On Affine Motions and Bar Frameworks in General Position

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Abstract

A configuration p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ that affinely span $\mathbb{R}^r$. A bar framework, denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph G on n vertices together with a configuration p in $\mathbb{R}^r$. A given bar framework $G(p)$ is said to be universally rigid if there does not exist another configuration q in any Euclidean space, not obtained from p by a rigid motion, such that $||q^i - q^j|| = ||p^i - p^j||$ for each edge (i, j) of G.

It is known [2, 6] that if configuration p is generic and bar framework $G(p)$ in $\mathbb{R}^r$ admits a positive semidefinite stress matrix S of rank ($n - r - 1$), then $G(p)$ is universally rigid. Connelly asked [8] whether the same result holds true if the genericity assumption of p is replaced by the weaker assumption of general position. We answer this question in the affirmative in this paper.

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1 Introduction

A *configuration* p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ in $\mathbb{R}^r$ that affinely span $\mathbb{R}^r$. A *bar framework* (or framework for short) in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration p in $\mathbb{R}^r$ together with a simple graph G on the vertices $1, 2, \ldots, n$. For a simple graph G, we denote its node set by $V(G)$ and its edge set by $E(G)$. To avoid trivialities, we assume throughout this paper that graph G is connected and not complete.

Framework $G(q)$ in $\mathbb{R}^r$ is said to be *congruent* to framework $G(p)$ in $\mathbb{R}^r$ if configuration q is obtained from configuration p by a rigid motion. That is, if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \ldots, n$, where $\|.\|$ denotes the Euclidean norm. We say that framework $G(q)$ in $\mathbb{R}^s$ is *equivalent* to framework $G(p)$ in $\mathbb{R}^r$ if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i, j) \in E(G)$. Furthermore, we say that framework $G(q)$ in $\mathbb{R}^r$ is *affinely-equivalent* to framework $G(p)$ in $\mathbb{R}^r$ if $G(q)$ is equivalent to $G(p)$ and configuration q is obtained from configuration p by an affine motion; i.e., $q^i = Ap^i + b$, for all $i = 1, \ldots, n$, for some $r \times r$ matrix A and an r-vector b.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if there does not exist a framework $G(q)$ in any Euclidean space that is equivalent, but not congruent, to $G(p)$. The notion of a stress matrix S of a framework $G(p)$ plays a key role in the problem of universal rigidity of $G(p)$.

1.1 Stress Matrices and Universal Rigidity

Let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$. An *equilibrium stress* of $G(p)$ is a real valued function ω on $E(G)$ such that

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n. \tag{1}$$

Let ω be an equilibrium stress of $G(p)$. Then the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin E(G), \\ \displaystyle\sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \tag{2}$$

is called the *stress matrix* associated with ω, or a stress matrix of $G(p)$. The following result provides a sufficient condition for the universal rigidity of a given framework.

Theorem 1.1 (Connelly [5, 6], Alfakih [1]) *Let $G(p)$ be a bar framework in $\mathbb{R}^r$, for some $r \leq n-2$. If the following two conditions hold:*

1. *There exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*
2. *There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.*

Then $G(p)$ is universally rigid.

Note that $(n-r-1)$ is the maximum possible value for the rank of the stress matrix S. In connection with Theorem 1.1, we mention the following result obtained in So and Ye [11] and Biswas et al. [4]: Given a framework $G(p)$ in $\mathbb{R}^r$, if there does not exist a framework $G(q)$ in $\mathbb{R}^s$ ($s \neq r$) that is equivalent to $G(p)$, then $G(p)$ is universally rigid. Moreover, if $G(p)$ contains a clique of $r+1$ points in general position, then the existence of a rank-$(n-r-1)$ positive semidefinite stress matrix implies that framework $G(p)$ is universally rigid, regardless whether the other non-clique points are in general position or not.

Condition 2 of Theorem 1.1 is satisfied if configuration p is assumed to be generic (see Lemma 2.2 below). A configuration p (or a framework $G(p)$) is said to be *generic* if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, \ldots, p^n) = 0$. Thus

Theorem 1.2 (Connelly [6], Alfakih [2]) *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leq n-2$. If there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$. Then $G(p)$ is universally rigid.*

The converse of Theorem 1.2 is also true.

Theorem 1.3 (Gortler and Thurston [10]) *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leq n-2$. If $G(p)$ is universally rigid, then there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*

Connelly [8] asked whether a result similar to Theorem 1.2 holds if the genericity assumption of $G(p)$ is replaced by the weaker assumption of general position. A configuration p (or a framework $G(p)$) in $\mathbb{R}^r$ is said to be in *general position* if no subset of the points $p^1, \ldots, p^n$ of cardinality $r+1$ is affinely dependent. For example, a set of points in the plane are in general position if no 3 of them lie on a straight line.

In this paper we answer Connelly's question in the affirmative. Thus the following theorem is the main result of this paper.

Theorem 1.4 *Let $G(p)$ be a bar framework on n nodes in general position in $\mathbb{R}^r$, for some $r \leq n-2$. If there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$. Then $G(p)$ is universally rigid.*

The proof of Theorem 1.4 will be given in Section 3. This paper and [3] are first steps toward the study of universal rigidity under the general position assumption. In [3], it was shown that the framework $G(p)$ on n nodes in general position in $\mathbb{R}^r$ for some $r \leq n-2$, where G is the $(r+1)$-lateration graph, admits a rank $(n-r-1)$ positive semi-definite stress matrix.

2 Preliminaries

To develop the ingredients needed for the proof of our main result, we review the necessary background on affine motions, stress matrices, and Gale matrices.

An affine motion in $\mathbb{R}^r$ is a map $f : \mathbb{R}^r \to \mathbb{R}^r$ of the form

$$f(p^i) = Ap^i + b,$$

for all p^i in $\mathbb{R}^r$, where A is an $r \times r$ matrix and b is an r-vector. A rigid motion is an affine motion where matrix A is orthogonal.

Vectors $v^1, \ldots, v^m$ in $\mathbb{R}^r$ are said to lie on a *quadratic at infinity* if there exists a non-zero symmetric $r \times r$ matrix Φ such that

$$(v^i)^T \Phi v^i = 0, \text{ for all } i = 1, \ldots, m. \tag{3}$$

Lemma 2.1 *(Connelly [7]) Let $G(p)$ be a bar framework on n vertices in $\mathbb{R}^r$. Then the following two conditions are equivalent:*

1. *There exists a framework $G(q)$ in $\mathbb{R}^r$ that is equivalent, but not congruent, to $G(p)$ such that $q^i = Ap^i + b$ for all $i = 1, \ldots, n$,*
2. *The vectors $p^i - p^j$ for all $(i,j) \in E(G)$ lie on a quadratic at infinity.*

Lemma 2.2 *(Connelly [7]) Let $G(p)$ be a generic bar framework on n vertices in $\mathbb{R}^r$. Assume that each node of G has degree at least r. Then the vectors $p^i - p^j$ for all $(i,j) \in E(G)$ do not lie on a quadratic at infinity.*

Therefore, under the genericity assumption, Condition 2 in Lemma 2.1 does not hold. Consequently, Theorem 1.2 follows as a simple corollary of Theorem 1.1.

Note that Condition 2 in Lemma 2.1 is expressed in terms of the edges of G. An equivalent condition in terms of the missing edges of G can also be obtained using Gale matrices. This equivalent condition turns out to be crucial for our proof of Theorem 1.4.

To this end, let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$. Then the following $(r+1) \times n$ matrix

$$\mathcal{A} := \begin{bmatrix} p^1 & p^2 & \ldots & p^n \\ 1 & 1 & \ldots & 1 \end{bmatrix} \tag{4}$$

has full row rank since $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$. Note that $r \leq n-1$. Let

$$\bar{r} = \text{the dimension of the null space of } \mathcal{A}; \text{ i.e., } \bar{r} = n-1-r. \tag{5}$$

Definition 2.1 *Suppose that the null space of $\mathcal{A}$ is nontrivial, i.e., $\bar{r} \geq 1$. Any $n \times \bar{r}$ matrix Z whose columns form a basis of the null space of $\mathcal{A}$ is called a* Gale matrix *of configuration p. Furthermore, the ith row of Z, considered as a vector in $\mathbb{R}^{\bar{r}}$, is called a* Gale transform *of p^i [9].*

Let S be a stress matrix of $G(p)$ then it follows from (2) and (4) that

$$\mathcal{A}S = 0. \tag{6}$$

Thus

$$S = Z\Psi Z^T, \tag{7}$$

for some $\bar{r} \times \bar{r}$ symmetric matrix Ψ, where Z is a Gale matrix of p. It immediately follows from (7) that rank S = rank Ψ. Thus, S attains its maximum rank of $\bar{r} = (n-1-r)$ if and only if Ψ is nonsingular, i.e., rank $\Psi = \bar{r}$.

Let e denote the vector of all 1's in $\mathbb{R}^n$, and let V be an $n \times (n-1)$ matrix that satisfies:

$$V^T e = 0, \; V^T V = I_{n-1}, \tag{8}$$

where I_{n-1} is the identity matrix of order $(n-1)$. Further, let E^{ij}, $i \neq j$, denote the $n \times n$ symmetric matrix with 1 in the (i,j)th and (j,i)th entries and zeros elsewhere, and let $\mathcal{E}(y) = \sum_{(i,j) \notin E(G)} y_{ij} E^{ij}$ where $y_{ij} = y_{ji}$. In other words, the (k,l) entry of matrix $\mathcal{E}(y)$ is given by

$$\mathcal{E}(y)_{kl} = \begin{cases} 0 & \text{if } (k,l) \in E(G), \\ 0 & \text{if } k = l, \\ y_{kl} & \text{if } k \neq l \text{ and } (k,l) \notin E(G). \end{cases} \tag{9}$$

Then we have the following result.

Lemma 2.3 *(Alfakih [2]) Let $G(p)$ be a bar framework on n vertices in $\mathbb{R}^r$ and let Z be any Gale matrix of p. Then the following two conditions are equivalent:*

1. *The vectors $p^i - p^j$ for all $(i,j) \in E(G)$ lie on a quadratic at infinity.*
2. *There exists a non-zero $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that:*

$$V^T \mathcal{E}(y) Z = \mathbf{0}, \tag{10}$$

where $\bar{m}$ is the number of missing edges of G, V is defined in (8), and $\mathcal{E}(y)$ is defined in (9). $\mathbf{0}$ *here is the zero matrix of dimension $(n-1) \times \bar{r}$.*

Condition 2 of Lemma 2.3 can be easily understood if a projected Gram matrix approach is used for the universal rigidity of bar frameworks (see [2] for details).

3 Proof of Theorem 1.4

The main idea of the proof is to show that Condition 2 of Lemma 2.3 does not hold under the general position assumption, and under the assumption that $G(p)$ admits a positive semidefinite stress matrix of rank $(n-r-1)$. The choice of the particular Gale matrix to be used in equation (10) is critical in this regard. We begin with a few necessary lemmas.

Lemma 3.1 *Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$ and let Z be any Gale matrix of configuration p. Then any $\bar{r} \times \bar{r}$ submatrix of Z is nonsingular.*

Proof. For a proof see e.g., [1].

□

Let $\bar{N}(i)$ denote the set of nodes of graph G that are non-adjacent to node i; i.e.,

$$\bar{N}(i) = \{j \in V(G) : j \neq i \text{ and } (i,j) \notin E(G)\}, \tag{11}$$

Lemma 3.2 *Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a stress matrix S of rank $(n-1-r)$. Then there exists a Gale matrix $\hat{Z}$ of $G(p)$ such that $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j+r+1)$.*

Proof. Let $G(p)$ be in general position in $\mathbb{R}^r$ and assume that it has a stress matrix S of rank $\bar{r} = (n-1-r)$. Let Z be any Gale matrix of $G(p)$, then $S = Z\Psi Z^T$ for some non-singular symmetric $\bar{r} \times \bar{r}$ matrix Ψ. Let us write Z as:

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \tag{12}$$

where Z_2 is $\bar{r} \times \bar{r}$. By Lemma 3.1, Z_2 is non-singular. Now let

$$\hat{Z} = (\hat{z}_{ij}) = Z\Psi {Z_2}^T. \tag{13}$$

Then $\hat{Z}$ is a Gale matrix of $G(p)$. This simply follows from the fact that the matrix obtained by multiplying any Gale matrix of $G(p)$ from the right by a non-singular $\bar{r} \times \bar{r}$ matrix, is also a Gale matrix of $G(p)$. Furthermore,

$$S = Z\Psi Z^T = Z\Psi\,[Z_1^T \;\; Z_2^T] = [Z\Psi Z_1^T \;\; \hat{Z}].$$

In other words, $\hat{Z}$ consists of the last $\bar{r}$ columns of S. Thus $\hat{z}_{ij} = s_{i,j+r+1}$. By the definition of S we have $s_{i,j+r+1} = 0$ for all $i \neq j+r+1$ and $(i, j+r+1) \notin E(G)$. Therefore, $\hat{z}_{ij} = 0$ for all $j = 1, \ldots, \bar{r}$ and $i \in \bar{N}(j+r+1)$.

□

Lemma 3.3 *Let the Gale matrix in (10) be $\hat{Z}$ as defined in (13). Then the system of equations (10) is equivalent to the system of equations*

$$\mathcal{E}(y)\hat{Z} = \mathbf{0}. \tag{14}$$

$\mathbf{0}$ *here is the zero matrix of dimension $n \times \bar{r}$.*

Proof. System of equations (10) is equivalent to the following system of equations in the unknowns, y_{ij} ($i \neq j$ and $(i,j) \notin E(G)$) and $\xi \in \mathbb{R}^{\bar{r}}$:

$$\mathcal{E}(y)\hat{Z} = e\,\xi^T, \tag{15}$$

Now for $j = 1, \ldots, \bar{r}$, we have that the $(j+r+1, j)$th entry of $\mathcal{E}(y)\hat{Z}$ is equal to ξ_j. But using (9) and Lemma 3.2 we have

$$(\mathcal{E}(y)\hat{Z})_{j+r+1,j} = \sum_{i=1}^{n} \mathcal{E}(y)_{j+r+1,i}\ \hat{z}_{ij} = \sum_{i:i\in\bar{N}(j+r+1)} y_{j+r+1,i}\ \hat{z}_{ij} = 0.$$

Thus, $\xi = 0$ and the result follows.

□

Now we are ready to prove our main theorem.

Proof of Theorem 1.4

Let $G(p)$ be a framework on n nodes in general position in $\mathbb{R}^r$. Assume that $G(p)$ has a positive semidefinite stress matrix S of rank $\bar{r} = n-1-r$. Then $\deg(i) \geq r+1$ for all $i \in V(G)$, i.e., every node of G is adjacent to at least $r+1$ nodes (for a proof see [1, Theorem 3.2]). Thus

$$|\bar{N}(i)| \leq n - r - 2 = \bar{r} - 1. \tag{16}$$

Furthermore, it follows from Lemmas 3.2, 3.3 and 2.3 that the vectors $p^i - p^j$ for all $(i,j) \in E(G)$ lie on a quadratic at infinity if and only if system of equations (14) has a non-zero solution y. But (14) can be written as

$$\sum_{j:\in\bar{N}(i)} y_{ij}\hat{z}^j = 0, \text{ for } i = 1, \ldots, n,$$

where $(\hat{z}^i)^T$ is the ith row of $\hat{Z}$. Now it follows from (16) that $y_{ij} = 0$ for all $(i,j) \notin E(G)$ since by Lemma 3.1 any subset of $\{\hat{z}^1, \ldots, \hat{z}^n\}$ of cardinality $\leq \bar{r} - 1$ is linearly independent.

Thus system (14) does not have a nonzero solution y. Hence the vectors $p^i - p^j$, for all $(i,j) \in E(G)$, do not lie on a quadratic at infinity. Therefore, by Lemma 2.1, there does not exist a framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$. Thus by Theorem 1.1, $G(p)$ is universally rigid.

□

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