

ON STRESS MATRICES OF CHORDAL BAR FRAMEWORKS IN GENERAL POSITION

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ABSTRACT. A bar framework in \mathbb{R}^r , denoted by $G(p)$, is a simple connected graph G whose vertices are points p^1, \dots, p^n in \mathbb{R}^r that affinely span \mathbb{R}^r , and whose edges are line segments between pairs of these points. In this paper, we use stress matrices to characterize the universal and global rigidities of chordal bar frameworks in general position in \mathbb{R}^r , i.e., bar frameworks where graph G is chordal and the points p^1, \dots, p^n are in general position in \mathbb{R}^r . We also prove that if a chordal bar framework in \mathbb{R}^r admits a stress matrix of rank $n - r - 1$ with generic rank profile, then it admits a positive semidefinite stress matrix of rank $n - r - 1$.

1. INTRODUCTION

A *bar framework* (a framework for short) in r -dimensional Euclidean space, denoted by $G(p)$, is a connected simple graph $G = (V, E)$ (a graph with no loops or multiple edges) whose vertices are points p^1, \dots, p^n in \mathbb{R}^r that affinely span \mathbb{R}^r , and whose edges are line segments between pairs of these points. The points p^1, \dots, p^n , which collectively are denoted by p , are referred to as the *configuration* of the framework. Throughout this paper, $V(G)$ and $E(G)$ denote the vertex set and the edge set of graph G respectively; and $\mathbf{0}$ denotes the zero matrix or the zero vector of the appropriate dimension.

We say that frameworks $G(p)$ and $G(q)$ in \mathbb{R}^r are *congruent* if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \dots, n$, where $\|\cdot\|$ denotes the Euclidean norm. For example, if configuration q is obtained from configuration p by a rigid motion, then $G(p)$ and $G(q)$ are congruent. On the other hand, frameworks $G(p)$ in \mathbb{R}^r and $G(q)$ in \mathbb{R}^s are said to be *equivalent* if the corresponding edges in the two frameworks have the same length, i.e., if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i, j) \in E(G)$.

A framework $G(p)$ in \mathbb{R}^r is said to be *globally rigid* if every framework $G(q)$ in the same dimension which is equivalent to $G(p)$, is in fact congruent

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to $G(p)$. Furthermore, if every framework $G(q)$ in any dimension which is equivalent to $G(p)$ is congruent to $G(p)$, then $G(p)$ is said to be *universally rigid*. Obviously, universal rigidity implies global rigidity.

A real valued function ω on the edge set $E(G)$ such that

$$\sum_{j:(i,j) \in E(G)} \omega_{ij}(p^i - p^j) = \mathbf{0} \text{ for all } i = 1, \dots, n \quad (1.1)$$

is called an *equilibrium stress* of framework $G(p)$. Furthermore, given an equilibrium stress ω , the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i, j) \in E(G), \\ 0 & \text{if } i \neq j \text{ and } (i, j) \notin E(G), \\ \sum_{k:(i,k) \in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \quad (1.2)$$

is called a *stress matrix* of $G(p)$.

The problems of universal and global rigidities of bar frameworks have many important applications in molecular conformations and sensor networks. As a result, they have received a great deal of attention. Furthermore, under the assumption of generic configurations, global and universal rigidities have nice characterizations in terms of stress matrices [7, 15, 2, 13, 8, 12]. A configuration p is said to be generic if the coordinates of p^1, \dots, p^n are algebraically independent over the integers.

The assumption of a generic configuration is quite strong. A study of the problem of universal rigidity under the weaker assumption of points in general position was initiated in [4, 3]. We say that a configuration p in \mathbb{R}^r is in *general position* if every $(r + 1)$ of the points p^1, \dots, p^n are affinely independent. Recall that points p^1, \dots, p^{r+1} are *affinely independent* if the trivial solution $\lambda_1 = \dots = \lambda_{r+1} = 0$ is the only solution for the system of equations

$$\sum_{i=1}^{r+1} \lambda_i \begin{bmatrix} p^i \\ 1 \end{bmatrix} = \mathbf{0}. \quad (1.3)$$

For example, points in \mathbb{R}^2 are in general position if no three of them are collinear. An $n \times n$ real symmetric matrix A is *positive semidefinite* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$.

The following two theorems are the main results in [4] and [3].

Theorem 1.1 (Alfakih and Ye [4]). *Let $G(p)$ be a bar framework on n vertices in general position in \mathbb{R}^r . Then $G(p)$ is universally rigid if $G(p)$ admits a positive semidefinite stress matrix S of rank $n - r - 1$.*

Theorem 1.2 (Alfakih et al [3]). *Let $G(p)$ be a bar framework on n vertices in general position in \mathbb{R}^r , where G is an $(r + 1)$ -lateration graph. Then $G(p)$ admits a positive semidefinite stress matrix S of rank $n - r - 1$.*

This paper builds on some of the ideas in [3] to characterize, the universal and global rigidities of chordal frameworks in general position, i.e., frameworks whose graphs are chordal, and whose configurations are in general

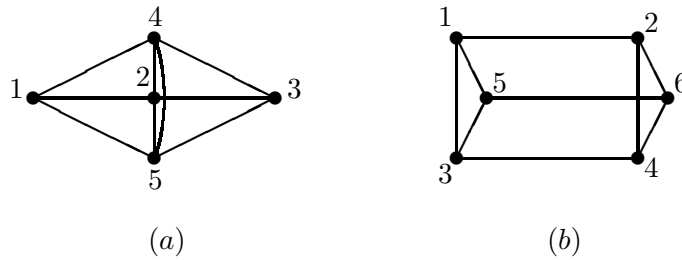


FIGURE 1. Two non-globally rigid frameworks in \mathbb{R}^2 . The edge (4, 5) in framework (a) is shown as an arc to make edges (2, 4) and (2, 5) visible.

position. A graph G is said to be *chordal* (also called *triangulated*, *rigid circuit*) if every cycle of G of length ≥ 4 has a chord, i.e., an edge that connects two non-consecutive vertices on the cycle.

A graph G is said to be *k -vertex connected* iff G is the complete graph on $k+1$ vertices, or $|V(G)| \geq k+2$ and the deletion of any $k-1$ vertices leaves G connected. The next two theorems are the main results of this paper.

Theorem 1.3. *Let $G(p)$ be a chordal bar framework on n vertices in general position in \mathbb{R}^r . Then the following statements are equivalent:*

- (1) G is $(r+1)$ -vertex connected.
- (2) $G(p)$ admits a positive semidefinite stress matrix S of rank $n-r-1$.
- (3) $G(p)$ is universally rigid.
- (4) $G(p)$ is globally rigid.

The proof of Theorem 1.3 is given in Section 3. We remark here that the equivalence of statements (1) and (3) in Theorem 1.3 also follows from a result by Bakonyi and Johnson concerning Euclidean distance matrix completions [5]. The assumptions of chordality and general position cannot be dropped in Theorem 1.3. Figure 1 depicts two frameworks in \mathbb{R}^2 that are not globally rigid. Framework (a) is 3-vertex connected and chordal, but it is not in general position; while framework (b) is in general position and 3-vertex connected, but it is not chordal.

The k th leading principal minor of an $n \times n$ matrix A is the determinant of the square submatrix obtained by deleting the last $(n-k)$ rows and columns of A . Let A be a given symmetric matrix of rank k , then A is said to have *generic rank profile* if its first k leading principal minors are nonzero.

Theorem 1.4. *Let $G(p)$ be a chordal bar framework on n vertices in general position in \mathbb{R}^r . If $G(p)$ admits a stress matrix S of rank $n-r-1$ with generic rank profile, then it admits a positive semidefinite stress matrix S' of rank $n-r-1$.*

The proof of Theorem 1.4 is given in Section 4.

2. PRELIMINARIES

In this section we present some necessary mathematical preliminaries. In particular we review some facts concerning chordal graphs and Gale matrices, which will be used in our proofs.

2.1. Chordal Graphs. A simple graph $G = (V, E)$ is *complete* if the nodes of G are pairwise adjacent. A complete subgraph is called a *clique*. For a vertex v in G , $N(v)$ denotes the set of nodes of G that are adjacent to v . A vertex v of G is said to be *simplicial* if $N(v)$ is a clique in G . A permutation of the vertices of G , $\pi(1), \pi(2), \dots, \pi(n)$, is called a *perfect elimination ordering (PEO)* of G if for each $i = 1, \dots, n-1$, $N(\pi(i))$ is simplicial in the graph induced by $\{\pi(i), \pi(i+1), \dots, \pi(n)\}$.

Among the many well-known characterizations of chordal graphs [6, 11], the one presented in the following lemma is the most useful for the purposes of this paper.

Theorem 2.1 (Fulkerson and Gross [9]). *A graph G is chordal if and only if G has a perfect elimination ordering.*

A PEO of a chordal graph G on n vertices can be found by using, for example, the *Maximum Cardinality Search (MCS)* algorithm [20, 21]. This algorithm runs in $O(n + |E(G)|)$ time, and obtains a PEO of G as follows. Initially all vertices of G are unlabeled. Arbitrarily select a vertex and label it $\pi(n)$. Then for $i = n-1, n-2, \dots, 1$, select a vertex which is adjacent to the largest number of already labeled vertices, breaking ties arbitrarily, and label it $\pi(i)$. One can then prove that $\pi(1), \dots, \pi(n)$ is a PEO of G . For ease of notation and without loss of generality, we assume throughout the paper that $1, 2, \dots, n$ is a PEO of G .

The following lemma will be needed in the sequel.

Lemma 2.1 (Lin et al [17]). *Let G be a chordal graph on n vertices and let $1, 2, \dots, n$ be a PEO of G . Furthermore, let $\hat{N}(j) = \{i : i > j \text{ and } (i, j) \in E(G)\}$. Then G is k -vertex connected if and only if $|\hat{N}(j)| \geq k$ for all $j = 1, \dots, n-k$, where $|\hat{N}(j)|$ denotes the cardinality of $\hat{N}(j)$.*

2.2. Gale Matrices and Stress Matrices. Gale matrices are intimately related to stress matrices. Given a framework $G(p)$ in \mathbb{R}^r , the $(r+1) \times n$ matrix

$$\mathcal{P} := \begin{bmatrix} p^1 & \cdots & p^n \\ 1 & \cdots & 1 \end{bmatrix}, \quad (2.1)$$

is called the *extended configuration matrix* of $G(p)$. \mathcal{P} has full row rank since p^1, \dots, p^n affinely span \mathbb{R}^r . Furthermore, under the general position assumption, every $(r+1) \times (r+1)$ submatrix of \mathcal{P} is nonsingular. Let \bar{r} be the nullity of \mathcal{P} , i.e., the dimension of its null space. Then

$$\bar{r} = n - r - 1. \quad (2.2)$$

For $\bar{r} \geq 1$, any $n \times \bar{r}$ matrix Z whose columns form a basis of the null space of \mathcal{P} is called a *Gale matrix* of $G(p)$ [10]. Gale matrices (or Gale transform) are widely used in polytope theory [14]. Two remarks are in order here. First, the columns of Z express the affine dependencies among the points p^1, \dots, p^n . Second, if Z is a Gale matrix of $G(p)$ and Q is any nonsingular $\bar{r} \times \bar{r}$ matrix. Then $Z' = ZQ$ is also a Gale matrix of $G(p)$. This fact will be used very often in the sequel.

It is not hard to see from (1.2) that an $n \times n$ symmetric matrix $S = (s_{ij})$ is a stress matrix of $G(p)$ if and only if

- (1) $\mathcal{P}S = \mathbf{0}$, and
- (2) $s_{ij} = 0$ for all $ij : i \neq j, (i, j) \notin E(G)$,

where \mathcal{P} is the extended configuration matrix of $G(p)$. The following lemma establishes the relationship between Gale matrices and stress matrices.

Lemma 2.2 (Alfakih [2]). *Let $G(p)$ be a bar framework on n vertices in \mathbb{R}^r , and let S and Z be, respectively, a stress matrix and a Gale matrix of $G(p)$. Then $S = Z\Psi Z^T$ for some $\bar{r} \times \bar{r}$ symmetric matrix Ψ . Conversely, let z^i denote the i th row of Z and let Ψ' be a symmetric matrix such that $(z^i)^T \Psi' z^j = 0$ for all $ij, i \neq j, (i, j) \notin E(G)$. Then $S' = Z\Psi' Z^T$ is a stress matrix of $G(p)$.*

It immediately follows from Lemma 2.2 that $\text{rank } S \leq \bar{r}$, and $\text{rank } S = \bar{r}$ if and only if Ψ is nonsingular.

The next lemma shows that Gale matrices have a useful property under the general position assumption.

Lemma 2.3. *Let $G(p)$ be a framework on n vertices in general position in \mathbb{R}^r , and let Z be a Gale matrix of $G(p)$. Then every $\bar{r} \times \bar{r}$ submatrix of Z is nonsingular.*

For a proof of Lemma 2.3 see, for example, [1]. The next theorem is an immediate corollary of Lemma 2.3.

Theorem 2.2. *Let $G(p)$ be a framework on n vertices in general position in \mathbb{R}^r and let S be a stress matrix of $G(p)$ of rank \bar{r} . Then every $\bar{r} \times \bar{r}$ submatrix of S is nonsingular.*

Proof. Let Z be a Gale matrix of $G(p)$. Then $S = Z\Psi Z^T$ for some $\bar{r} \times \bar{r}$ symmetric matrix Ψ . Moreover, Ψ is nonsingular since $\text{rank } S = \bar{r}$. Let α and β be two subsets of $\{1, 2, \dots, n\}$ of cardinality \bar{r} , and let $S_{\alpha, \beta}$ be the submatrix of S whose rows and columns are indexed by α and β

respectively. Further, let Z_α be the submatrix of Z of order \bar{r} whose rows are indexed by α . Then

$$S_{\alpha,\beta} = Z_\alpha \Psi Z_\beta^T.$$

Thus it follows from Lemma 2.3 that Z_α and Z_β^T are nonsingular and hence $S_{\alpha,\beta}$ is nonsingular. \square

2.3. Gale Matrices and Chordal Frameworks. Let $G(p)$ be a chordal framework and let $1, 2, \dots, n$ be a PEO of G . As we saw earlier, $G(p)$ does not admit a unique Gale matrix. Of particular interest to our purposes is a Gale matrix $Z = (z_{ij})$ that satisfies the following property:

$$(A) \quad z_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i < j, \\ 0 & \text{if } i > j \text{ and } (i, j) \notin E(G). \end{cases}$$

That is, Z is of the form

$$\begin{bmatrix} 1 & & & & & \\ * & 1 & & & & \\ & * & 1 & & & \\ * & * & * & \ddots & & \\ * & & * & & 1 & \\ & & * & * & * & \\ * & * & & * & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ & * & * & * & * & \end{bmatrix},$$

where zero entries are left blank and possibly nonzero entries, i.e., entries ij where $i > j$ and $(i, j) \in E(G)$, are indicated with (*).

The importance of Property (A) is illustrated in the next Lemma.

Lemma 2.4. *Let $G(p)$ be a chordal framework on n vertices in \mathbb{R}^r , and assume that $1, 2, \dots, n$ is a PEO of G . If $G(p)$ admits a Gale matrix Z that satisfies Property (A), then $G(p)$ admits a positive semidefinite stress matrix S with rank \bar{r} .*

Proof. Let Z be a Gale matrix of $G(p)$ that satisfies Property (A) and let $S = ZZ^T$. Then obviously S is symmetric, positive semidefinite, and of rank \bar{r} . Thus, it suffices to show that $s_{ij} = 0$ for all $ij : i \neq j, (i, j) \notin E(G)$. To this end, let $(i, j) \notin E(G)$ and $i < j$. Then

$$s_{ij} = \sum_{k=1}^{\bar{r}} z_{ik} z_{jk} = \sum_{k=1}^{\min(i, \bar{r})} z_{ik} z_{jk}$$

since $z_{ik} = 0$ for $k > i$. But for any $k : 1 \leq k < i < j$ and $(i, j) \notin E(G)$, either $i \notin \hat{N}(k)$ (which implies that $z_{ik} = 0$) or $j \notin \hat{N}(k)$ (which implies

that $z_{jk} = 0$) since $\hat{N}(k)$ is a clique. Therefore,

$$\sum_{k=1}^{\min(i, \bar{r})} z_{ik} z_{jk} = \begin{cases} 0 & \text{if } \bar{r} < i, \\ z_{ii} z_{ji} & \text{if } \bar{r} \geq i. \end{cases}$$

But $z_{ji} = 0$ since $(j, i) \notin E(G)$ and $i < j$. Therefore, $s_{ij} = 0$ for all $ij : i < j, (i, j) \notin E(G)$ and the result follows since S is symmetric. \square

3. PROOF OF THEOREM 1.3

The following lemma is needed for our proof.

Lemma 3.1. *Let $G(p)$ be a chordal framework on n vertices in general position in \mathbb{R}^r and assume that $1, 2, \dots, n$ is a PEO of G . If G is $(r+1)$ -vertex connected then $G(p)$ admits a Gale matrix Z that satisfies Property (A).*

Proof. Recall that $\hat{N}(j) = \{i : i > j \text{ and } (i, j) \in E(G)\}$. Since G is $(r+1)$ -vertex connected, it follows from Lemma 2.1 that $|\hat{N}(j)| \geq r+1$ for all $j = 1, \dots, \bar{r}$. Under the general position assumption, for each $j = 1, \dots, \bar{r}$, the following system of equations

$$\begin{bmatrix} p^j \\ 1 \end{bmatrix} + \sum_{i \in \hat{N}(j)} x_{ij} \begin{bmatrix} p^i \\ 1 \end{bmatrix} = \mathbf{0} \quad (3.1)$$

has a solution \hat{x}_{ij} . Thus, let $Z = (z_{ij})$ be the $n \times \bar{r}$ matrix where

$$z_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \hat{x}_{ij} & \text{if } i \in \hat{N}(j), \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Therefore, Z is a Gale matrix of $G(p)$ that satisfies Property (A). \square

Proof of Theorem 1.3.

(1) \Rightarrow (2) follows from Lemmas 3.1 and 2.4, (2) \Rightarrow (3) follows from Theorem 1.1, (3) \Rightarrow (4) is obvious, (4) \Rightarrow (1) follows from Hendrickson's necessary conditions for global rigidity [15]. If G is not $(r+1)$ -vertex connected, then there exists a vertex cut X of size $\leq r$ whose removal disconnects $G(p)$ into at least two subframeworks. Since the vertices in X are contained in an $(r-1)$ -dimensional hyperplane, one could obtain another framework $G(q)$ that is equivalent, but not congruent, to $G(p)$ by reflecting one of the subframeworks with respect to this hyperplane. Hence, $G(p)$ is not globally rigid. \square

Remark 3.1. *One might be tempted to simplify the proof that (1) \Rightarrow (2) in Theorem 1.3 as follows. A k -tree is either the complete graph on k vertices, or a graph obtained from a k -tree H by adding a new vertex adjacent to*

applied to A without row exchanges, and hence $A^{(k)}$ is of the form given in (4.1) [16, 19].

Lemma 4.1. *Let G be a chordal graph on n vertices and let A be an $n \times n$ symmetric matrix of rank k with generic rank profile such that $a_{ij} = 0$ for all $ij : i \neq j$ and $(i, j) \notin E(G)$. Further, assume that $1, 2, \dots, n$ is a PEO of G . Let $A^{(t)} = (a_{ij}^{(t)})$ denote the matrix A after t steps of Gauss elimination. Then, $a_{ij}^{(t)} = 0$ for all $ij : i \neq j$ and $(i, j) \notin E(G)$, and for all $t = 1, \dots, k$.*

Proof. Since A has generic rank profile, no row exchanges are needed during Gauss elimination, i.e., $a_{11} \neq 0$, $a_{22}^{(1)} \neq 0, \dots, a_{kk}^{(k-1)} \neq 0$. Therefore,

$$a_{ij}^{(t)} = \begin{cases} a_{ij}^{(t-1)} & \text{if } i < t, \\ a_{ij}^{(t-1)} / a_{tt}^{(t-1)} & \text{if } i = t, \\ 0 & \text{if } i > t, j = 1, \dots, t, \\ a_{ij}^{(t-1)} - \frac{a_{it}^{(t-1)} a_{tj}^{(t-1)}}{a_{tt}^{(t-1)}} & \text{if } i > t, j > t, \end{cases} \quad (4.2)$$

where $A^{(0)} = A$.

Since G is chordal and since $1, 2, \dots, n$ is a PEO of G , it follows that for all $ij : i \neq j$ and $(i, j) \notin E(G)$, and for all $s < \min\{i, j\}$, either $(s, i) \notin E(G)$ or $(s, j) \notin E(G)$ since if $(s, i) \in E(G)$ and $(s, j) \in E(G)$, then $(i, j) \in E(G)$, an obvious contradiction.

The proof is by induction on t . Let $i \neq j$ and $(i, j) \notin E(G)$ then $a_{ij} = 0$. Thus for $t = 1$ we have

$$a_{ij}^{(1)} = \begin{cases} a_{ij} / a_{11} & \text{if } i = 1, \\ 0 & \text{if } i > 1, j = 1, \\ a_{ij} - \frac{a_{i1} a_{1j}}{a_{11}} & \text{if } i > 1, j > 1. \end{cases}$$

Therefore, $a_{ij}^{(1)} = 0$ since $a_{ij} = 0$, and since if $i > 1$ and $j > 1$, then either $a_{1i} = 0$ or $a_{1j} = 0$. Therefore, the statement of the lemma is true for $t = 1$. Now assume that the statement of the lemma is true for $t = m - 1$ for some $m : 2 \leq m \leq k$. That is, $a_{ij}^{(m-1)} = 0$ for all $ij : i \neq j$ and $(i, j) \notin E(G)$. Then it follows from (4.2) that $a_{ij}^{(m)} = 0$ since $a_{ij}^{(m-1)} = 0$, and since if $i > m$ and $j > m$, then either $a_{im}^{(m-1)} = 0$ or $a_{mj}^{(m-1)} = 0$. Thus the result follows. \square

Proof of Theorem 1.4

Let $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ be a stress matrix of $G(p)$ of rank \bar{r} with generic rank profile, where S_1 is $\bar{r} \times n$ and S_2 is $(r + 1) \times n$. Then the columns of S span the null space of \mathcal{P} defined in (2.1). But since S has generic rank profile, it follows that the first \bar{r} rows of S are linearly independent. Thus S_1^T is a Gale matrix of $G(p)$.

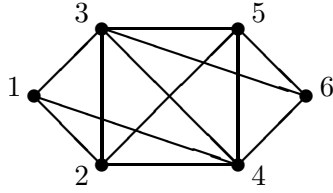


FIGURE 3. The framework $G(p)$ of Example 4.1.

Now let $S^{(\bar{r})} = \begin{bmatrix} S_1^{(\bar{r})} \\ \mathbf{0} \end{bmatrix}$ be the matrix S after \bar{r} steps of Gauss elimination, where $S_1^{(\bar{r})}$ is $\bar{r} \times n$ of rank \bar{r} , and $\mathbf{0}$ is the $(r+1) \times n$ zero matrix; and let $Z^T = S_1^{(\bar{r})}$. Then Z is a Gale matrix of $G(p)$ since the rows of Z^T are linear combinations of the rows of S_1^T . Furthermore, since $s_{ij} = 0$ whenever $i \neq j$ and $(i, j) \notin E(G)$, it follows from Lemma 4.1 that if $i < j$, $i \leq \bar{r}$ and $(i, j) \notin E(G)$, then $z_{ij}^T = 0$. Furthermore, $z_{ii}^T = 1$ for $i = 1, \dots, \bar{r}$ and $z_{ij}^T = 0$ if $i > j$. Hence, Z satisfies Property (A). Therefore, the result follows from Lemma 2.4. □

Example 4.1. Consider the chordal framework in general position in \mathbb{R}^2 shown in Figure 3, where $p^1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $p^2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $p^3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $p^4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $p^5 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $p^6 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then

$$S = \begin{bmatrix} 10 & -10 & -5 & 5 & 0 & 0 \\ -10 & 8 & 7 & -3 & -2 & 0 \\ -5 & 7 & 1 & -5 & 1 & 1 \\ 5 & -3 & -5 & 1 & 3 & -1 \\ 0 & -2 & 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 & -2 & 2 \end{bmatrix}$$

is an indefinite stress matrix of $G(p)$ of rank 3 with generic rank profile since the first 3 leading principal minors are nonzero. Applying Gauss elimination

on S leads to

$$S^{(3)} = \begin{bmatrix} 1 & -1 & -0.5 & 0.5 & 0 & 0 \\ & 1 & -1 & -1 & 1 & 0 \\ & & 1 & -1 & -2 & 2 \end{bmatrix}.$$

Hence,

$$Z = \begin{bmatrix} 1 \\ -1 & 1 \\ -0.5 & -1 & 1 \\ 0.5 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix},$$

is a Gale matrix of $G(p)$ that satisfies Property (A). Furthermore,

$$ZZ^T = \begin{bmatrix} 1 & -1 & -0.5 & 0.5 & 0 & 0 \\ -1 & 2 & -0.5 & -1.5 & 1 & 0 \\ -0.5 & -0.5 & 2.25 & -0.25 & -3 & 2 \\ 0.5 & -1.5 & -0.25 & 2.25 & 1 & -2 \\ 0 & 1 & -3 & 1 & 5 & -4 \\ 0 & 0 & 2 & -2 & -4 & 4 \end{bmatrix}$$

is a positive semidefinite stress matrix of $G(p)$ of rank 3.

5. SUMMARY AND OPEN PROBLEM

This paper is a continuation of the study, initiated in [4, 3], of the universal rigidity of bar frameworks under the general position assumption. In particular, we studied stress matrices of chordal frameworks in general position. We characterized such frameworks in terms of stress matrices, and we showed that for such frameworks universal and global rigidities are equivalent. Furthermore, we showed that if a chordal framework on n vertices in general position in r -dimensions admits a stress matrix of rank $n - r - 1$ with generic rank profile, then it admits a positive semidefinite stress matrix of rank $n - r - 1$. These results suggest the following problems:

- (1) Characterize frameworks $G(p)$ in general position for which universal and global rigidities are equivalent.
- (2) Characterize frameworks $G(p)$ in general position having the property that if $G(p)$ has a stress matrix of rank $n - r - 1$ with generic rank profile, then it has a positive semidefinite stress matrix of rank $n - r - 1$.

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