follows from Lemma 4.1.

□

The following theorem classifies frameworks with four missing edges.

Theorem 4.4 *Let $G(p)$ be a generic framework with $r+4$ vertices in $\mathbb{R}^r$. Assume that G is not the complete graph and let $\bar{G}$ denote the complement graph of G. Then*

1. *If $\bar{G} = P_5 \cup (r-1)K_1$, then $G(p)$ is globally rigid.*
2. *If $\bar{G} = P_4 \cup K_2 \cup (r-2)K_1$, then $G(p)$ is globally rigid.*
3. *If $\bar{G} = 2P_3 \cup (r-2)K_1$, then $G(p)$ is globally rigid.*
4. *If $\bar{G} = P_3 \cup 2K_2 \cup (r-3)K_1$, then $G(p)$ is globally rigid.*
5. *If $\bar{G} = 4K_2 \cup (r-4)K_1$, then $G(p)$ is globally rigid.*

6. *If* $\bar{G} = K_3 \cup K_2 \cup (r-1)K_1$*, then* $G(p)$ *is globally rigid.*

7. *If* $\bar{G} = C_4 \cup rK_1$*, then* $G(p)$ *is not globally rigid.*

8. *If* $\bar{G} = K_{1,3} \cup K_2 \cup (r-2)K_1$ *or,* $\bar{G} = K_{1,4} \cup (r-1)K_1$ *or,* $\bar{G}$ *consists of a paw and* r *isolated nodes or,* $\bar{G}$ *consists of a chair and* $r-1$ *isolated nodes, then* $G(p)$ *is not globally rigid.*

Proof. Recall that $\bar{r} = 3$ and $z^1 = (1\ 0\ 0)^T$, $z^2 = (0\ 1\ 0)^T$ and $z^3 = (0\ 0\ 1)^T$.

1. Assume that the edges of P_5 are $(1,2)$,$(2,3)$, $(3,4)$ and $(4,5)$. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \psi_{33} \end{bmatrix}$, where $\psi_{33} = -z^4_1/z^4_3$ and

$$\psi_{11} = -\frac{z^4_3 z^5_1 + z^4_2 z^5_2}{z^4_1 z^5_1}$$

satisfies Condition (5). Note that $\psi_{33}\ \psi_{11} \neq 1$. Hence, $G(p)$ is globally rigid.

2. Assume that the edges of P_4 are $(1,2)$, $(2,3)$, $(3,4)$, and that the edge of K_2 is $(5,6)$. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & \psi_{13} \\ 0 & 1 & 0 \\ \psi_{31} & 0 & 1 \end{bmatrix}$, where $\psi_{13} = \psi_{31} = -z^4_3/z^4_1$ and

$$\psi_{11} = -\frac{z^5_2 z^6_2 + z^5_3 z^6_3 + \psi_{13}(z^5_1 z^6_3 + z^5_3 z^6_1)}{z^5_1 z^6_1}$$

satisfies Condition (5). Hence, $G(p)$ is globally rigid.

3. Assume that the edges of the two P_3's are $(1,2)$, $(2,3)$, and $(4,5)$, $(5,6)$ respectively. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & 0 \\ 0 & \psi_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where

$$\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \begin{bmatrix} z^4_1 z^5_1 & z^4_2 z^5_2 \\ z^5_1 z^6_1 & z^5_2 z^6_2 \end{bmatrix}^{-1} \begin{bmatrix} -z^4_3 z^5_3 \\ -z^5_3 z^6_3 \end{bmatrix}$$

satisfies Condition (5). Hence, $G(p)$ is globally rigid.

4. Assume that the edges of the P_3's are $(1,2)$, $(2,3)$, and the edges of the two K_2's are $(4,5)$ and $(6,7)$ respectively. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & 0 \\ 0 & \psi_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where

$$\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \begin{bmatrix} z_1^4 z_1^5 & z_2^4 z_2^5 \\ z_1^6 z_1^7 & z_2^6 z_2^7 \end{bmatrix}^{-1} \begin{bmatrix} -z_3^4 z_3^5 \\ -z_3^6 z_3^7 \end{bmatrix}$$

satisfies Condition (5). Hence, $G(p)$ is globally rigid.

5. Assume that the 4 nonadjacent edges are $(1,2)$, $(3,4)$, $(5,6)$ and $(7,8)$. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & 1 \\ 0 & \psi_{22} & 0 \\ 1 & 0 & \psi_{33} \end{bmatrix}$, where $\psi_{33} = -z_1^4/z_3^4$, and

$$\begin{bmatrix} \psi_{11} \\ \psi_{22} \end{bmatrix} = \begin{bmatrix} z_1^5 z_1^6 & z_2^5 z_2^6 \\ z_1^7 z_1^8 & z_2^7 z_2^8 \end{bmatrix}^{-1} \begin{bmatrix} -z_3^5 z_3^6 \psi_{33} - z_1^5 z_3^6 - z_3^5 z_1^6 \\ -z_3^7 z_3^8 \psi_{33} - z_1^7 z_3^8 - z_3^7 z_1^8 \end{bmatrix}$$

satisfies Condition (5). Hence, $G(p)$ is globally rigid.

6. Assume that the edges of K_3 are $(1,2)$, $(2,3)$, $(3,1)$, and that the edge of K_2 is $(4,5)$. Then $\Psi = \begin{bmatrix} \psi_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where

$$\psi_{11} = -\frac{z_2^4 z_2^5 + z_3^4 z_3^5}{z_1^4 z_1^5}$$

satisfies Condition (5). Hence, $G(p)$ is globally rigid.

7. Assume that the edges of C_4 are $(1,2)$, $(2,3)$, $(3,4)$ and $(4,1)$. Then any 3×3 symmetric matrix $\Psi = (\psi_{ij})$ satisfying ${z^1}^T \Psi z^2 = {z^2}^T \Psi z^3 = {z^3}^T \Psi z^4 = {z^4}^T \Psi z^1 = 0$ must have $\psi_{12} = \psi_{21} = \psi_{23} = \psi_{32} = 0$, and

$$\begin{bmatrix} \psi_{11} & \psi_{13} \\ \psi_{31} & \psi_{33} \end{bmatrix} \begin{bmatrix} z_1^4 \\ z_3^4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

But $z_1^4 \neq 0$ and $z_3^4 \neq 0$ since $G(p)$ is generic. Thus $\psi_{11}\psi_{33} - \psi_{31}\psi_{13} = 0$. Hence, Ψ is singular. Therefore, $G(p)$ is not globally rigid.

8. Follows from Lemma 4.1.

□

5 Concluding Remarks

In this paper we classified r-dimensional bar frameworks with $r+k$ vertices as universally rigid, globally rigid or non-globally rigid; where $k \leq 4$. The analysis presented in this paper can in principle be extended to frameworks with more vertices and more missing edges. However, the cases where the framework has 5 or more missing edges, or where it has $r+5$ vertices or more becomes tedious.

Finally we consider the interesting framework $G(p)$ in $\mathbb{R}^3$ where $G = K_{5,5}$. This framework was shown by Connelly [4] to be generically non-globally rigid even though it satisfies both of Hendrickson's necessary conditions for global rigidity. Using the Gale transform approach of this paper, it can be easily shown that this framework is indeed not generically globally rigid.

Let us write the Gale matrix Z corresponding to the framework $K_{5,5}$ with a generic configuration in $\mathbb{R}^3$ in block form as

$$Z = \begin{bmatrix} I_5 & 0 \\ 0 & 1 \\ B & b \end{bmatrix}$$

where I_5 is the identity matrix of order 5, B is 4×5 and b is 4×1 (in this case $\bar{r} = 6$, i.e., Z is 10×6). Let $\Psi = \begin{bmatrix} \phi & \xi \\ \xi^T & \alpha \end{bmatrix}$, where ϕ is 5×5, be any 6×6 symmetric matrix such that $Z \Psi Z^T = S$ is a stress matrix of $G(p)$. Wlog assume that $K_{5,5}$ has the bipartition $\{1,2,3,4,5\}, \{6,7,8,9,10\}$. Since each entry s_{ij}, $(i \neq j)$, for $i,j = 1,\ldots,5$ and for $i,j = 6,\ldots,10$, of any stress matrix associated with $K_{5,5}$ must be zero, it follows that matrix ϕ is diagonal, say ϕ=Diag(x) for some $x \in \mathbb{R}^5$, $B\xi + \alpha b = 0$, and

$$\text{off-diagonal entries of } (B \text{Diag}(x) B^T - \alpha b b^T) = 0. \tag{7}$$

(7) is a homogeneous system of 6 equations in 6 unknowns (x, α). Thus the trivial solution is the only solution of (7) since the entries of B and b are algebraically independent. Therefore, $\phi = 0$ and $\alpha = 0$. Thus rank $\Psi = 2$, and hence it is singular. Therefore, $G(p)$ does not satisfy Condition (5).

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