1 Introduction

In the matrix completion problem we are given a partial symmetric real matrix A with certain elements specified or *fixed* and the rest are unspecified or *free*; and, we are asked whether A can be completed to satisfy a given property (P) by assigning certain values to its free elements. In this chapter, we are interested in the following two completion problems: the *positive semidefinite matrix completion problem* corresponding to (P) being the positive semidefiniteness (PSD) property; and the *Euclidean distance matrix completion problem* corresponding to (P) being the Euclidean distance (ED) property. (We concentrate on the latter problem. A survey of the former is given in [**?**]. The relationships between the two is discussed in e.g. [**?**, **?**, **?**].) These problems can be generalized to allow for approximate completion by formulating the completion problems as weighted closest matrix problems:

$$\min \{ f(X) = || \; H \circ (A - X) \; ||_F^2 : \text{for all } X \text{ satisfying (P)} \}, \tag{1}$$

where $|| \; . \; ||_F$ denotes the *Frobenius* norm defined as $|| \; A \; ||_F = \sqrt{\text{trace } A^T A}$, $(\circ)$ denotes the *Hadamard* or the element-wise product, and the weight matrix H is symmetric with nonnegative elements. If each element of the weight matrix H is set to 0 or 1 depending respectively on whether the corresponding element of A is free or fixed, then matrix A has an exact completion if and only if the optimal objective value of problem (**??**) is zero. Problem (**??**) will be referred to as CPM when (P) is the PSD property, and as CDM when (P) is the ED property. Both CPM and CDM are examples of semidefinite programming with the former problem being a special case of the latter.

An $n \times n$ symmetric matrix D with nonnegative elements and zero diagonal is called a *dissimilarity matrix* or a *pre-distance matrix*. A dissimilarity matrix D is said to be a *Euclidean distance matrix* (EDM) if there exist points $x^1, x^2, \ldots, x^n$ in some Euclidean space $\Re^r$ such that

$$D_{ij} = \|x^i - x^j\|^2, \qquad i, j = 1, 2, \ldots, n, \tag{2}$$

where $|| \; . \; ||$ denotes the usual Euclidean norm. The *embedding dimension* of D is given by r where r is the dimension of the smallest space containing $x^1, x^2, \ldots, x^n$. Note that r is always less than or equal to $n - 1$.

Euclidean distance matrices have received much attention in recent years both for their elegance and for their many important applications. Applications of EDMs abound: e.g. molecular conformation problems in chemistry

[?, ?]; multidimensional scaling and multivariate analysis problems in statistics [?, ?]; genetics, geography, and others, [?]. In many of these applications, a low embedding dimension, e.g. $r = 3$, is required.

Theoretical properties of EDMs can be found in e.g. [?, ?, ?, ?, ?, ?, ?]. This includes characterizations as well as graph theoretic conditions for existence of completions. More information can be found in the recent survey article by Laurent [?].

2 Weighted Closest Euclidean Distance Matrix

Consider the objective function

$$f(D) := \| H \circ (A - D) \|_F^2 .$$

The weighted, *closest Euclidean distance matrix problem* is

$$(CDM_0) \quad \mu^* := \begin{array}{cc} \min & f(D) \\ \text{subject to} & D \in \mathcal{E}, \end{array}$$

where $\mathcal{E}$ is the set of all EDMs. In the following section, it is shown that $\mathcal{E}$ is in fact a convex cone whose structure is similar to that of the cone of positive semidefinite matrices $\mathcal{P}$. This relationship between $\mathcal{E}$ and $\mathcal{P}$ will allow us to formulate (CDM_0) as a semidefinite program.

2.1 Distance Geometry

It is well known, e.g. [?, ?, ?], that a pre-distance matrix D is a EDM if and only if D is negative semidefinite on

$$M := \left\{ x \in \Re^n : x^T e = 0 \right\},$$

the orthogonal complement of e where e is the vector of all ones. Thus $\mathcal{E}$ is a convex cone. We exploit this result and the fact that faces of $\mathcal{P}_n$, the cone of positive semidefinite matrices of order n, are isomorphic to $\mathcal{P}_r$, for some $0 \le r \le n$, to translate the cone $\mathcal{E}$ to the cone $\mathcal{P}_{n-1}$.

Let V be the $n \times (n-1)$ matrix whose columns form an orthonormal basis for the subspace M. i.e., V satisfies:

$$V^T e = 0 \ , \text{ and } V^T V = I. \tag{3}$$

Then,

$$J := VV^T = I - \frac{ee^T}{n} \tag{4}$$

is the orthogonal projection onto M. Now define the *centered* and *hollow* subspaces of $\mathcal{S}^n$, the space of $n \times n$ symmetric matrices, as

$$\begin{array}{rcl} \mathcal{S}_C & := & \{B \in \mathcal{S}^n : Be = 0\}, \\ \mathcal{S}_H & := & \{D \in \mathcal{S}^n : \mathrm{diag(D)} = 0\}, \end{array} \tag{5}$$

where $\mathrm{diag}(D)$ denotes the column vector formed from the diagonal of D. Following [**?**], we define the two linear operators $\mathcal{K} : \mathcal{S}_C \to \mathcal{S}_H$ and $\mathcal{T} : \mathcal{S}_H \to \mathcal{S}_C$ such that

$$\mathcal{K}(B) \quad := \quad \mathrm{diag}(B)\, e^T + e\, \mathrm{diag}(B)^T - 2B, \tag{6}$$

and

$$\mathcal{T}(D) \quad := \quad -\tfrac{1}{2} JDJ. \tag{7}$$

Theorem 2.1 *The linear operators $\mathcal{K}$ and $\mathcal{T}$ are inverses of each other.*

Proof. See [**?**, **?**].

□

It can easily be verified that

$$\mathcal{K}^*(D) \quad = \quad 2\,(\mathrm{Diag(De)} - \mathrm{D}) \tag{8}$$

is the adjoint operator of $\mathcal{K}$, where Diag(De) denotes the diagonal matrix formed from the vector De. In addition, a hollow matrix D is EDM if and only if $B = \mathcal{T}(D) \succeq 0$. Equivalently, D is EDM if and only if $D = \mathcal{K}(B)$, for some B with $Be = 0$ and $B \succeq 0$. In this case the embedding dimension r is given by the rank of B. If $B = XX^T$ for some $n \times r$ matrix X, then the rows of X are the coordinates of the points $x^1, x^2, \ldots, x^n$ that generate D. Furthermore, since $Be = 0$, it follows that the origin coincides with the centroid of these points. For these and other basic results on EDM see e.g. [**?**, **?**, **?**, **?**, **?**].

We now introduce the composite operators $\mathcal{K}_V : \mathcal{S}^{n-1} \to \mathcal{S}_H$ and $\mathcal{T}_V : \mathcal{S}_H \to \mathcal{S}^{n-1}$ such that

$$\mathcal{K}_V(X) \quad := \quad \mathcal{K}(VXV^T), \tag{9}$$

and

$$\mathcal{T}_V(D) \quad := \quad V^T \mathcal{T}(D) V = -\tfrac{1}{2} V^T D V, \tag{10}$$

where V is defined in (**??**).

Lemma 2.1 *The linear operators $\mathcal{K}_V$ and $\mathcal{T}_V$ are inverses of each other.*

Proof. This immediately follows from Theorem **??** and the definition of V. $\square$

From (**??**), we get that

$$\mathcal{K}_V^*(D) \quad = \quad V^T \mathcal{K}^*(D) V \tag{11}$$

is the adjoint operator of $\mathcal{K}_V$. The following corollary summarizes the relationships between the cones $\mathcal{E}$ and $\mathcal{P}_{n-1}$.

Corollary 2.1 *Suppose that V is defined as in (**??**). Then:*

$$\begin{array}{rcl} \mathcal{K}_V(\mathcal{P}_{n-1}) & = & \mathcal{E}, \\ \mathcal{T}_V(\mathcal{E}) & = & \mathcal{P}_{n-1}. \end{array}$$

Proof. We saw earlier that D is EDM if and only if $D = \mathcal{K}(B)$ with $Be = 0$ and $B \succeq 0$. Let $X = V^T B V$, then since $Be = 0$ we have $B = VXV^T$. Therefore, $VXV^T \succeq 0$ if and only if $X \succeq 0$; and the result follows using (**??**) and Lemma **??**. $\square$

2.2 Program Formulations

It can be assumed without loss of generality that $\mathrm{diag}(H) = 0$ since both $\mathrm{diag}(A)$ and $\mathrm{diag}(D)$ are 0. Also, all unspecified elements of A can be originally set to 0. Elements D_{ij} corresponding to $H_{ij} = 0$ are free, while elements D_{ij} corresponding to $H_{ij} > 0$ are approximately fixed since the weight $H_{ij} > 0$ forces D_{ij} to be approximately equal to A_{ij}. Hence, if the optimal value of the objective function $f(\bar{D})$ is 0, then an exact completion of A is found. i.e., $\bar{D}$ differs from A only in free elements i.e., elements with zero weight. If D_{ij} is fixed i.e., it is required to be equal to A_{ij} exactly, then a linear constraint can be added to the program. (See below.)

Recall that the graph of H is *connected* if for all indices $i \neq j$ there is a *path* of indices $i_1, i_2, ..., i_k$ such that $H_{i,i_1} \neq 0, H_{i_1,i_2} \neq 0, \ldots, H_{i_{k-1},i_k} \neq 0, H_{i_k,j} \neq 0$, see e.g. [**?**]. Thus, it is assumed that the graph of H is connected or the problem can be solved more simply as two or more smaller problems one for each component of H, see Lemma **??** below. In particular, it is assumed that H does not have a row (hence a column) of all zeros; otherwise the corresponding row and column in A and D consists of free (independent) elements and the problem can be posed in a lower dimensional space.

By abuse of notation, let the function

$$f(X) := \| H \circ (A - \mathcal{K}_V(X)) \|_F^2 = \| H \circ \mathcal{K}_V(B - X) \|_F^2,$$

where $B = \mathcal{T}_V(A)$. Using Corollary **??**, (CDM_0) is equivalent to the following problem

$$\begin{array}{lccc} & \mu^* := & \min & f(X) \\ (CDM) & & \text{subject to} & \mathcal{A}(X) = b \\ & & & X \succeq 0. \end{array}$$

An additional constraint using the linear operator $\mathcal{A} : \mathcal{S}^{n-1} \longrightarrow \Re^m$ is allowed here. The addition of this linear constraint could represent some of the fixed elements in the given matrix A, e.g. adding the constraint $(K_V(X))_{ij} = A_{ij}$ fixes the ij element of D. Also, note that $X \in \mathcal{S}^{n-1}$. It is in this lower dimensional space that the problem is solved. The optimal distance matrix is recovered by using the optimal X and the relation

$$D = \mathcal{K}_V(X).$$

Using finite precision, one can never solve the approximation problem exactly. In addition, one needs to calculate the embedding dimension. The following lemma shows that little is lost in the objective function if a small embedding dimension is chosen using a numerical rank approach i.e., if only very small eigenvalues are discarded, then the objective function changes very little.

Lemma 2.2 *Suppose that X^* solves (CDM). Let $\bar{X}$ be the closest symmetric matrix to X^* with rank k, i.e. we set the smallest $n-k$ eigenvalues of X^* to 0, $\lambda_{k+1} = \ldots \lambda_n = 0$. Then,*

$$\sqrt{f(\bar{X})} \leq \sqrt{f(X^*)} + 2\gamma\left(\sqrt{n}+1\right)\sqrt{\sum_{i=k+1}^{n} \lambda_i^2}, \tag{12}$$

where $\gamma := \max_{ij} H_{ij}$.

Proof. See [**?**]

□

2.3 Duality and Optimality

Duality theory and optimality conditions needed for a primal-dual interior-point approach are derived in this section.

For $\Lambda \in \mathcal{S}^{n-1}$ and $y \in \Re^m$, let

$$L(X, y, \Lambda) = f(X) + \langle y, b - \mathcal{A}(X) \rangle - \text{trace}\Lambda X \tag{13}$$

denote the *Lagrangian* of (CDM). It is easy to see that the primal program (CDM) is equivalent to

$$\mu^* = \min_{X} \max_{\substack{y \\ \Lambda \succeq 0}} L(X, y, \Lambda) = \min_{X \succeq 0} \max_{\substack{y \\ \Lambda \succeq 0}} L(X, y, \Lambda). \tag{14}$$

We assume that the generalized Slater's constraint qualification,

$$\exists\ X \succ 0 \text{ with } \mathcal{A}(X) = b,$$

holds for (CDM). Slater's condition implies that strong duality holds, i.e.,

$$\mu^* = \max_{\substack{y \\ \Lambda \succeq 0}} \min_{X} L(X, y, \Lambda) = \max_{\substack{y \\ \Lambda \succeq 0}} \min_{X \succeq 0} L(X, y, \Lambda), \tag{15}$$

and μ^* is attained for some y and $\Lambda \succeq 0$, see e.g. [**?**]. Since the semidefinite constraint on X can be treated as redundant, the inner minimization of the convex, in X, Lagrangian is unconstrained and we can differentiate to get the equivalent problem

$$\mu^* = \max_{\substack{\nabla f(X) - \mathcal{A}^* y = \Lambda \\ \Lambda \succeq 0}} f(X) + \langle y, b - \mathcal{A}(X) \rangle - \text{trace}\Lambda X. \tag{16}$$

We can now state the dual problem

$$\text{(DCDM)} \quad \mu^* := \begin{array}{cc} \max & f(X) + \langle y, b - \mathcal{A}(X) \rangle - \text{ trace } \Lambda X \\ \text{subject to} & \nabla f(X) - \mathcal{A}^* y - \Lambda = 0 \\ & \Lambda \succeq 0, (X \succeq 0). \end{array} \tag{17}$$

The semidefinite constraint on X is kept in brackets to emphasize that it is a hidden constraint in the dual, though it is a given constraint in the primal. The above pair of dual programs, (CDM) and (DCDM), provides an optimality criteria in terms of feasibility and complementary slackness. This provides the basis for many algorithms including primal-dual interior-point algorithms.

In particular, we see that the duality gap, in the case of primal and dual feasibility, is given by the complementary slackness condition:

$$\text{trace } X(\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(\bar{X} - B)) - \mathcal{A}^*\bar{y}) = 0, \tag{18}$$

or equivalently

$$X(\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(\bar{X} - B)) - \mathcal{A}^*\bar{y}) = 0.$$

Theorem 2.2 *Suppose that Slater's condition holds. Then $\bar{X} \succeq 0$, and $\bar{y}$, $\bar{\Lambda} \succeq 0$ solve (CDM) and (DCDM), respectively, if and only if the following three equations hold.*

$$\begin{array}{cc} \mathcal{A}(\bar{X}) = b & \textit{primal feasibility} \\ 2\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(\bar{X} - B)) - \mathcal{A}^*\bar{y} - \bar{\Lambda} = 0 & \textit{dual feasibility} \\ \textit{trace } \bar{\Lambda}\bar{X} = 0 & \textit{complementary slackness} \end{array}$$

2.4 Primal-Dual Interior-Point Algorithm

In this section, a primal-dual interior point method is derived using the log-barrier approach [**?**]. This is an alternative way of deriving the optimality conditions in Theorem **??**. For simplicity, the unconstrained problem i.e., (CDM) without the linear equality constraint $\mathcal{A}X = b$, is treated. In this case, the Slater constraint qualification holds for both primal and dual problems, as can be seen from the following lemma. (In the case that some of the elements of A are definitely fixed, then one needs to re-derive the algorithm and include the constraint $\mathcal{A}X = b$.)

Lemma 2.3 *Let H be an $n \times n$ symmetric matrix with nonnegative elements and 0 diagonal such that the graph of H is connected. Then*

$$\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(I)) \succ 0,$$

where $I \in \mathcal{S}^{n-1}$ is the identity matrix.

Proof. See [**?**]

□

Strict primal feasibility follows since every $X \succeq 0$ is feasible. Strict dual feasibility, i.e. $\Lambda \succ 0$, follows in the case of a connected graph, since a positvie scalar α can be chosen such that

$$X = B + \alpha I \succ 0; \tag{19}$$

one can then apply Lemma **??** to get

$$\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(X - B)) \succ 0. \tag{20}$$

Corollary 2.2 *Slater's constraint qualification holds for the dual problem (**??**) if and only if the graph of H is connected.*

Proof. See [**?**]

□

The log-barrier problem for (CDM) is

$$\min_{X \succ 0} B_\mu(X) := f(X) - \mu \log\det(X),$$

where $\mu \downarrow 0$. For each $\mu > 0$, we take one Newton step for solving the stationarity condition

$$\nabla B_\mu(X) = 2\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(X - B)) - \mu X^{-1} = 0. \tag{21}$$

Let

$$C := 2\mathcal{K}_V^*(H^{(2)} \circ \mathcal{K}_V(B)) = 2\mathcal{K}_V^*(H^{(2)} \circ A). \tag{22}$$

Then the stationarity condition is equivalent to

$$\nabla B_\mu(X) = 2\mathcal{K}_V^*\left(H^{(2)} \circ \mathcal{K}_V(X)\right) - C - \mu X^{-1} = 0. \tag{23}$$

By equating $\Lambda = \mu X^{-1}$ and multiplying through by X, one gets the optimality conditions, $F := \begin{pmatrix} F_d \\ F_c \end{pmatrix} = 0$,

$$\begin{array}{lllll} F_d & := 2\mathcal{K}_V^*\left(H^{(2)} \circ \mathcal{K}_V(X)\right) - C - \Lambda & = & 0 & \text{dual feas.} \\ F_c & := \Lambda X - \mu I & = & 0 & \text{perturbed compl. slack.,} \end{array} \tag{24}$$

and the estimate of the barrier parameter

$$\mu = \frac{1}{n-1} \operatorname{trace} \Lambda X. \tag{25}$$

The linearization of the optimality conditions (**??**) yield

$$\begin{array}{rcl} 2\mathcal{K}_V^*\left(H^{(2)} \circ \mathcal{K}_V(\delta X)\right) - \delta\Lambda & = & -F_d \\ \Lambda \;\; \delta X + \delta\Lambda \;\; X & = & -F_c. \end{array} \tag{26}$$

which is written in block form as

$$F' \begin{pmatrix} \delta X \\ \delta \Lambda \end{pmatrix} = \begin{bmatrix} F'_{u1} & F'_{u2} \\ F'_{l1} & F'_{l2} \end{bmatrix} \begin{pmatrix} \delta X \\ \delta \Lambda \end{pmatrix} = -F = \begin{pmatrix} -F_d \\ -F_c \end{pmatrix}, \tag{27}$$

where F' denotes the derivative of F. Following, we present the p-d i-p framework used in [**?**]. This is a common framework for both linear and semidefinite programming, see e.g. [**?**]. A centering parameter σ_k is included rather than the customary predictor-corrector approach. Let $\mathcal{F}^0$ denote the set of strictly feasible primal-dual points.

Algorithm 2.1 *(p-d i-p framework:)*
Given $(X^0, \Lambda^0) \in \mathcal{F}^0$
for $k = 0, 1, 2 \ldots$
solve *for the search direction (in a least squares sense)*

$$F'(X^k, \Lambda^k) \begin{pmatrix} \delta X^k \\ \delta \Lambda^k \end{pmatrix} = \begin{pmatrix} -F_d \\ \Lambda^k X^k + \sigma_k \mu_k I \end{pmatrix}$$

where σ_k *centering,* $\mu_k = \operatorname{trace} X^k \Lambda^k / (n-1)$

$$(X^{k+1}, \Lambda^{k+1}) = (X^k, \Lambda^k) + \alpha_k (\delta X^k, \delta \Lambda^k), \quad \alpha_k > 0,$$

so that $(X^{k+1}, \Lambda^{k+1}) \succ 0$
end (for).

The linear system for the search direction in Algorithm **??** is over-determined. Therefore, it is not clear what is meant by solving this system. There are many different search directions that have been used for SDP, see e.g. [**?**]. Two approaches to find the least squares solution of the over-determined linear system (**??**) can be used. First, one can solve the large least squares problem. This is called the Gauss-Newton (GN) method. Second, one can restrict dual feasibility and substitute for $\delta \Lambda^k$ in the second equation. This is called the restricted Gauss-Newton (RGN) method. The Gauss-Newton direction, first introduced in [**?**], always exists and it has many other good properties. See [**?**] for more details.

There does not seem to be an easy way to exploit the sparsity of H without using an iterative solution technique, e.g. a Lanczos approach to solving the least squares problem. In fact, such an approach (with a pre-conditioner) does exploit sparsity and can be very effective. Alternatively, one can exploit the fact that the sparsity pattern does not change and use a predetermined pivot order from a symbolic factorization.

Numerical test were done on randomly generated problems. The program was written in MATLAB and ran on sun SPARC 20 with 64 megs of RAM and SPECint 65.3, SPECfp 53.1. Table 1 shows the results of problems with matrices up to dimension $n = 42$. the Algorithm has proven to be efficient and robust, i.e., it has not failed on any of the test problems. An important observation is that the ranks of the optimal solution X, for sparse problems where no completion existed, were typically small. i.e., a small embedding dimension was obtained without any original rank restriction. However, when a completion existed, the typically the embedding dimension was very high. For more details see [**?**].

dim	toler	H dens.	$rank(X)$	iterations	lss cpu-time	
					GN	RGN
8	10^{-13}	.8	2	25	.16	.1
9	10^{-13}	.8	2	23	.24	.13
10	10^{-13}	.8	3	25	.34	.18
12	10^{-9}	.5	3	17	.73	.32
15	10^{-9}	.5	2	20	2.13	.79
18	10^{-9}	.5	4	20	6.15	1.9
20	10^{-9}	.3	2	20	11.35	3.3
24	10^{-9}	.3	2	20	34.45	8.4
30	10^{-9}	.3	4	20	138.	31.5
35	10^{-9}	.2	3	19	373.	77.
38	10^{-9}	.2	3	19	634.	127
40	10^{-8}	.1	2	20	845.9	181.7
42	10^{-8}	.1	4	18	1118.	232.02

Table 1: data for closest distance matrix: dimension; tolerance for duality gap; density of nonzeros in H; rank of optimal X; number of iterations; cpu-time for one least squares solution of the GN and restricted GN directions.

3 Weighted Closest Positive Semidefinite Matrix

Consider the objective function

$$f(P) := \|H \circ (A - P)\|_F^2,$$

The weighted, *closest positive semidefinite matrix problem* is

$$
\begin{array}{lccc}
 & \mu^* := & \min & f(P) \\
(CPM) & & \text{subject to} & \mathcal{A}P = b \\
 & & & P \succeq 0.
\end{array}
$$

As in the case of CDM, the linear constraint $\mathcal{A}P = b$ is added in case some elements of P are required to be exactly equal to the corresponding elements of A. However, unlike CDM, the weight matrix H here is assumed to have positive diagonal elements since otherwise if $H_{kk} = 0$, then P_{kk} is free and by setting $P_{kj} = A_{kj}$ for all j, and by taking P_{kk} large enough, P will maintain its positive semi-definiteness. Also note that CDM reduces to CPM if the linear operator $\mathcal{K}_V$ in CDM is defined as the identity operator. Thus, optimality conditions for the CPM follows from (**??**) by replacing $\mathcal{K}_V$ and $\mathcal{K}_V^*$ with the identity operator.

Theorem 3.1 *Suppose that Slater's condition holds. Then* $\bar{P} \succeq 0$, *and* $\bar{y}$, $\bar{\Lambda} \succeq 0$ *solve (CPM) and (DCPM), respectively, if and only if the following three equations hold.*

$$
\begin{array}{cc}
\mathcal{A}(\bar{P}) = b & \textit{primal feasibility} \\
2H^{(2)} \circ (\bar{P} - A) - \mathcal{A}^* \bar{y} - \bar{\Lambda} = 0 & \textit{dual feasibility} \\
\textit{trace } \bar{\Lambda}\bar{P} = 0 & \textit{complementary slackness}
\end{array}
$$

In the case that there is no linear operator $\mathcal{A}$, and in some special cases, explicit solutions can be found for CPM as can be shown by the following theorem and its corollaries.

Theorem 3.2 *Suppose that (CPM) has no linear constraint* $\mathcal{A}$. *The matrix* $\bar{P} \succeq 0$ *solves (CPM) if and only if*

$$
\textit{trace } (H^{(2)} \circ (\bar{P} - A)) \, (P - \bar{P}) \geq 0, \quad \forall P \succeq 0, \tag{28}
$$

where $H^{(2)} = H \circ H$.

Proof. See [**?**].

Let $A_{\succ}$ denote the positive semidefinite part of A.

Corollary 3.1 *If* $H = E$, *the matrix of ones, then the (unique) optimal solution of CPM is* $\bar{P} = A_{\succ}$.

Proof. See [**?**].

Corollary 3.2 *If $H \succ 0$, then the (unique) optimal solution of CPM is the solution of*

$$H \circ \bar{P} = (H \circ A)_{\succ}.$$

Proof. See [?].

Whereas a theorem similar to Theorem **??** is known for CDM [?], no explicit solutions, similar to those in the previous two corollaries, are known for the CDM.

3.1 Primal-Dual Interior-Point Algorithms

As in the case of CDM, the optimality conditions of CPM (Theorem **??**) can be derived using the log-barrier approach. The stationarity condition for the CPM is obtained from (**??**) by setting both $\mathcal{K}_V$ and $\mathcal{K}_V^*$ to the identity operator.

$$\nabla B_\mu(X) = 2H^{(2)} \circ (P - A) - \mu P^{-1} = 0. \tag{29}$$

By equating $\Lambda = \mu P^{-1}$ we get the optimality conditions,

$$\begin{array}{rcll} 2H^{(2)} \circ (P - A) - \Lambda & = & 0 & \text{dual feasibility} \\ -P + \mu\Lambda^{-1} & = & 0 & \text{perturbed complementary slackness,} \end{array} \tag{30}$$

and the estimate of the barrier parameter

$$\mu = \frac{2}{n} \operatorname{trace} PH^{(2)} \circ (P - A). \tag{31}$$

Note that the perturbed complementary slackness in (**??**) could have been written as $\Lambda P - \mu I = 0$ as was done in the CDM case. However, unlike the CDM case where there was no easy way for exploiting the sparsity of H, the current form (**??**) for CPM allows one to take full advantage of the sparsity of H. Furthermore, both dual feasibility and perturbed complementary slackness maintain symmetry in (**??**). Thus, (**??**) is a square system which can be solved using Newton's method.

The linearization of equations (**??**) yield

$$\begin{array}{rcl} 2H^{(2)} \circ \delta P - \delta\Lambda & = & -(2H^{(2)} \circ (P - A) - \Lambda) \\ P + \delta P & = & \mu\Lambda^{-1} - \mu\Lambda^{-1} \ \delta\Lambda \ \Lambda^{-1}. \end{array} \tag{32}$$

From the second equation in (**??**) it follows that

$$\delta P = \mu\Lambda^{-1} - \mu\Lambda^{-1} \ \delta\Lambda \ \Lambda^{-1} - P \tag{33}$$

and

$$\delta\Lambda = \frac{1}{\mu}(-\Lambda(P+\delta P)\Lambda) + \Lambda. \tag{34}$$

Substituting (**??**) into the first equation in (**??**), one arrives at Newton equation in terms of the dual step $\delta\Lambda$

$$2H^{(2)} \circ (\mu\Lambda^{-1} \ \ \delta\Lambda \ \ \Lambda^{-1}) + \delta\Lambda = 2H^{(2)} \circ (\mu\Lambda^{-1} - A) - \Lambda. \tag{35}$$

One can solve (**??**) for $\delta\Lambda$ and substitute in (**??**) to recover δP (*the dual-step-first algorithm*). Note that (**??**) is equivalent to

$$\begin{aligned} &[\ 2\mathrm{Diag}\,(\mathrm{vec}\,(\mathrm{H}^{(2)}))\ \mu\ (\Lambda^{-1}\otimes\Lambda^{-1}) + \mathrm{I}\ ]\ \ \mathrm{vec}\,(\delta\Lambda) \\ &\qquad\qquad = \mathrm{vec}\,(2H^{(2)}\circ(\mu\Lambda^{-1} - A) - \Lambda). \end{aligned} \tag{36}$$

In this final form possible sparsity of H can be exploited. In particular, $H_{ij} = 0$ implies that $\delta\Lambda_{ij} = 0$. Therefore, the corresponding rows and columns of the Kronecker product can be deleted when solving (**??**). This simplifies the construction of the linear system (**??**) considerably when H is sparse. On the other hand, if H has many significantly large elements. i.e., many elements of A are specified (or fixed). Then it is more efficient to substitute (**??**) into the first equation in (**??**) to obtain the following Newton equation in terms of the primal step δP

$$2H^{(2)} \circ \delta P + \frac{1}{\mu}\Lambda \ \ \delta P \ \ \Lambda = \Lambda - \frac{1}{\mu}\Lambda P \Lambda - (2H^{(2)} \circ (P - A) - \Lambda) \tag{37}$$

or equivalently,

$$\begin{aligned} &[\ 2\mathrm{Diag}\,(\mathrm{vec}\,(\mathrm{H}^{(2)})) + \tfrac{1}{\mu}(\Lambda\otimes\Lambda)\ ]\ \ \mathrm{vec}\,(\delta \mathrm{P}) \\ &= \mathrm{vec}\,(\Lambda - \tfrac{1}{\mu}\Lambda P\Lambda - (2H^{(2)}\circ(P - A) - \Lambda)). \end{aligned} \tag{38}$$

Then, this system is solved for δP and substitute in (**??**) to recover $\delta\Lambda$ (*the primal-step-first algorithm).* Note that if P_{ij} is fixed, i.e., if H_{ij} is sufficiently large, then $\delta P_{ij} = 0$. Therefore, the system (**??**) can be very small if H has many elements which are sufficiently large.

In [**?**], numerical tests on randomly generated dense and sparse problems were done on a SUN SPARC 1 with 16megs RAM and with SPECint18.2 and SPECfp 17.9. The program was written in MATLAB. Tables 2 and 3 summarize the results of these test for the dual-step-first and for the primal-step-first algorithms respectively. For more details see [**?**].

dim	toler	H dens. / infty	$A \succeq 0$	cond(A)	$H \succ 0$	min/max	iters
60	10^{-6}	.01 / .001	yes	79.738	no	15/23	16.8
65	10^{-6}	.015 / .001	yes	49.8611	yes	18/24	21.25
83	10^{-6}	.007 / .001	no	235.1547	no	24/29	25.45
85	10^{-5}	.008 / .001	yes	94.6955	no	11/17	13.05
85	10^{-6}	.0075 / .001	no	299.8517	no	23/27	25.25
87	10^{-6}	.006 / .001	yes	74.163	yes	14/19	16.85
89	10^{-6}	.006 / .001	no	179.33	no	23/28	15.2
110	10^{-6}	.007 / .001	yes	172.255	yes	15/20	17.8
155	10^{-6}	.01 / 0	yes	643.9619	yes	14/18	15.3

Table 2: data for dual-step-first (20 problems per test); dimension; tolerance for duality gap; density of nonzeros in H/ density of infinite values in H; positive semidefiniteness of A; positive definiteness of H; min and max number of iterations; average number of iterations.

4 Other Completion Problems

The list of papers on completion problems and applications is large and varied. The different types of completion problems include: maximum rank completions; M-matrices and inverse M-matrices; partially upper triangular matrices; partial Jordan matrices; completions to normal matrices and to P-matrices. One can see the vast assortment of further problems and applications by searching MathSciNet with the keywords *matrix* and *completion*.

dim	toler	H dens. / infty	$A \succeq 0$	cond(A)	$H \succ 0$	min/max	iters
15	10^{-5}	.0751 / .02	yes	222.52	no	8/17	10.25
15	10^{-6}	.1 / .95	yes	19.6102	no	10/23	15.55
15	10^{-6}	.01 / .95	yes	21.0473	no	10/20	13.2
19	10^{-6}	.005 / .1	yes	14.811	no	10/18	12.95
21	10^{-6}	.005 / .1	yes	20.3077	no	8/24	15
38	10^{-6}	1 / .99	yes	49.6068	yes	14/24	16.3
45	10^{-6}	1 / .99	yes	46.8396	yes	15/22	17.1
55	10^{-6}	1 / .99	yes	37.1883	yes	15/30	17.45
85	10^{-5}	.0219 / .02	yes	1374.54	no	16/23	18.95
95	10^{-5}	.0206 / .02	yes	2.6983	no	8/14	11.05
95	10^{-6}	1 / .999	yes	196.0130	yes	14/18	16.8
145	10^{-6}	.01 / .997	yes	658.5103	yes	13/17	14.95

Table 3: data for primal-step-first (20 problems per test): dimension; tolerance for duality gap; density of nonzeros in H/ density of infinite values in H; positive semidefiniteness of A; positive definiteness of H; min and max number of iterations; average number of iterations.