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On Bar Frameworks, Stress Matrices and Semidefinite Programming

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Abstract A bar framework $G(p)$ in r-dimensional Euclidean space is a graph G on the vertices $1, 2, \ldots, n$, where each vertex i is located at point p^i in $\mathbb{R}^r$. Given a framework $G(p)$ in $\mathbb{R}^r$, a problem of great interest is that of determining whether or not there exists another framework $G(q)$, not obtained from $G(p)$ by a rigid motion, such that $||q^i - q^j||^2 = ||p^i - p^j||^2$ for each edge (i, j) of G. This problem is known as either the global rigidity problem or the universal rigidity problem depending on whether such a framework $G(q)$ is restricted to be in the same r-dimensional space or not. The stress matrix S of a bar framework $G(p)$ plays a key role in these and other related problems.

In this paper, semidefinite programming (SDP) theory is used to address, in a unified manner, several problems concerning universal rigidity. New results are presented as well as new proofs of previously known theorems. In particular, we use the notion of SDP non-degeneracy to obtain a sufficient condition for universal rigidity, and we show that this condition yields the previously known sufficient condition for generic universal rigidity. We present new results concerning positive semidefinite stress matrices and we use a semidefinite version of Farkas lemma to characterize bar frameworks that admit a nonzero positive semidefinite stress matrix S.

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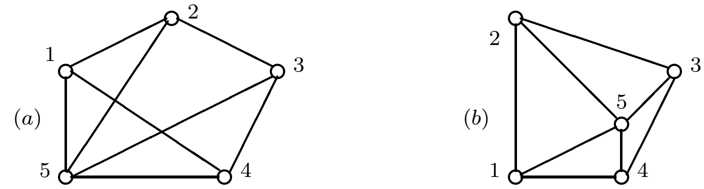

Fig. 1 Two bar frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^2$. Both frameworks have the same underlying graph G.

1 Introduction

A *configuration* p in r-dimensional Euclidean space is a finite collection of points $p^1, \ldots, p^n$ in $\mathbb{R}^r$ that affinely span $\mathbb{R}^r$. A *bar framework* in $\mathbb{R}^r$ (or a framework [1] for short), denoted by $G(p)$, is a configuration p in $\mathbb{R}^r$ together with a simple graph G on the vertices $1, 2, \ldots, n$, where each vertex i of G is located at p^i. Figure 1 depicts two frameworks in the plane. The vertices of G are represented by little circles, while the edges of G are represented as line-segments (or bars). To avoid trivialities we assume throughout this paper that G is connected and $G \neq K_n$, i.e., G is not the complete graph.

Two frameworks $G(p)$ and $G(q)$ in $\mathbb{R}^r$ are said to be *congruent* if $||q^i - q^j|| = ||p^i - p^j||$ for all $i, j = 1, \ldots, n$, where $||.||$ denotes the Euclidean norm. That is, $G(p)$ and $G(q)$ are congruent if configuration q can be obtained from configuration p by applying a rigid motion such as a translation or a rotation in $\mathbb{R}^r$. On the other hand, two frameworks $G(p)$ in $\mathbb{R}^r$ and $G(q)$ in $\mathbb{R}^s$ are said to be *equivalent* if $||q^i - q^j|| = ||p^i - p^j||$ for each edge (i, j) of graph G. The term "bar" is used to denote such frameworks since in any two equivalent frameworks, every two adjacent vertices of $G(p)$ stay the same distance apart. Thus one can think of each edge of $G(p)$ as a stiff bar.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be *globally rigid* if there does not exist a framework $G(q)$ in the same r-dimensional Euclidean space that is equivalent, but not congruent, to $G(p)$. Furthermore, if there does not exist a framework $G(q)$ in any Euclidean space that is equivalent, but not congruent, to $G(p)$, then $G(p)$ is said to be *universally rigid* [2]. Obviously, universal rigidity implies global rigidity, however, the converse need not be true. Using Theorems 1 and 2 below, one can show (see the example in Section 5 of [4]) that framework (a) in Figure 1 is universally rigid while framework (b) is globally (but not universally) rigid.

The global and universal rigidity problems of frameworks can also be posed in the context of the *graph realization problem (GRP)*. Given an edge-weighted graph G where each edge (i, j) has a positive weight d_{ij}, a realization of G in $\mathbb{R}^r$ is a mapping of the vertices $1, 2, \ldots, n$ of G into points $p^1, p^2, \ldots, p^n$ in $\mathbb{R}^r$ such that $||p^i - p^j||^2 = d_{ij}$ for each edge (i, j) of graph G. The GRP is the problem of determining whether or not G has a realization in $\mathbb{R}^r$. Thus, the problems of global rigidity and universal rigidity can be stated as the problems of determining whether or not a given realization of G in $\mathbb{R}^r$ is unique, up to a rigid motion, in $\mathbb{R}^r$ or in all Euclidean spaces.

[1] Only bar frameworks are considered here. Tensegrity frameworks fall outside the scope of this paper.

[2] There are many other notions of rigidity such as rigidity, infinitesimal rigidity, dimensional rigidity etc. However, these notions will not be discussed in this paper.

The GRP and the global and universal rigidity of frameworks have important applications in molecular conformations [11], multidimensional scaling [17,10] and wireless sensor networks [23,13]. In particular, the wireless sensor network localization problem is a special case of the GRP where $r = 2$ or 3, and where G has a clique of size at least $r + 1$.

The GRP is well known to be NP-hard [21]. Semidefinite programming (SDP) was successfully used in [5] to solve a relaxation of the GRP by asking whether a given edge-weighted graph G has a realization in some Euclidean space, not necessarily in a given r-dimensional space. Also, SDP was successfully used in [7,12,23,19,24] to solve the wireless senor network localization problem, and in [22] to solve some problems in tensegrity theory. A tensegrity framework (see [9] and the references therein) is a generalization of a bar framework where the edges of graph G are labeled as bars, cables or struts. If an edge (i, j) is labeled as a cable (strut), then $||p^i - p^j||$ is constrained to be $\leq$ ($\geq$) a certain given value.

The stress matrix S of a framework $G(p)$ plays a key role in the characterization of generic global rigidity and generic universal rigidity of $G(p)$. It is closely related to the Gale matrix of $G(p)$. Gale matrix or Gale transform is a well-known technique in polytope theory [14].

In this paper, SDP theory is used to address, in a unified manner, several problems concerning universal rigidity. New results are presented as well as new proofs of previously known theorems. In particular, we use the notion of SDP non-degeneracy to obtain a sufficient condition for universal rigidity, and we show that this condition yields the previously known sufficient condition for generic universal rigidity. We present new results concerning positive semidefinite stress matrices and we use a semidefinite version of the Farkas lemma to characterize frameworks that admit a nonzero positive semidefinite stress matrix S.

We denote by $\mathcal{S}_n$ the subspace of symmetric matrices of order n. The positive semi-definiteness (definiteness) of a symmetric matrix A is denoted by $A \succeq 0$ ($A \succ 0$). e denotes the vector of all ones in $\mathbb{R}^n$, and I_n denotes the identity matrix of order n. E^{ij} denotes the $n \times n$ symmetric matrix with 1's in the (i, j)th and (j, i)th entries and zeros elsewhere. For a matrix A, diag(A) denotes the vector consisting of the diagonal entries of A. Finally, $E(G)$ denotes the set of edges of a simple graph G.

2 Preliminaries

2.1 The Stress Matrix S and the Gale Matrix Z

An *equilibrium stress* of a framework $G(p)$ is a real-valued function ω on $E(G)$ such that

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n. \tag{1}$$

One may think of $\omega_{ij}(p^i - p^j)$ as the force exerted by the edge (bar) (i, j) on node i. This force is called a *tension* in the bar (i, j) if $\omega_{ij} < 0$, and it is called a *compression* in the bar (i, j) if $\omega_{ij} > 0$. Thus equation (1) is equivalent to the statement that the net force acting on each node i is equal to zero.

Let ω be an equilibrium stress of $G(p)$. Then the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } (i,j) \notin E(G), \\ \sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j, \end{cases}$$

is called the *stress matrix* associated with ω, or a stress matrix of $G(p)$.

Given a configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$, the $n \times r$ matrix

$$P := \begin{bmatrix} {p^1}^T \\ {p^2}^T \\ \vdots \\ {p^n}^T \end{bmatrix} \tag{2}$$

is called the *configuration matrix* of p. Let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$ and let P be the configuration matrix of $G(p)$. Then it immediately follows that the following $(r+1) \times n$ matrix

$$A := \begin{bmatrix} P^T \\ e^T \end{bmatrix}, \tag{3}$$

has full row rank since $p^1, \ldots, p^n$ affinely span $\mathbb{R}^r$. Note that $r \leq n-1$. Let $\bar{r}$ be the dimension of the nullspace of A; i.e., $\bar{r} = n - 1 - r$. For $\bar{r} \geq 1$, let Λ be an $n \times \bar{r}$ matrix whose columns form a basis for the nullspace of A. Λ is called a Gale matrix corresponding to $G(p)$. Obviously Λ is not unique. However, we define, next, a unique and sparse Gale matrix Z which will be referred to, in the sequel, as "the" Gale matrix of $G(p)$. The sparsity of Z is particularly convenient for the purposes of this paper.

Let us write Λ in block form as

$$\Lambda = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix},$$

where Λ_1 is $\bar{r} \times \bar{r}$ and Λ_2 is $(r+1) \times \bar{r}$. Since Λ has full column rank, we can assume without loss of generality that Λ_1 is nonsingular. Then *the Gale matrix* Z is defined by

$$Z := \Lambda {\Lambda_1}^{-1} = \begin{bmatrix} I_{\bar{r}} \\ \Lambda_2 {\Lambda_1}^{-1} \end{bmatrix}. \tag{4}$$

Let $(z^i)^T$ denote the ith row of Z. Then the $\bar{r}$-vector z^i is called *the Gale transform* of p^i [14].

The following theorem shows that the Gale matrix Z of framework $G(p)$ is closely related to the stress matrix S associated with an equilibrium stress ω of $G(p)$.

Lemma 1 (Alfakih [4]) *Given a framework $G(p)$ with n vertices in $\mathbb{R}^r$, let Z be the Gale matrix of $G(p)$ and recall that $\bar{r} = n - 1 - r$. Further, let S be a stress matrix of $G(p)$. Then there exists an $\bar{r} \times \bar{r}$ symmetric matrix Ψ such that*

$$S = Z \Psi Z^T. \tag{5}$$

On the other hand, let Ψ' be any $\bar{r} \times \bar{r}$ symmetric matrix such that $(z^i)^T \Psi' z^j = 0$ for all $(i,j) \notin E(G)$, where $(z^i)^T$ denotes the ith row of Z. Then $S' = Z \Psi' Z^T$ is a stress matrix of $G(p)$.

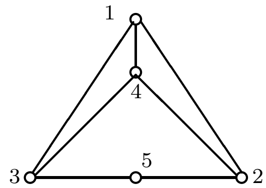

Fig. 2 The non-generic framework in $\mathbb{R}^2$ of Example 1.

Example 1 *Consider the framework $G(p)$ in Figure 2, where $p^1 = [0 \;\; 2]^T$, $p^2 = [2 \;\; -1]^T$, $p^3 = [-2 \;\; -1]^T$, $p^4 = [0 \;\; 1]^T$ and $p^5 = [0 \;\; -1]^T$. Thus*

$$A = \begin{bmatrix} p^1 & p^2 & p^3 & p^4 & p^5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2 & 0 & 0 \\ 2 & -1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Hence, the nullspace of A is spanned by the columns of

$$\Lambda = \begin{bmatrix} -2 & 0 \\ -1/2 & -1 \\ -1/2 & -1 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix}.$$

Accordingly, the Gale matrix Z of $G(p)$ is:

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -3/2 & 0 \\ 1/2 & -2 \end{bmatrix},$$

where Z is obtained by multiplying Λ from the right with ${\Lambda_1}^{-1}$.

It is easy to show that $G(p)$ has an equilibrium stress $\omega = (\omega_{12} = -1,\ \omega_{13} = -1,\ \omega_{14} = 6,\ \omega_{24} = 3/2,\ \omega_{25} = -1/2,\ \omega_{34} = 3/2,\ \omega_{35} = -1/2)$, and a stress matrix

$$S = \begin{bmatrix} 4 & 1 & 1 & -6 & 0 \\ 1 & 0 & 0 & -3/2 & 1/2 \\ 1 & 0 & 0 & -3/2 & 1/2 \\ -6 & -3/2 & -3/2 & 9 & 0 \\ 0 & 1/2 & 1/2 & 0 & -1 \end{bmatrix} = Z \Psi Z^T,$$

where $\Psi = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$.

2.2 Generic Frameworks

The problems of framework global rigidity and universal rigidity become easier if we consider frameworks with a "typical" or generic configuration. A configuration p (or a framework $G(p)$) in $\mathbb{R}^r$ is said to be *generic* if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, $G(p)$ is generic if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, \ldots, p^n) = 0$.

The notion of generic frameworks can be weakened to that of frameworks in general position. A framework $G(p)$ in $\mathbb{R}^r$ is said to be in *general position* if no subset of the points $p^1, \ldots, p^n$ of cardinality $r+1$ is affinely dependent. For example, a set of points in the plane are in general position if no 3 of them lie on a straight line.

The following are two recently obtained results concerning generic framework global and universal rigidity.

Theorem 1 (Connelly [8] Gortler et al [15]) *Let $G(p)$ be a generic framework on n vertices in $\mathbb{R}^r$, $r \leq n-2$. Then $G(p)$ is globally rigid if and only if*

$$\exists \text{ a stress matrix } S \text{ of } G(p) \text{ such that rank } S = \bar{r} = n - 1 - r. \tag{6}$$

Theorem 2 *Let $G(p)$ be a generic framework on n vertices in $\mathbb{R}^r$, $r \leq n-2$. Then $G(p)$ is universally rigid if and only if*

$$\exists \text{ a stress matrix } S \text{ of } G(p) \text{ such that } S \succeq 0 \text{ and rank } S = \bar{r} = n - 1 - r. \tag{7}$$

The "if" part of Theorem 2 was independently proved by Alfakih in [4] and Connelly in [9], while the "only if" part was conjectured by Alfakih in [4] and proved by Gortler and Thurston in [16].

Few comments are in order here. First, if $r = n-1$, i.e., if $G(p)$ is a framework in $\mathbb{R}^{n-1}$. Then $G(p)$ is not globally rigid (see the paragraph after equation (16)). Second, in light of Lemma 1, Conditions (6) and (7) are equivalent, respectively, to the following two conditions, where $z^1, z^2, \ldots, z^n$ are the Gale transforms of $p^1, p^2, \ldots, p^n$.

$$\exists \ \bar{r} \times \bar{r} \text{ nonsingular sym. matrix } \Psi : (z^i)^T \Psi z^j = 0, \forall \, (i,j) \not\in E(G), \tag{8}$$

$$\exists \ \bar{r} \times \bar{r} \text{ sym. matrix } \Psi \succ 0 : (z^i)^T \Psi z^j = 0, \forall \, (i,j) \not\in E(G). \tag{9}$$

Third, the assumption in Theorems 1 and 2 that the framework is generic can not be dropped. The framework of Example 1 depicted in Figure 2, which is clearly non-generic, is not globally rigid even though it satisfies Condition (6). Similarly, the non-generic framework of Example 2 is not universally rigid even though it satisfies Condition (7).

3 A Characterization of Equivalent Frameworks

The first step in making the universal rigidity problem amenable to semidefinite programming is to use a Gram matrix, or more accurately a projected Gram matrix, to represent a configuration p. This enables us to characterize the set of all frameworks that are equivalent to a given framework $G(p)$.

Given a configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$, the $n \times n$ symmetric matrix $B := PP^T$, where P is the configuration matrix of p defined in (2), is called the *Gram matrix* associated with p.

Note that in addition to being positive semidefinite of rank r, matrix B is invariant under orthogonal transformations. Furthermore, wlog we assume that in any configuration p, the origin coincides with the centroid of the points $p^1, \ldots, p^n$; i.e., $Be = 0$.

Then, B becomes also invariant under translations. Consequently, all congruent frameworks have the same Gram matrix. Hence, representing frameworks by Gram matrices, instead of configuration matrices, allows us to identify all congruent frameworks. Therefore, in the sequel we don't distinguish between congruent frameworks.

The set $\{C \in \mathcal{S}_n : C \succeq 0, Ce = 0\}$ forms a face of the cone of $n \times n$ symmetric positive semidefinite matrices. This face is isomorphic to the cone of positive semidefinite matrices of order $n-1$. For our purposes, it is more convenient to work in this latter cone. To this end, let V be an $n \times (n-1)$ matrix such that

$$V^T e = 0, \quad V^T V = I_{n-1}. \tag{10}$$

Definition 1 *Given a configuration $p = (p^1, p^2, \ldots, p^n)$ in $\mathbb{R}^r$. The projected Gram matrix X associated with p is the $(n-1) \times (n-1)$ matrix*

$$X := V^T B V, \tag{11}$$

where B is the Gram matrix associated with p.

Note that X is an $(n-1) \times (n-1)$ positive semidefinite matrix of rank r.

Given a configuration p, the projected Gram matrix associated with p can be easily computed using equation (11). On the other hand, given a projected Gram matrix X, configuration p can be recovered as follows. Compute the Gram matrix B which is given by $B = VXV^T$. Then factorize B as $B = PP^T$. This can be done since B is positive semidefinite. Furthermore, since rank $B = r$ and $Be = 0$, it follows that P is an $n \times r$ matrix such that $P^T e = 0$. Thus P is the configuration matrix of p, i.e., the points $p^1, \ldots, p^n$ are given by the rows of P and their centroid coincides with the origin. Recall that we do not distinguish between congruent configurations. Thus, the configuration matrix P, the Gram matrix B and the projected Gram matrix X uniquely determine one another. Hence, the terms "framework $G(p)$" and "framework $G(X)$" can be used interchangeably. For more details see [3].

Let $G(\hat{p})$ be a given framework in $\mathbb{R}^r$ and let $\bar{m}$ denote the number of missing edges of G. For each $(i,j) \not\in E(G)$ define the matrix

$$M^{ij} := -\frac{1}{2} V^T E^{ij} V. \tag{12}$$

Recall that E^{ij} is the $n \times n$ matrix with 1's in the (i,j)th and (j,i)th entries and 0's elsewhere. Let $\hat{X}$ be the projected Gram matrix corresponding to $G(\hat{p})$ and let

$$\Omega := \{y \in \mathbb{R}^{\bar{m}} : X(y) := \hat{X} + \sum_{(i,j) \not\in E(G)} y_{ij} M^{ij} \succeq 0\}. \tag{13}$$

and

$$\Omega_r := \{y \in \Omega : \text{ rank } X(y) = r\}. \tag{14}$$

Note that the origin ($y = 0$) always belongs to Ω and to Ω_r since $\hat{X}$ is positive semidefinite of rank r. It was shown in [1] that the set of all frameworks $G(q)$ in $\mathbb{R}^s$, $1 \leq s \leq n-1$, that are equivalent to $G(\hat{p})$ is given by

$$\{G(X(y)) : y \in \Omega_s\}, \tag{15}$$

and that the set of all frameworks $G(q)$ in all Euclidean spaces that are equivalent to $G(\hat{p})$ is given by

$$\{G(X(y)) : y \in \Omega\}. \tag{16}$$

For more details on set Ω see [2]. It is clear that framework $G(\hat{p})$ is globally rigid if and only if set Ω_r is a singleton, and $G(\hat{p})$ is universally rigid if and only if set Ω is a singleton. Note that Ω is a closed convex set. This makes the universal rigidity problem amenable to SDP. On the other hand, the global rigidity problem is much harder to tackle since Ω_r is, in general, non-convex due to the rank constraint. Note that if $r = n-1$, i.e., if $G(\hat{p})$ is a framework in $\mathbb{R}^{n-1}$, then $\hat{X} \succ 0$. Thus Ω_r is not a singleton and hence $G(\hat{p})$ is not globally rigid.

Next, we provide new insights into set Ω by presenting an equivalent definition to (13). An $n \times n$ symmetric matrix $D = (d_{ij})$ is said to be a *Euclidean distance matrix (EDM)* if there exist points $p^1, \ldots, p^n$ in some Euclidean space such that $d_{ij} = ||p^i - p^j||^2$ for all $i, j = 1, \ldots, n$. Let $\mathcal{K}_V : \mathcal{S}_{n-1} \to \mathcal{S}_n$ be the linear mapping defined by

$$\mathcal{K}_V(C) = \operatorname{diag}(VCV^T)\, e^T + e\,(\operatorname{diag}(VCV^T))^T - 2VCV^T, \tag{17}$$

where V is the matrix defined in (10) and e is the vector of all 1's in $\mathbb{R}^n$. It is well known (see [5] and the references therein) that an $n \times n$ symmetric matrix D with a zero diagonal is EDM if and only if $D = \mathcal{K}_V(X)$ for some positive semidefinite matrix X in $\mathcal{S}_{n-1}$. Moreover, $\mathcal{K}_V(M^{ij}) = E^{ij}$ for all $(i,j) \notin E(G)$. Thus for all $(k,l) \in E(G)$ and $(i,j) \notin E(G)$ it follows that $(\mathcal{K}_V(M^{ij}))_{kl} = 0$.

Let $\hat{D} = (\hat{d}_{ij})$ be the EDM corresponding to $\hat{X}$. i.e., Let $\hat{D} = \mathcal{K}_V(\hat{X})$. Then set Ω can be equivalently defined by

$$\Omega = \{X \in \mathcal{S}_{n-1} : X \succeq 0, (\mathcal{K}_V(X))_{kl} = \hat{d}_{kl} \text{ for all } (k,l) \in E(G).\} \tag{18}$$

Note that the edges of G are used in the definition of set Ω in (18) while the missing edges of G are used in (13). We conclude this section with the following technical lemma that establishes the connection between the projected Gram matrix $\hat{X}$ and the Gale matrix Z of a given framework $G(\hat{p})$.

Lemma 2 (Alfakih [3]) *Let $G(\hat{p})$ be a given framework with n vertices in $\mathbb{R}^r$ for some $r \leq n-2$; and let Z be the Gale matrix of $G(\hat{p})$. Further, let U and W be the matrices whose columns form orthonormal bases of the nullspace and the rangespace of $\hat{X}$, where $\hat{X}$ is the projected Gram matrix associated with $\hat{p}$. Then*

1. *$VU = ZQ$ for some nonsingular matrix Q, i.e., VU is a Gale matrix.*
2. *$VW = PQ'$ for some nonsingular matrix Q', where P is the configuration matrix of $\hat{p}$.*

4 Positive Semidefinite Stress Matrices

Using the results of the previous section, we focus in this section on stress matrices that are positive semidefinite. Consider the following pair of dual SDP problems, where $\hat{X}$ is the projected Gram matrix of a given framework $G(\hat{p})$ in $\mathbb{R}^r$, and M^{ij} is as defined in (12).

$$\begin{array}{lll} \text{(P):} & \displaystyle\max_{y} & 0^T y \\ & \text{subject to} & \hat{X} + \displaystyle\sum_{(i,j)\notin E(G)} y_{ij} M^{ij} \succeq 0. \end{array} \tag{19}$$

$$
\begin{array}{llll}
\text{(D):} & \min\limits_{Y} & \text{trace } (\hat{X}Y) & \\
& \text{subject to} & \text{trace } (YM^{ij}) = 0 & \text{for } (i,j) \notin E(G), \\
& & Y \succeq 0. &
\end{array} \tag{20}
$$

Two remarks concerning problems (19) and (20) are in order here. First, it follows from (13) that the set of all frameworks $G(X(y))$ that are equivalent to $G(\hat{p})$, is equal to the the set of feasible solutions of the primal problem (19). Furthermore, since the objective function of problem (19) is identically equal to zero, the set of all frameworks that are equivalent to $G(\hat{p})$, is also equal to the set of optimal solutions of problem (19).

Second, let W and U be the $(n-1) \times r$ and $(n-1) \times \bar{r}$ matrices whose columns form orthonormal bases for the rangespace and the nullspace of $\hat{X}$ respectively. Since $\hat{X} \succeq 0$, any optimal solution of the dual problem (20) can be written in the form $Y = U\Psi U^T$ for some $\Psi \succeq 0$. Furthermore, using (12), trace $(YM^{ij}) = 0$ implies that trace $(VU\Psi U^T V^T E^{ij}) = 0$. Thus by Lemma 2 we have $(Z\Psi' Z^T)_{ij} = 0$ for all $(i,j) \notin E(G)$, where $\Psi' = Q\Psi Q^T \succeq 0$, for some nonsingular matrix Q. Hence we have the following theorem.

Theorem 3 *Let S be a positive semidefinite stress matrix of framework $G(\hat{p})$. Then Y is an optimal solution of the dual problem (20), where*

$$
Y = V^T S V \ \textit{and}\ S = VYV^T. \tag{21}
$$

That is, Y is a "projected stress matrix" of $G(\hat{p})$.

The connection between the stress matrix and the optimal dual solutions of certain SDP problems in tensegrity theory was first observed in [22].

A main ingredient in Connelly's proof of the "if" part of Theorem 1 is the following theorem.

Theorem 4 (Connelly [8]) *Let $G(\hat{p})$ be a given framework in $\mathbb{R}^r$ and let S be a stress matrix of $G(\hat{p})$. If $G(\hat{p})$ is generic, then S is a stress matrix of any framework $G(q)$ in $\mathbb{R}^r$ that is equivalent to $G(\hat{p})$.*

A similar result is presented next, where the assumption of the genericity of the framework is replaced by the assumption of the positive semi-definiteness of the stress matrix. This result is a simple consequence of SDP complementary slackness.

Theorem 5 *Let $G(\hat{p})$ be a given framework, generic or otherwise, in $\mathbb{R}^r$ and let S be a stress matrix of $G(\hat{p})$. If $S \succeq 0$, then S is a stress matrix of any framework $G(q)$ that is equivalent to $G(\hat{p})$.*

Proof. Let $\hat{X}$ and X be the projected Gram matrices of frameworks $G(\hat{p})$ and $G(q)$ respectively. It suffices to show that $XY = 0$, where $Y = V^T SV$. But from (19) we have that $XY = \hat{X}Y + \sum_{(i,j)\notin E(G)} y_{ij} M^{ij} Y = \sum_{(i,j)\notin E(G)} y_{ij} M^{ij} Y$. Thus it follows from (20) that trace $(XY) = \sum_{(i,j)\notin E(G)} y_{ij}$ trace $(M^{ij}Y) = 0$. Therefore $XY = 0$. This follows from the well-known fact (see, e.g., [6]) that for any two positive semidefinite matrices A and B, trace $(AB) = 0$ if and only if $AB = 0$.

□

Next we characterize frameworks that admit a non-zero positive semidefinite stress matrix.

5 Frameworks with Positive Semidefinite Stress Matrices

In this section, we use the following well-known semidefinite version of Farkas lemma to characterize frameworks that admit a non-zero positive semidefinite stress matrix. A proof is added for completeness.

Lemma 3 *Let $A^0, A^1, A^2, \ldots, A^k$ be given symmetric matrices of order n. Then exactly one of the following two statements holds.*

1. *$\exists\, y \in \mathbb{R}^k$ such that $A^0 + \sum_{i=1}^k y_i A^i \succ 0$,*
2. *$\exists\, Y \succeq 0, Y \neq 0$, trace $(YA^0) \leq 0$ and trace $(YA^i) = 0$ for $i = 1, \ldots, k$.*

Proof. First we prove that Statements 1 and 2 cannot hold at the same time. Assume that there exist $y \in \mathbb{R}^k$ and $Y \succeq 0$, $Y \neq 0$ such that $A^0 + \sum_{i=1}^k y_i A^i \succ 0$, trace $(YA^0) \leq 0$, and trace $(YA^i) = 0$ for $i = 1, \ldots, k$. Then $0 <$ trace $(Y(A^0 + \sum_{i=1}^k y_i A^i))$ $=$ trace $(YA^0) \leq 0$, a contradiction.

Now assume that Statement 1 does not hold and let $\mathcal{L} = \{C \in \mathcal{S}_n : C = A^0 + \sum_{i=1}^k y_i A^i$ for some $y \in \mathbb{R}^k\}$. Then $\mathcal{L} \cap \{C : C \succ 0\} = \emptyset$. By the separation theorem [20, Theorem 11.2, page 96], there exist $Y \in \mathcal{S}_n$, $Y \neq 0$ and scalar α such that trace $(YC) = \alpha$ for all $C \in \mathcal{L}$; and trace $(YC) > \alpha$ for all $C \succ 0$. Now $\mu^* =$ inf $\{$ trace $(YC) : C \succ 0\}$ is finite iff $Y \succeq 0$, in which case $\mu^* = 0$. Thus $Y \succeq 0$ and $\alpha \leq 0$. Similarly, trace (YC) is finite for all $C \in \mathcal{L}$ iff trace $(YA^i) = 0$ for $i = 1, \ldots, k$. Hence trace $(YC) = \alpha$ for all $C \in \mathcal{L}$ implies that trace $(YA^i) = 0$ for $i = 1, \ldots, k$ and trace $(YA^0) = \alpha \leq 0$. Therefore Statement 2 holds.

□

Before we present our characterization, in the following theorem, of frameworks that admit a non-zero positive semidefinite stress matrix, we remark that a framework on n vertices in $\mathbb{R}^{n-1}$ is a framework where its vertices are located at affinely independent points.

Theorem 6 *Let $G(\hat{p})$, $G \neq K_n$, be a given framework on n vertices in $\mathbb{R}^r$. Then $G(\hat{p})$ admits a non-zero positive semidefinite stress matrix S if and only if there does not exist a framework $G(q)$ in $\mathbb{R}^{n-1}$ that is equivalent to $G(\hat{p})$.*

Proof. Let $\hat{X}$ be the projected Gram matrix of framework $G(\hat{p})$ and let U be the matrix whose columns form an orthonormal basis of the nullspace of $\hat{X}$. Then there does not exist a framework $G(q)$ in $\mathbb{R}^{n-1}$ that is equivalent to $G(\hat{p})$ if and only if there does not exist a $y \in \mathbb{R}^{\bar{m}}$ such that $X(y) = \hat{X} + \sum_{(i,j) \notin E(G)} y_{ij} M^{ij} \succ 0$, if and only if there exists a $Y \succeq 0, Y \neq 0$, such that trace $(Y\hat{X}) \leq 0$ and trace $(YM^{ij}) = 0$ for all $(i,j) \notin E(G)$. The first equivalence follows from (15) since rank $X(y) = n-1$ if and only if $X(y) \succ 0$. The second equivalence follows from Lemma 3. Now since $\hat{X} \succeq 0$, it follows that trace $(Y\hat{X}) \leq 0$ is equivalent to trace $(Y\hat{X}) = 0$. It also follows that $Y = U\Psi U^T$ for some positive semidefinite matrix Ψ.

Therefore, there does not exist a framework $G(q)$ in $\mathbb{R}^{n-1}$ that is equivalent to $G(\hat{p})$ if and only if there exists a non-zero positive semidefinite matrix Ψ such that trace $(U\Psi U^T M^{ij}) = 0$ for all $(i,j) \notin E(G)$. But, it follows from the definition of M^{ij} in (12) and from Lemma 2 that -2 (trace $(U\Psi U^T M^{ij})$) $=$ trace $(U\Psi U^T V^T E^{ij} V)$ $=$ trace $(Z\Psi' Z^T E^{ij})$ where $\Psi' = Q\Psi Q^T$ for some nonsingular matrix Q. The result follows from Lemma 1 since the matrix $Z\Psi' Z^T$ whose ijth entries vanish for all $(i,j) \notin E(G)$ is a stress matrix of $G(\hat{p})$ and since Ψ' is nonzero positive semidefinite if and only if Ψ is nonzero positive semidefinite.

□

6 A Sufficient Condition for Universal Rigidity

We apply the notion of SDP non-degeneracy to the pair of dual problems (19) and (20) to obtain a sufficient condition for universal rigidity of frameworks. We also show that Condition (7) in Theorem 2 follows from this sufficient condition when the given framework is generic.

Let S, rank $S = s$, be a given positive semidefinite stress matrix of framework $G(\hat{p})$. Let W' and U' be the $(n-1)\times s$ and $(n-1)\times(n-1-s)$ matrices whose columns form orthonormal bases for the rangespace and the nullspace of $Y = V^T S V$ respectively. Following [6], let

$$\mathcal{L} = \text{ span } \{M^{ij} : (i,j) \notin E(G)\},$$

and let

$$\mathcal{T}_Y = \{C \in \mathcal{S}_{n-1} : C = [W' \; U'] \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_2^T & 0 \end{bmatrix} \begin{bmatrix} {W'}^T \\ {U'}^T \end{bmatrix} \},$$

where Φ_1 is a symmetric matrix of order s and Φ_2 is $s \times (n-1-s)$. $\mathcal{T}_Y$ is the tangent space, at Y, to the set of $(n-1)\times(n-1)$ symmetric matrices of rank s.

Definition 2 (Alizadeh et al [6]) *Let Y be a feasible solution of problem (20). Y is said to be non-degenerate if*

$$\mathcal{T}_Y + \mathcal{L}^{\perp} = \mathcal{S}_{n-1}. \tag{22}$$

Otherwise, Y is called degenerate.

Equation (22) is equivalent to

$$\mathcal{T}_Y^{\perp} \cap \mathcal{L} = \{0\}, \tag{23}$$

where $\mathcal{T}_Y^{\perp}$ is the orthogonal complement of $\mathcal{T}_Y$, namely

$$\mathcal{T}_Y^{\perp} = \{C \in \mathcal{S}_{n-1} : C = U' \Phi {U'}^T\}.$$

The following theorem is well known.

Theorem 7 (Alizadeh et al [6]) *Consider a pair of dual SDP problems, where A^0, $A^1, \ldots, A^k$ are given linearly independent matrices in $\mathcal{S}_{n-1}$, and b is a given vector in $\mathbb{R}^k$.*

$$\begin{array}{llll} \textit{(P):} & \max\limits_{y} & b^T y & \\ & \textit{subject to} & A^0 - \sum_{i=1}^{k} y_i A^i \succeq 0. & \end{array}$$

$$\begin{array}{llll} \textit{(D):} & \min\limits_{Y} & \textit{trace } (A^0 Y) & \\ & \textit{subject to} & \textit{trace } (Y A^i) = b_i & \textit{for } i = 1, \ldots, k, \\ & & Y \succeq 0. & \end{array}$$

If the dual problem has a non-degenerate optimal solution, then the primal optimal solution is unique.

Applying this theorem to the pair of dual problems (19) and (20) we get the following theorem.

Theorem 8 *Given a framework $G(\hat{p})$ on n vertices in $\mathbb{R}^r$, $r \leq n-2$, let S be a positive semidefinite stress matrix of $G(\hat{p})$ and let U' be the matrix whose columns form an orthonormal basis of the nullspace of $Y = V^T S V$. If the trivial solution, $\Phi = 0$ and $y_{ij} = 0$ for all $(i,j) \notin E(G)$, is the only solution of the equation:*

$$U' \Phi {U'}^T + \sum_{(i,j) \notin E(G)} y_{ij} M^{ij} = 0. \tag{24}$$

Then $G(\hat{p})$ is universally rigid.

Proof. Assume that the only solution of (24) is the trivial solution. Then $\mathcal{T}_Y^{\perp} \cap \mathcal{L} = \{0\}$. Hence, Y is a non-degenerate optimal solution of (dual) problem (20). Thus, the set of optimal solution of (primal) problem (19) is a singleton. Therefore, $G(\hat{p})$ is universally rigid.

□

It is worth noting here that in equation (24), $\Phi = 0$ if and only if $y_{ij} = 0$ for all $(i,j) \notin E(G)$ since the set $\{M^{ij} : (i,j) \notin E(G)\}$ is linearly independent, and since U' has full column rank. Also, equation (24) is equivalent to a homogeneous system of $n(n-1)/2$ equations in $\bar{m} + (n-s)(n-s-1)/2$ unknowns, where s is the rank of the stress matrix S and $\bar{m}$ is the number of missing edges of G. Hence, the problem of determining whether or not equation (24) has a non-trivial solution reduces to that of computing the rank of the matrix of coefficients of this system.

Next we focus our attention on generic frameworks. The following lemmas are needed in the proof of Theorem 9 below.

Lemma 4 *Let $G(p)$ be a framework in general position on n vertices in $\mathbb{R}^r$, $r \leq n-2$, and let $z^1, \ldots, z^n$ be the Gale transform of $p^1, \ldots, p^n$ respectively. Recall that $\bar{r} = n - 1 - r$. Then any subset of $z^1, \ldots, z^n$ of cardinality $\bar{r}$ is linearly independent.*

Lemma 5 (Alfakih [4]) *Let $G(p)$ be a generic framework on n vertices in $\mathbb{R}^r$, $r \leq n-2$, and let each vertex of G have a degree at least r. Further, let Z be the Gale matrix of $G(p)$. Then there does not exist a non-zero $y = (y_{ij}) \in \mathbb{R}^{\bar{m}}$ such that*

$$\sum_{(i,j) \notin E(G)} y_{ij} V^T E^{ij} Z = 0.$$

Theorem 9 *Theorem 2 follows as a corollary of Theorem 8 .*

Proof. Assume that framework $G(p)$ in $\mathbb{R}^r$ is generic and that $S = Z \Psi Z^T$ is a stress matrix of $G(p)$ where matrix Ψ is $\bar{r} \times \bar{r}$ positive definite (recall that $\bar{r} = n-1-r$). We will show that, in this case, the only solution of (24) is the trivial solution. Hence it would follow from Theorem 8 that $G(p)$ is universally rigid.

First, we show that every vertex of G has a degree at least $r+1$. For assume, to the contrary, that the degree of one node of G, say node 1, is $\leq r$, and wlog assume that nodes $2, 3, \ldots, \bar{r}+1$ are not adjacent to node 1. Thus it follows from Lemma 4 that $z^2, z^3, \ldots, z^{\bar{r}+1}$ form a basis in $\mathbb{R}^{\bar{r}}$. Hence there exist $\lambda_2, \lambda_3, \ldots, \lambda_{\bar{r}+1}$, not all of which are zeros, such that $z^1 = \lambda_2 z^2 + \lambda_3 z^3 + \cdots + \lambda_{\bar{r}+1} z^{\bar{r}+1}$. Therefore, $(z^1)^T \Psi z^i = 0$ for $i = 2, \ldots, \bar{r}+1$ implies that $(z^1)^T \Psi z^1 = 0$ and therefore, Ψ is singular contradicting our assumption that $\Psi \succ 0$.

Now, the stress matrix $S = Z \Psi Z^T$ with $\Psi \succ 0$ also implies that the columns of the matrix $[P \;\; e]$ form a basis for the nullspace of S since in this case, rank $S = n-1-r$.

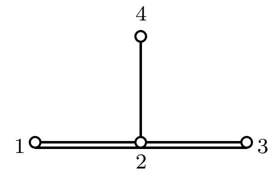

Fig. 3 The non-generic framework $G(p)$ in $\mathbb{R}^2$ of Example 2. Here, the missing edges of G are $(4,1)$ and $(4,3)$.

Consequently, it follows from Lemma 2 that the columns of V^TP form a basis for the nullspace of $Y = V^TSV$. Thus, in this case equation (24) reduces to

$$V^T(P\Phi'P^T + \sum_{(i,j)\notin E(G)} y'_{ij}E^{ij})V = 0. \tag{25}$$

But equation (25) is equivalent to

$$P\Phi'P^T + \sum_{(i,j)\notin E(G)} y'_{ij}E^{ij} = ae^T + ea^T, \tag{26}$$

for some n-vector a.

Thus it suffices to show that the only solution of equation (26) is the trivial solution.

Assume, to the contrary, that (26) has a solution $\Phi' \neq 0$, $y' = (y'_{ij}) \neq 0$. Then by multiplying (26) from the left by V^T and from the right by Z we get $\sum_{(i,j)\notin E(G)} y'_{ij}V^TE^{ij}Z = 0$. But this contradicts, from Lemma 5, our assumption that $G(p)$ is generic. Thus the result follows.

□

Finally, we end this section with the following numerical example that illustrates some of the results of this paper.

Example 2 *Consider the non-generic framework $G(p)$ in $\mathbb{R}^2$ depicted in Figure 3. $G(p)$ has two missing edges $(4,1)$ and $(4,3)$, and $\bar{r} = n - 1 - r = 1$ in this case. It is easy to show that the Gale matrix Z and a stress matrix S of $G(p)$ are*

$$Z = [1 \ \ -2 \ \ 1 \ \ 0]^T \text{ and } S = ZZ^T.$$

Note that S is positive semidefinite with rank $1 = \bar{r}$. Thus $G(p)$ satisfies Condition (7) of Theorem 2. However, $G(p)$ is obviously not universally rigid (in fact it is not even globally rigid). This shows that the assumption in Theorem 2 that the framework is generic cannot be dropped.

On the other hand, since stress matrix S is non-zero positive semidefinite, it follows from Theorem 6 that there does not exist a framework in $\mathbb{R}^3$ that is equivalent to $G(p)$. Indeed, this is obviously the case. In fact, $G(p)$ has an infinite number of equivalent frameworks in $\mathbb{R}^2$, and it has two equivalent frameworks in $\mathbb{R}^1$: One where node 4 coincides with node 1, and one where node 4 coincides with node 3.

7 Summary and Concluding Remarks

In this paper we presented a unified approach based on semidefinite programming theory for addressing several problems concerning universal rigidity. The salient feature of this approach is the use of projected Gram matrices for representing point configurations in Euclidean space. As a result, the set of all frameworks that are equivalent to a given framework in $\mathbb{R}^r$ was characterized in terms of a convex closed set formed by the intersection of the positive semidefinite cone with an affine subspace.

We characterized frameworks that admit non-zero positive semidefinite stress matrices, and we obtained some new results concerning such matrices. We used the notion of semidefinite programming non-degeneracy to obtain a sufficient condition for universal rigidity, and we showed that this condition yields the known sufficient condition for generic universal rigidity.

The fact that Ω_r, the set of all frameworks in $\mathbb{R}^r$ that are equivalent to a given framework in $\mathbb{R}^r$, is non-convex, due to the rank constraint, makes the global rigidity problem much harder to tackle. This also provides a challenge to extend the techniques used in the paper to global rigidity and to indefinite stress matrices. Perhaps a first step in that direction would be to use Moreau Theorem [18] to express an indefinite stress matrix S as the difference between two positive semidefinite matrices S_1 and S_2 such that trace $(S_1 S_2) = 0$.

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References

1. A. Y. Alfakih. Graph rigidity via Euclidean distance matrices. *Linear Algebra Appl.*, 310:149–165, 2000.
2. A. Y. Alfakih. On the set of realizations of edge-weighted graphs in Euclidean spaces, 2005. Research Report, Mathematics and Statistics, U of Windsor.
3. A. Y. Alfakih. On dimensional rigidity of bar-and-joint frameworks. *Discrete Appl. Math.*, 155:1244–1253, 2007.
4. A. Y. Alfakih. On the universal rigidity of generic bar frameworks. *Contrib. Discrete Math*, 5:7–17, 2010.
5. A. Y. Alfakih, A. Khandani, and H. Wolkowicz. Solving Euclidean distance matrix completion problems via semidefinite programming. *Comput. Optim. Appl.*, 12:13–30, 1999.
6. F. Alizadeh, J. A. Haeberly, and M. L. Overton. Complementarity and nondegeneracy in semidefinite programming. *Math. Programming, Ser. B*, 77:111–128, 1997.
7. P. Biswas and Y. Ye. Semidefinite programming for ad hoc wireless sensor network localization. Technical report, Dept. of Management Science and Engineering, Stanford University, 2003.
8. R. Connelly. Generic global rigidity. *Discrete Comput. Geom.*, 33:549–563, 2005.
9. R. Connelly. Tensegrity structures: Why are they stable?. In M. F. Thorpe and P. M. Duxbury, Editors, *Rigidity Theory and Applications*, pages 47–54, Kluwer academic/ Plenum Publishers, 1999.
10. T.F. Cox and M.A. Cox. *Multidimensional scaling*. Chapman and Hall, 2001.
11. G. M. Crippen and T. F. Havel. *Distance Geometry and Molecular Conformation*. Wiley, New York, 1988.
12. Y. Ding, N. Krislock, J. Qian, and H. Wolkowicz. Sensor network localization, Euclidean distance matrix completions, and graph realization, 2006. CORR 2006-23, Dept of Combinatorics and Optimization, U. of Waterloo.
13. T. Eren, D.K. Goldenberg, W. Whiteley, Y.R. Yang, A.S. Morse, B.D.O. Anderson, and P.N. Belhumeur. Rigidity, computation, and randomization in network localization, 2004. IEEE INFOCOM.

14. D. Gale. Neighboring vertices on a convex polyhedron. In *Linear inequalities and related system*, pages 255–263. Princeton University Press, 1956.
15. S. J. Gortler, A. D. Healy, and D. P. Thurston. Characterizing generic global rigidity, 2007. arXiv/0710.0926v4.
16. S. J. Gortler and D. P. Thurston. Characterizing the universal rigidity of generic frameworks, 2009. arXiv/1001.0172v1.
17. J. De Leeuw and W. Heiser. Theory of multidimensional scaling. In P. R. Krishnaiah and L. N. Kanal, editors, *Handbook of Statistics*, volume 2, pages 285–316. North-Holland, 1982.
18. J. J. Moreau. Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires. *C. R. Acad. Sci Paris*, 255:238–240, 1962.
19. T.K. Pong and P. Tseng. Robust edge-based semidefinite programming relaxation of sensor network localization. Technical report, U. of Washington, 2009.
20. R. T. Rockafellar. *Convex analysis*. Princeton University Press, 1970.
21. J. B. Saxe. Embeddability of weighted graphs in k-space is strongly NP-hard. *Proc. 17th Allerton Conf. in Communications, Control, and Computing*, pages 480–489, 1979.
22. A. M-C So and Y. Ye. A semidefinite programming approach to tensegrity theory and realizability of graphs. In *17th annual ACM-SIAM symposium on discrete algorithms*, pages 766–775, 2006.
23. A. M-C So and Y. Ye. Theory of semidefinite programming for sensor network localization. *Math. Program. Ser. B*, pages 367–384, 2007.
24. Z. Wang, S. Zheng, Y. Ye, and S. Boyd. Further relaxations of the semidefinite programming approach to sensor network localization. *SIAM J. Optim.*, pages 655–673, 2008.