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# The accountable function: a new approach to scheduling problems

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# The accountable function: a new approach to scheduling problems

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**Abstract**

We study cooperative cost-sharing in scheduling problems where agents differ in job lengths but share identical waiting costs. While the optimistic and pessimistic cost functions are natural approaches to compute the waiting cost of a coalition, their symmetry—well established for queueing problems—breaks down for scheduling. In particular, the pessimistic Shapley value violates equity by rewarding longer jobs. To address this, we propose an alternative formulation, termed "accountable cost" of serving a coalition. Under this, the coalitional members are served first, but they internalize the externality imposed on the non-coalitional members. We show that the accountable cost game is concave and the Shapley value of this game coincides with the decreasing serial rule, thereby restoring the lost symmetry. We also provide axiomatic characterizations of both the optimistic and accountable Shapley values, explore extensions to multi-server environments, and discuss connections with general sequencing problems. These results position the accountable function as a principled counterpart to the optimistic approach, unifying efficiency and fairness in the cooperative analysis of scheduling problems.

**Keywords:** scheduling, cooperative game, cost sharing, Shapley value, serial (decreasing serial) rule

**JEL codes:** C71, D63, D71

# 1 Introduction

Waiting lines are ubiquitous in modern life, from passengers boarding public transit to patients awaiting medical appointments or data packets queued for digital processing. How to organize such queues efficiently, and how to share the resulting waiting costs fairly, are classical problems at

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the intersection of operations research and economics. Operations research has focused largely on optimizing the order in which jobs are processed (or agents are served), while economics has been concerned with implementing these efficient orders through incentive-compatible and equitable cost-sharing mechanisms. Cooperative game theory provides a natural framework to formalize this question: agents can form coalitions to minimize total waiting costs jointly, and the challenge is to allocate the aggregate cost among them in a manner consistent with fairness, stability, and efficiency.

Within this literature, two specific classes of problems have been widely studied: queueing problems and scheduling problems. In queueing problems, all agents require identical service times but differ in their unit waiting costs, i.e., their opportunity cost of time per unit time period. In scheduling problems, agents have heterogeneous job lengths but share an identical per-unit waiting cost. We study the implications of this inversion of heterogeneity on cost-sharing games. In queueing, the well-known optimistic and pessimistic coalitional cost functions, which respectively assume that a coalition is served first or last, exhibit a remarkably elegant symmetry. The associated Shapley values coincide with the popular serial and decreasing serial rules in the cost-sharing literature. (Maniquet (2003), Chun (2006)). In scheduling problems, however, this symmetry collapses (Banerjee and Trudeau (2025)). The pessimistic approach, which is well-behaved in queueing environments, produces inequitable outcomes in scheduling: the pessimistic Shapley value rewards longer jobs with lower cost shares, violating the basic equity principles. It has been shown by Moulin (2007) that the pessimistic Shapley value for scheduling problems must fail the *ranking* property if there are more than five agents. Banerjee and Trudeau (2025) show that the pessimistic Shapley is always such that if $i$ has a strictly longer job than $j$, he never pays more than $j$, with examples with strict inequality with three or more agents. The absence of a fairness-preserving counterpart to the optimistic approach leaves a theoretical gap in the cooperative analysis of scheduling. This paper fills that gap.

We propose a new coalitional cost formulation called the "accountable" function, that restores the lost symmetry. Under this approach, a coalition is assumed to be served first, but it must internalize the externality imposed on the agents behind them in the queue. In essence, the coalition "pays forward" the additional waiting time it causes to non-members. We show that the accountable cost game is concave, implying that the Shapley value of this game lies in its core, rendering it stable. Moreover, we prove that the Shapley value of the accountable game coincides exactly with the decreasing serial rule, thereby restoring the equivalence that was previously lost. Together with the known coincidence between the optimistic Shapley value and the serial rule, this result reestablishes a complete symmetry between these two cost-sharing rules for queueing and scheduling environments.

Our analysis also bridges the axiomatic foundations of the two settings. We provide axiomatic characterizations for the optimistic and accountable Shapley values. The optimistic Shapley value uniquely satisfies Equal Treatment of Equals and Independence of Longer Jobs, while the accountable value uniquely satisfies Equal Treatment of Equals and Independence of Shorter Jobs. We also provide symmetric characterizations of the optimistic and the accountable Shapley values based on identical length upper bound, monotonicity, and proportional-effect properties which are analogous to those in Maniquet (2003) and Chun (2006). These parallel axiomatizations underscore that the accountable approach is not merely a technical fix but a normative counterpart that mirrors the optimistic formulation in structure and spirit.

We discuss the extension of the framework to multi-server environments, distinguishing between divisible and indivisible jobs. We demonstrate that the coincidence between the accountable Shapley value and the decreasing serial rule persists under indivisible tasks but breaks under divisibility. This mirrors the behavior of the optimistic formulation. Finally, we situate the accountable function within the broader class of waiting-line problems, demonstrating that it provides a coherent generalization, even though, like the pessimistic function, it exhibits order reversal in queueing by making an agent with a lower unit waiting cost bear a relatively higher cost burden.

### 1.1 Literature

Job scheduling is a central problem in operations research, with an extensive body of literature. Following the standard terminology, the problems considered in this paper are *single-stage job scheduling problems*, where each job must be processed exactly once on a single machine. Moreover, we study the case of *identical-machine scheduling*, where all servers operate at the same speed for all jobs. For comprehensive reviews of this field, see Graham et al. (1979), Lawler et al. (1993), and Chen et al. (1998).

In economics, the study of queueing problems was initiated by Maniquet (2003), who introduced the *optimistic* approach, defining the cost of a coalition under the assumption that it gains first access to the servers. The *optimistic* Shapley value was axiomatized by Maniquet (2003), while Katta and Sethuraman (2006) suggested alternative interpretations of fairness. In contrast, Chun (2006) and Klijn and Sanchez (2006) defined the *pessimistic* approach, where members of a coalition are served last (after the non-coalitional members). The *pessimistic* Shapley value was axiomatized by Chun (2006). Chun (2011) allowed for variable job sizes, extended both solutions to this context, and offered parallel characterizations. A comparative review of the optimistic and pessimistic standpoints was provided by Chun (2016).

Scheduling problems, on the other hand, were examined by Moulin (2007, 2008). In these settings, the server can monitor the length of a job but not the identity of the job's user; merging, splitting, or partially transferring jobs offers cooperative strategic opportunities. Moulin (2007) showed that the optimistic Shapley value is mergeproof while the pessimistic Shapley value is splitproof. The companion paper Moulin (2008) discussed the same strategic maneuvers when the server, instead of cash transfers, used randomization. The proportional rule has been characterized by the combination of mergeproofness, splitproofness, and a couple of natural properties of invariance and fairness. Banerjee and Trudeau (2025) explored the relationship between the Shapley values of the optimistic and pessimistic games and the corresponding *serial* and *decreasing serial* allocation methods, in both queueing and scheduling contexts.

The aforementioned studies address situations where waiting lines are constructed from scratch. An alternative line of research focuses on the *reconfiguration of existing waiting lines*, as discussed in Curiel et al. (1989) and Curiel et al. (1993). Finally, the *job scheduling problem* (Okamoto, 2008; Bahel and Trudeau, 2019) provides a complementary perspective: rather than allocating agents to a fixed number of servers, the objective is to determine the minimum number of servers required so that no agent must wait.

The serial rule, which is naturally defined for scheduling, was originally defined when agents are sharing the cost of production using a joint technology (Moulin and Shenker, 1992). The decreasing serial rule was first defined by de Frutos (1998).

# 2 The model

A finite set of agents $N = \{1, 2, \dots, n\}$ arrive at a service facility with their jobs. Each agent has one job to process, but the length of the jobs can vary between agents. Let $l_i \in \mathbb{N}$ be the *job length of agent* $i$. For $S \subseteq N$, let $l_S = \sum_{i \in S} l_i$ be the aggregate job length of the agents in $S$. Each agent also incurs a waiting cost per unit period of time that he spends at the facility. We assume that all agents have identical *per-unit-time waiting costs*, which are normalized to 1. The facility has a single server that processes jobs sequentially (one job at a time). An agent exits the facility only when his job is fully processed. If an agent $i$ exits the facility at time $t_i \in \mathbb{N}$, his waiting cost in monetary terms is given by $t_i$. A scheduling problem is specified by $(N, l)$ where $l := (l_1, l_2, \dots, l_n) \in \mathbb{N}^n$ is the vector of job lengths. In most of what we do the agent set $N$ is fixed, in which case we often define a scheduling problem by the vector of job lengths.

Our framework assumes that jobs are processed *efficiently*, i.e., agents are served in an order that minimizes their aggregate waiting cost. From Smith et al. (1956), it is well established that efficiency in scheduling problems requires arranging agents in a non-decreasing order of their job lengths. An order is said to be efficient if and only if for any pair $i, j \in N$, if $l_i < l_j$ then agent $i$ is served before agent $j$. Without loss of generality, we may therefore assume that agents are indexed such that $l_i \leq l_{i+1}$ for $1 \leq i \leq n-1$; equivalently, if $i < j$, it is always efficient to process $i$'s job before $j$'s. We accordingly introduce the following notations.

For a set of agents $S \subseteq N$, let $r_i(S) = |\{j \in S \,|\, j \leq i\}|$ be the rank of agent $i$ in $S$. We define $F_i^S = \{j \in S \,|\, j > i\}$ to be the set of agents that follow agent $i$ when the members in $S$ are served efficiently. Similarly, let $P_i^S = \{j \in S \,|\, j < i\}$. be the set of agents that precede agent $i$ when members in $S$ are served efficiently. Note that, $r_i(S) = \left|P_i^S\right| + 1 = |S| - \left|F_i^S\right|$. If we have $l_i = l_j$ for any $i, j \in N$, we say that agent $i$ is identical to agent $j$. We study a scheduling problem as a cost-sharing game and employ standard axioms of efficiency and equal treatment of equals to characterize desirable cost-sharing rules. The axiom of equal treatment of equals requires that the minimum aggregate waiting cost be distributed such that identical agents incur identical cost shares. Therefore, despite ties yielding multiple efficient orders, the sequence in which identical agents are served in any such order ceases to matter.

**Example 1** *Consider a facility with a single server. Let* $N = \{1, 2, 3\}$ *and* $l = (2, 2, 4)$.

*Note that serving agents in a non-decreasing order of their job lengths minimizes the aggregate waiting cost. Moreover, since agents* 1 *and* 2 *are identical, interchanging their positions does not affect the minimum cost.*

For each $l \in \mathbb{N}^N$, we write $c(N, l) = \sum_{i \in N} (n - i + 1) l_i$ as the minimum cost to serve agents in $N$.

Table 1: Waiting times under different serving orders

| Order | $t_1$ | $t_2$ | $t_3$ | Total Waiting Cost |
|---|---|---|---|---|
| $(1,2,3)$ | 2 | 4 | 8 | 14 |
| $(2,1,3)$ | 4 | 2 | 8 | 14 |
| $(3,2,1)$ | 8 | 6 | 4 | 18 |
| $(1,3,2)$ | 2 | 8 | 6 | 16 |
| $(2,3,1)$ | 8 | 2 | 6 | 16 |
| $(3,1,2)$ | 6 | 8 | 4 | 18 |

### 2.1 Preliminary definitions (Cooperative Game Theory)

Consider a finite set of players $N = \{1, \ldots, n\}$. A *coalitional cost function* is a mapping $c : 2^N \to \mathbb{R}_+$ that assigns a nonnegative cost to each coalition $S \subseteq N$, with the convention that $c(\emptyset) = 0$.

**Definition 1** A cost function $c$ is said to be *convex* if, for all coalitions $S, T \subseteq N$,

$$c(S) + c(T) \leq c(S \cup T) + c(S \cap T)$$

Equivalently, for all $S \subseteq T \subseteq N \setminus \{i\}$,

$$c(S \cup \{i\}\ ) - c(S) \leq c(T \cup \{i\}\ ) - c(T).$$

For concavity, both the inequalities reverse from at most as ($\leq$) to at least as ($\geq$).

**Definition 2** An *allocation* is a vector $y \in \mathbb{R}^N$ such that $\sum_{i \in N} y_i = c(N)$.

**Definition 3** An allocation $y$ belongs to the *core* of $c$, denoted $\mathcal{C}(c)$, if

$$\sum_{i \in S} y_i \leq c(S) \quad \text{for all } S \subset N.$$

Similarly, $y$ belongs to the *anticore* of $c$, denoted $\mathcal{A}(c)$, if

$$\sum_{i \in S} y_i \geq c(S) \quad \text{for all } S \subset N.$$

**Definition 4** A *rule* (or *allocation method*) is a mapping that assigns an allocation to each coalitional cost function. A canonical and widely used rule is the *Shapley value*, defined for each player $i \in N$ as

$$Sh_i(c) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} \big(c(S \cup i) - c(S)\big).$$

**Proposition 1** If $c$ is concave, then $Sh(c) \in \mathcal{C}(c)$. Conversely, if $c$ is convex, then $Sh(c) \in \mathcal{A}(c)$.

# 3 Coalitional cost funtions

In order to identify solutions to scheduling problems that satisfy fairness properties, the tools of cooperative game theory are invoked. This requires modeling scheduling problems as coalitional games. We apply the solution concepts of Shapley value (for fairness) and core (for stability) to these games. There are multiple ways of defining such a game. The existing literature has discussed two extreme scenarios based on whether the coalitional members are served first (optimistic) or last (pessimistic).

## 3.1 The optimistic cost and Shapley value

We discuss the optimistic approach in this section which assumes that a coalition has access to the facility first and ignores the presence of other agents when calculating its cost. In other words, the cost of coalition $S$, denoted by $c^o(S,l)$, is the minimum waiting cost of $S$ when its members are served first.

Formally, for each problem $(N,l)$ and each $S \subseteq N$ :

$$\begin{aligned} c^o(S,l) &= \sum_{i \in S} (|F_i^S| + 1) l_i. \\ &= \sum_{i \in S} (|S| - r_i(S) + 1) l_i \end{aligned}$$

**Theorem 1** For each $(N,l)$ the game $c^o(\cdot,l)$ is convex.

**Proof.** We fix $(N,l)$ and write $c^o(S)$ instead of $c^o(S,l)$. Take $i \in N$ and $S \subset T \subseteq N \backslash i$. We show that $c^o(S \cup i) - c^o(S) \leq c^o(T \cup i) - c^o(T)$.

Notice that if $j \in P_i^{S \cup i}$, then $\left|P_j^{S \cup i}\right| = \left|P_j^S\right|$ while if $j \in F_i^{S \cup i}$, then $\left|P_j^{S \cup i}\right| = \left|P_j^S\right| + 1$. Thus,

$$c^o(S \cup i) - c^o(S) = \sum_{j \in P_i^{S \cup i}} l_j + (|S| + 1 - \left|P_i^{S \cup i}\right|) l_i$$

and

$$c^o(T \cup i) - c^o(T) = \sum_{j \in P_i^{T \cup i}} l_j + (|T| + 1 - \left|P_i^{T \cup i}\right|) l_i.$$

First, notice that

$$\sum_{j \in P_i^{S \cup i}} l_j \leq \sum_{j \in P_i^{T \cup i}} l_j$$

as $S \subset T$ implies that $P_i^{S \cup i} \subseteq P_i^{T \cup i}$.

Second, notice that

$$(|S| + 1 - \left|P_i^{S \cup i}\right|) \leq (|T| + 1 - \left|P_i^{T \cup i}\right|)$$

as $\left(P_i^{T \cup i} \setminus P_i^{S \cup i}\right) \subseteq T \setminus S$.

Combining, $c^o(S \cup i) - c^o(S) \leq c^o(T \cup i) - c^o(T)$.

■

The optimistic Shapley value is defined as $y^o(N,l) = Sh(c^o(\cdot,l))$. It is easy to show that

$$\begin{aligned} y_i^o &= \left(\frac{n-i}{2}+1\right) l_i + \sum_{j<i} \frac{l_j}{2} \\ &= (n-i+1)\, l_i + \sum_{j<i} \frac{l_j}{2} - \frac{\left|F_i^N\right|}{2} l_i \end{aligned}$$

This has been termed as the minimal transfer rule by Maniquet (2003) in the queueing context.

**Corollary 1** For all $(N,l)$, $y^o(N,l) \in \mathcal{A}(C^o(\cdot,l))$.

**Example 2** *Revisiting Example 1, we obtain* $c^o(\{1\},l) = c^o(\{2\},l) = 2$, $c^o(\{3\},l) = 4$, $c^o(\{1,2\},l) = 6$, $c^o(\{1,3\},l) = c^o(\{2,3\},l) = 8$ *and* $c^o(N,l) = 14$.

*We get* $y^o(l) = (4,4,6) \in \mathcal{A}(c^o(\cdot,l))$.

## 3.2 The pessimistic function and Shapley value

The pessimistic approach assumes that the coalitional members are served after the non-coalitional members. The pessimistic cost of a coalition $S$, denoted by $c^p(S,l)$, as the minimum waiting cost of $S$ when its members are served last. Formally, for each problem $(N,l)$ and each $S \subseteq N$ :

$$\begin{aligned} c^p(S,l) &= \sum_{i\in N\setminus S} |S| l_i + \sum_{i\in S} (|F_i^S|+1) l_i. \\ &= \sum_{i\in N\setminus S} |S| l_i + \sum_{i\in S} (|S| - r_i(S) + 1) l_i \end{aligned}$$

The pessimistic Shapley value is defined as $y^p(N,l) = Sh(c^p(\cdot,l))$. It can be easily checked that:

$$y_i^p = \left(\frac{3-i}{2}\right) l_i + \sum_{j<i} l_j + \sum_{j>i} \frac{l_j}{2}$$

For queueing problems, the pessimistic game exhibits a well-behaved symmetry with the optimistic game, and pessimistic Shapley value, termed as the maximum transfer rule by Chun (2006), can be seen as a natural counterpart to the optimistic Shapley value. For scheduling problems, Banerjee and Trudeau (2025) show that the pessimistic Shapley value satisfies a rather counterintuitive property called the *order reversal* in which agents with shorter job lengths bear a higher cost share burden. This is strongly undesirable from the standpoint of equitable cost sharing.

**Theorem 2** (Banerjee and Trudeau, 2025) For all $(N,l)$, if $l_i \leq l_j$ then $y_i^p(l) \geq y_j^p(l)$.[1]

[1] As seen in Example 3 the inequality can be strict as soon as $n > 2$.

Observe that the Shapley value expression above puts a larger weight on predecessors' lengths than followers' lengths, and that the weight on an agent's length is decreasing in his rank.

**Example 3** *We revisit Example 1 and calculate* $c^p(\{1\},l) = c^p(\{2\},l) = c^p(\{3\},l) = 8$, $c^p(\{1,2\},l) = 14$, $c^p(\{1,3\},l) = c^p(\{2,3\},l) = 12$ *and* $c^p(N,l) = 14$. *We obtain* $y^p(l) = (5,5,4)$.

Note that agent 3, who has the longest job, ends up paying less than others.

## 3.3 The accountable cost function

In this section, we introduce a novel way of defining the cost of serving a coalition. It is built upon the assumption that although the coalitional members are served first, they are held responsible for the externalities imposed on the non-coalitional members. Alternatively, agents in a coalition $S$ arrive after $N \backslash S$ but pay them to move in front of them in queue.

For scheduling problems, the externality that a coalition $S$ imposes on $N \backslash S$ is precisely $\sum_{i \in S} l_i(n - |S|)$, as each agent in $N \backslash S$ is displaced by $\sum_{i \in S} l_i$ periods. Adding this to the optimistic cost, we obtain the *accountable* cost function $c^a$. For a problem $(N,l)$ and each $S \subseteq N$, we have that

$$\begin{aligned} c^a(S,l) &= c^o(S,l) + \sum_{i \in S} l_i(n - |S|) \\ &= \sum_{i \in S} (|S| - r_i(S) + 1) l_i + \sum_{i \in S} l_i(n - |S|) \\ &= \sum_{i \in S} (n - r_i(S) + 1) l_i \\ &= \sum_{i \in S} (n - |P_i^S|) l_i \end{aligned}$$

It can be verified that $c^a(N,l) = c^o(N,l)$.

**Example 4** *We revisit Example 1. We obtain* $c^a(\{1\},l) = c^a(\{2\},l) = 6$, $c^a(\{3\},l) = 12$, $c^a(\{1,2\},l) = 10$, $c^a(\{1,3\},l) = c^a(\{2,3\},l) = c^a(N,l) = 14$.

**Theorem 3** *For each* $(N,l)$, *the game* $c^a(\cdot,l)$ *is concave.*

**Proof.** Fix $(N,l)$ and write $c^a(S)$ instead of $c^a(S,l)$. Let $i \in N$ and $S \subset T \subseteq N \setminus \{i\}$. We show that

$$c^a(S \cup i) - c^a(S) \geq c^a(T \cup i) - c^a(T).$$

Observe that if $j \in P_i^{S \cup i}$, then $|P_j^{S \cup i}| = |P_j^S|$, whereas if $j \in F_i^{S \cup i}$, then $|P_j^{S \cup i}| = |P_j^S| + 1$. Hence,

$$c^a(S \cup i) - c^a(S) = - \sum_{j \in F_i^{S \cup i}} l_j + \left(n - |P_i^{S \cup i}|\right) l_i,$$

and similarly,

$$c^a(T \cup i) - c^a(T) = -\sum_{j \in F_i^{T\cup i}} l_j + \big(n - |P_i^{T\cup i}|\big) l_i.$$

First, note that

$$-\sum_{j \in F_i^{S\cup i}} l_j \geq -\sum_{j \in F_i^{T\cup i}} l_j,$$

since $S \subset T$ implies $F_i^{S\cup i} \subseteq F_i^{T\cup i}$.

Second, we have

$$\big(n - |P_i^{S\cup i}|\big) l_i \geq \big(n - |P_i^{T\cup i}|\big) l_i,$$

because $S \subset T$ implies $P_i^{S\cup i} \subseteq P_i^{T\cup i}$.

Combining these two inequalities yields

$$c^a(S \cup i) - c^a(S) \geq c^a(T \cup i) - c^a(T),$$

which proves the claim. ■

Next, we show the relationship between the core of the accountable game and the anticore of the optimistic game. Atay and Trudeau (2026) discuss, as an example of a more general model, a similar relationship between the core of the pessimistic game and the anticore of the optimistic game for the queueing game. Their general results, while closely related to the theorem below, do not quite apply to the accountable game.

**Theorem 4** *For all $(N, l)$, it holds that*

$$\mathcal{A}(c^o(\cdot, l)) \subseteq \mathcal{C}(c^a(\cdot, l)).$$

**Proof.** Fix $(N, l)$, and write $c^a(S)$ and $c^o(S)$ instead of $c^a(S, l)$ and $c^o(S, l)$, respectively. We have

$$c^o(S) = \sum_{i \in S} \big(|S| - r_i(S) + 1\big) l_i, \qquad c^a(S) = \sum_{i \in S} \big(n - r_i(S) + 1\big) l_i.$$

Moreover,

$$c^o(N) = c^a(N) = \sum_{i \in N} \big(n - r_i(N) + 1\big) l_i \;\equiv\; c(N).$$

We can verify that $x \in \mathcal{C}(c^a)$ if, for all $S \subseteq N$,

$$c(N) - c^a(N \setminus S) \leq x_S \leq c^a(S),$$

$$\sum_{i \in N} (n - r_i(N) + 1) l_i - \sum_{i \in N \setminus S} (n - r_i(N \setminus S) + 1) l_i \leq x_S \leq \sum_{i \in S} (n - r_i(S) + 1) l_i.$$

Similarly, $x \in \mathcal{A}(c^o)$ if, for all $S \subseteq N$,

$$c^o(S) \leq x_S \leq c(N) - c^o(N \setminus S),$$
$$\sum_{i\in S}(|S| - r_i(S) + 1)l_i \leq x_S \leq \sum_{i\in N}(n - r_i(N) + 1)l_i - \sum_{i\in N\setminus S}(n - |S| - r_i(N \setminus S) + 1)l_i.$$

Therefore, $\mathcal{A}(c^o) \subseteq \mathcal{C}(c^a)$ if, for all $S \subseteq N$,

$$\sum_{i\in N}(n - r_i(N) + 1)l_i - \sum_{i\in N\setminus S}(n - r_i(N \setminus S) + 1)l_i \leq \sum_{i\in S}(|S| - r_i(S) + 1)l_i, \tag{1}$$
$$\sum_{i\in N}(n - r_i(N) + 1)l_i - \sum_{i\in N\setminus S}(n - |S| - r_i(N \setminus S) + 1)l_i \leq \sum_{i\in S}(n - r_i(S) + 1)l_i. \tag{2}$$

We now verify inequality (1). It can be rewritten as

$$\sum_{i\in N}(n - r_i(N) + 1)l_i \leq \sum_{i\in N\setminus S}(n - r_i(N \setminus S) + 1)l_i + \sum_{i\in S}(|S| - r_i(S) + 1)l_i.$$

The right-hand side corresponds to the total cost incurred if the agents in $N \setminus S$ are served first (in increasing order of their ranks), followed by the agents in $S$ (also in increasing rank order within that group). This constitutes a feasible, but not necessarily optimal, service order. Therefore, its total cost is at least as large as $c(N)$, which corresponds to the optimal order of serving all agents in $N$. Hence, inequality (1) holds.

The argument for inequality (2) is analogous. It can be rewritten as

$$\sum_{i\in N}(n - r_i(N) + 1)l_i \leq \sum_{i\in N\setminus S}(n - |S| - r_i(N \setminus S) + 1)l_i + \sum_{i\in S}(n - r_i(S) + 1)l_i.$$

Here, the right-hand side represents the total cost when the coalition $S$ is served before $N \setminus S$. Again, this order is feasible but (possibly) suboptimal, and thus the corresponding total cost cannot be smaller than $c(N)$.

Since both inequalities (1) and (2) are satisfied, it follows that

$$\mathcal{A}(c^o) \subseteq \mathcal{C}(c^a),$$

as claimed. ■

## 3.4 The accountable Shapley value

### 3.4.1 Closed-form expression and the core

We define the accountable Shapley value as $y^a(N, l) = Sh(C^a(\cdot, l))$. It is easy to show that

$$\begin{aligned} y_i^a &= \left(n - \frac{i}{2} + \frac{1}{2}\right) l_i - \sum_{j>i} \frac{l_j}{2} \\ &= (n - i + 1)l_i - \sum_{j>i} \frac{l_j}{2} + \frac{\left|P_i^N\right|}{2} l_i \end{aligned}$$

We can see that it exhibits a remarkable symmetry to $y^o$, in a manner similar to the relationship between the optimistic and pessimistic Shapley rules (minimum and maximum transfer rules) in queueing problems.

**Corollary 2** *For all $(N,l)$, $y^o(N,l) \in \mathcal{C}(C^a(\cdot,l))$.*

**Example 5** *Applying the Shapley value to the accountable game computed in Example 4, we obtain $y^a(l) = (3,3,8) \in \mathcal{C}(c^a(\cdot,l))$.*

*In this example, $y^a(l) \in \mathcal{A}(c^o(\cdot,l))$, but this is not always the case. Consider now $l' = (1,2,4)$. We obtain the following values:*

| | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $N$ |
|---|---|---|---|---|---|---|---|
| $c^o(S,l')$ | 1 | 2 | 4 | 4 | 6 | 8 | 11 |
| $c^a(S,l')$ | 3 | 6 | 12 | 7 | 11 | 14 | 11 |

*We have $y^a(l') = (0,3,8)$. Since $y^a_1(l') = 0 < 1 = c^o(\{1\},l')$, $y^a(l') \notin \mathcal{A}(c^o(\cdot,l'))$.*

### 3.4.2 Coincidence with the decreasing serial rule

We explain the serial principle. First, arrange all the agents in a non-decreasing order of their job lengths. Start by supposing that all agents are assigned the shortest length value, compute the optimal cost in this case, and divide this cost equally among all $n$ agents. Next, fix the length of the first agent and increase the lengths of the remaining $n-1$ agents to the second shortest length. Calculate the increment in total cost and divide this equally among $n-1$ agents. Continue in this manner, at step $k$ supposing that all agents $i < k$ have their own lengths, while others are assigned the length of agent $k$. The increment in the total cost relative to the previous step is shared among the last $|N| - k + 1$ agents.

Like the serial principle, it is also possible to divide the cost burden among agents using the "decreasing serial" principle. We begin by supposing that all agents have the largest length. The optimal cost is computed and distributed equally among all $n$ agents. The subsequent steps are similar to the serial principle described above, with the only difference being that the process moves from the largest length to the second largest, and so on.

Formally, let $\underline{l}^i$ be such that $\underline{l}^i_j = \min(l_i, l_j)$ and $\overline{l}^i$ be such that $\overline{l}^i_j = \max(l_i, l_j)$. For any problem $(N,l)$, we have that $Ser_1(N,l) = \frac{c(N,\underline{l}^1)}{n}$ and for all $i = 2, ..., n$,

$$Ser_i(N,l) = Ser_{i-1}(N,l) + \frac{c(N,\underline{l}^i) - c(N,\underline{l}^{i-1})}{n-i+1}.$$

We also have that $DSer_n(N,L) = \frac{c(N,\overline{l}^n)}{n}$ and for all $i = 1, ..., n-1$,

$$DSer_i(N,l) = DSer_{i+1}(N,l) + \frac{c(N,\overline{l}^i) - c(N,\overline{l}^{i+1})}{i}.$$

It is known that $y^o(N,l) = Ser(N,l)$.

**Theorem 5** *For all $(N,l)$, $y^a(N,l) = DSer(N,l)$.*

**Proof.** We fix $(N,l)$ and write $c^a(S)$ instead of $c^a(S,l)$ and $y^a$ and $DSer$ instead of $y^a(N,l)$ and $DSer(N,l)$

First, we have that $y^a_n = \frac{n+1}{2}l_n$. Notice that if all agents have length $l_n$, total cost is $\frac{n(n+1)}{2}l_n$. Thus, $DSer_n$ is $\frac{1}{n}$ of that, or $\frac{n+1}{2}l_n$. Thus, $y^a_n = DSer_n$.

Let $k \in \{1,...,k-1\}$. We have that

$$\begin{aligned}
y^a_k - y^a_{k+1} &= \left(n - \frac{k}{2} + \frac{1}{2}\right) l_k - \frac{l_{k+1}}{2} - \sum_{j>k+1} \frac{l_j}{2} - \left(n - \frac{k+1}{2} + \frac{1}{2}\right) l_{k+1} - \sum_{j>k+1} \frac{l_j}{2} \\
&= \left(n - \frac{k}{2} + \frac{1}{2}\right)(l_k - l_{k+1}) \\
&= \left(\frac{2n-k+1}{2}\right)(l_k - l_{k+1})
\end{aligned}$$

Next, notice that by definition, we have $DSer_k - DSer_{k+1} = \frac{c(N,\bar{l}^i) - c(N,\bar{l}^{i+1})}{k}$.
Generally, we have that

$$c(N,\bar{l}^k) = \sum_{m=1}^{k} (n-m+1)l_k + \sum_{m=k+1}^{n} (n-m+1)l_m.$$

Thus,

$$\begin{aligned}
c(N,\bar{l}^k) - c(N,\bar{l}^{k+1}) &= \sum_{m=1}^{k}(n-m+1)l_k + \sum_{m=k+1}^{n}(n-m+1)l_m \\
&\quad - \sum_{m=1}^{k+1}(n-m+1)l_{k+1} - \sum_{m=k+2}^{n}(n-m+1)l_m \\
&= \sum_{m=1}^{k}(n-m+1)(l_k - l_{k+1})
\end{aligned}$$

Now, the sum of integers between $a$ and $b$ is $\frac{(b-a)(a+b)}{2}$, so we obtain that

$$\begin{aligned}
c(N,\bar{l}^k) - c(N,\bar{l}^{k+1}) &= \frac{k(n+n-k+1}{2}(l_k - l_{k+1}) \\
&= k\left(\frac{2n-k+1}{2}\right)(l_k - l_{k+1})
\end{aligned}$$

Thus,

$$\begin{aligned}
DSer_k - DSer_{k+1} &= \frac{c(N,\bar{l}^k) - c(N,\bar{l}^{k+1})}{k} \\
&= \left(\frac{2n-k+1}{2}\right)(l_k - l_{k+1}) \\
&= y^a_k - y^a_{k+1}.
\end{aligned}$$

Combining with $y^a_n = DSer_n$, we obtain $y^a = DSer$. ■

### 3.4.3 Axiomatizations

We propose two axiomatizations—one for the optimistic and one for the accountable Shapley value—that highlight the symmetry between these two solutions. The first set of axiomatizations builds on their coincidence with the serial and decreasing serial methods, following Banerjee and Trudeau (2025) and the classic axiomatization of the serial rule (Moulin and Shenker, 1992).

The first property is the well-known *Equal Treatment of Equals*.

**Equal treatment of equals**: For all $(N, l)$, and all $i, j \in N$, if $l_i = l_j$ then $y_i(N, l) = y_j(N, l)$.

The next property states that the share of an agent should remain unchanged if the job of a follower becomes longer.

**Independence of longer jobs**: For all $(N, l)$, $(N, l')$ such that $l_i = l'_i$ for all $i \in N \setminus j$ and $l_j < l'_j$ then if $l_i \leq l_j$, $y_i(N, l) = y_i(N, l')$.

We define a symmetric property with respect to shorter jobs.

**Independence of shorter jobs**: For all $(N, l), (N, l')$ such that $l_i = l'_i$ for all $i \in N \setminus j$ and $l_j > l'_j$ then if $l_i \geq l_j$, $y_i(N, l) = y_i(N, l')$.

**Corollary 3** *(Banerjee and Trudeau (2025)) On the domain of scheduling problems (with a single machine), we have that:*

1. *$y^o$ is the only rule satisfying Equal Treatment of Equals and Independence of Longer Jobs.*

2. *$y^a$ is the only rule satisfying Equal Treatment of Equals and Independence of Shorter Jobs.*

The second set of axiomatizations is adapted from Maniquet (2003) and Chun (2006), who offer symmetric axiomatizations for the optimistic and pessimistic Shapley values in queueing problems.

The first property sets an upper bound on what an agent should pay—namely, an equal share of the total cost if everyone had her job length.

**Identical length upper bound:** For all $(N, l)$ and all $i \in N$,

$$y_i(N, l) \leq \frac{(n+1)}{2} l_i.$$

The next two properties describe how the cost shares of other agents respond when one agent's job length increases. *Length Monotonicity* expresses solidarity: as one job becomes longer, the shares of the others weakly increase. Conversely, *Length Antimonotonicity* captures competition: as one job becomes longer, others' shares weakly decrease.

**Length monotonicity:** For $(N, l)$ and $(N, l')$ such that $l_i = l'_i$ for all $i \in N \setminus j$ and $l_j < l'_j$, we have

$$y_i(N, l) \leq y_i(N, l') \quad \text{for all } i \in N \setminus j.$$

**Length antimonotonicity:** For $(N, l)$ and $(N, l')$ such that $l_i = l'_i$ for all $i \in N \setminus j$ and $l_j < l'_j$, we have

$$y_i(N, l) \geq y_i(N, l') \text{ for all } i \in N \setminus j.$$

Our last two properties concern the removal of agents at the extremes of the length distribution. The *First Agent Proportional Effect* pertains to the removal of the agent with the shortest job, while the *Last Agent Proportional Effect* pertains to the removal of the agent with the longest job. In both cases, the shares of the remaining agents should adjust proportionally to their job lengths.

**First agent proportional effect:** Let $(N, l)$ and $i \in N$ be such that $l_i \le l_j$ for all $j \in N$. Then, for all $j, k \in N \setminus i$,

$$\frac{y_j(N,l) - y_j(N \setminus i, l^{-i})}{l_j} = \frac{y_k(N,l) - y_k(N \setminus i, l^{-i})}{l_k},$$

where $l^{-i}$ is the vector of job lengths excluding $l_i$.

**Last agent proportional effect:** Let $(N, l)$ and $i \in N$ be such that $l_i \ge l_j$ for all $j \in N$. Then, for all $j, k \in N \setminus i$,

$$\frac{y_j(N,l) - y_j(N \setminus i, l^{-i})}{l_j} = \frac{y_k(N,l) - y_k(N \setminus i, l^{-i})}{l_k}.$$

We are now ready for our second set of axiomatizations.

**Theorem 6** *On the domain on scheduling problems with a single machine, we have:*

1. *$y^o$ is the only rule satisfying Identical length upper bound, Length monotonicity and Last agent proportional effect.*

2. *$y^a$ is the only rule satisfying Identical length upper bound, Length antimonotonicity and First agent proportional effect.*

**Proof.** The proof is strongly inspired by Chun (2006). We show ii). Proof of i) is done in the same way.

Suppose that agents are ranked in non-decreasing order of lengths. Let $y$ be a rule that satisfies all properties. We first show that we must have $y_n(N,l) = \frac{(n+1)}{2} l_n$.

By Identical length upper bound, we must have $y_n(N,l) \le \frac{(n+1)}{2} l_n$.

By contradiction, suppose that $y_n(N,l) < \frac{(n+1)}{2} l_n$. Let $l^n = (l_n, ..., l_n)$. By multiple applications of Length antimonotonicity, by moving from $l$ to $l^n$, we must have $y_n(N,l^n) \le y_n(N,l) < \frac{(n+1)}{2} l_n$. By Identical length upper bound, for all $i < n$, we have $y_i(N,l^n) \le \frac{(n+1)}{2} l_n$.

Summing up, we obtain

$$\begin{aligned} \sum_{i \in N} y_i(N,l^n) &< \frac{n(n+1)}{2} l_n \\ &= c(N,l^n) \end{aligned}$$

and thus we fail Efficiency, a contradiction. Thus, $y_n(N,l) = \frac{(n+1)}{2} l_n$.

Start with the problem $(\{n\}, (l_n))$. By the above, $y_n(\{n\}, (l_n)) = l_n$. Suppose that we have shown that we have a unique solution for $y(\{k, ..., n\}, (l_k, ..., l_n))$ for all $k = m, ..., n$, with $1 < m \le n$. We show that it implies that $y(\{m-1, ..., n\}, (l_{m-1}, ..., l_n))$ is also unique. Using $y_n(N,l) = \frac{(n+1)}{2} l_n$, we have that $y_n(\{m, ..., n\}, (l_m, ..., l_n)) = \frac{n-m+2}{2} l_n$ and $y_n(\{m-1, ..., n\}, (l_{m-1}, ..., l_n)) = \frac{n-m+3}{2} l_n$.

By First agent proportional effect, for any $m \leq i < n$,

$$\begin{aligned}
\frac{\begin{array}{c} y_i(\{m-1,...,n\},(l_{m-1},...,l_n)) \\ -y_i(\{m,...,n\},(l_m,...,l_n)) \end{array}}{l_i} &= \\
& \frac{\begin{array}{c} y_n(\{m-1,...,n\},(l_{m-1},...,l_n)) \\ -y_n(\{m,...,n\},(l_m,...,l_n)) \end{array}}{l_n} \\
&= \frac{n-m+3}{2} - \frac{n-m+2}{2} = \frac{1}{2}
\end{aligned}$$

Thus,

$$y_i(\{m-1,...,n\},(l_{m-1},...,l_n)) = y_i(\{m,...,n\},(l_m,...,l_n)) + \frac{l_i}{2}.$$

By efficiency, $y_{m-1}(\{m-1,...,n\},(l_{m-1},...,l_n)) = c(\{m-1,...,n\},(l_{m-1},...,l_n)) - \sum_{j=m}^{n} y_j(\{m-1,...,n\},(l_{m-1},...,l_n))$ is uniquely defined. Thus, $y(\{m-1,...,n\},(l_{m-1},...,l_n))$ is uniquely defined.

To complete the proof, we show that $y^a$ satisfies all properties. Recall that

$$y_i^a(N,l) = \left(n - \frac{i}{2} + \frac{1}{2}\right) l_i - \sum_{j>i} \frac{l_j}{2}.$$

It is easy to see that it satisfies Efficiency and Length antimonotonicity.

We show that it satisfies the Identical length upper bound. We need to show that

$$\left(n - \frac{i}{2} + \frac{1}{2}\right) l_i - \sum_{j>i} \frac{l_j}{2} \leq \frac{(n+1)}{2} l_i$$

or

$$(n-i)\, l_i \leq \sum_{j=i+1}^{n} l_j$$

which is verified since the sum contains $n-i$ terms, all at least as large as $l_i$.

Finally, we show that it satisfies the First agent proportional effect. Take $2 \leq i \leq n$. We have that

$$\begin{aligned}
y_i^a(N,l) - y_i^a(N \backslash i, l^{-1}) &= \left(\left(n - \frac{i}{2} + \frac{1}{2}\right) l_i - \sum_{j>i} \frac{l_j}{2}\right) \\
&\quad - \left(\left(n - 1 - \frac{i-1}{2} + \frac{1}{2}\right) l_i - \sum_{j>i} \frac{l_j}{2}\right) \\
&= \left(n - \frac{i}{2} + \frac{1}{2} - n + 1 + \frac{i-1}{2} - \frac{1}{2}\right) l_i \\
&= \frac{1}{2} l_i
\end{aligned}$$

Thus, for any $2 \leq i, j \leq n$,

$$\frac{y_i^a(N,l) - y_i^a(N \backslash i, l^{-1})}{l_i} = \frac{y_j^a(N,l) - y_j^a(N \backslash i, l^{-1})}{l_j} = \frac{1}{2}.$$

■

# 4 Discussions

We discuss two extensions of our work.

## 4.1 Multiple servers

How does the accountable Shapley value behave if we have more than one server? While we do not show the formal model, we can extend most of our results to that framework. As discussed in Banerjee and Trudeau (2025), we then need to distinguish between cases in which the jobs are either divisible (can be split into multiple jobs of unit length, processed simultaneously on multiple servers) or indivisible (must be processed from start to finish on a single server).

Banerjee and Trudeau (2025) show that the optimistic Shapley value coincides with the serial method if jobs are indivisible, but the coincidence breaks down for divisible jobs. For the accountable Shapley value and the reverse serial rule, we can show parallel results: the coincidence carries over for indivisible jobs, but not for divisible ones.

Suppose that $N = \{1,2,3\}$, $l = (1,2,3)$ and that we have two servers. To see how the divisibility affects the cost, if jobs are not divisible, the best we can is to start serving agents 1 and 2 in the first period, and agent 3 is being served after the job of agent 1 has completed, at the beginning of the second period. Thus, agent 1 is served after 1 period, agent 2 is served after 2 periods, and agent 3 after 4 periods, for a cost of 7.

If the jobs are divisible[2], we still start by serving agents 1 and 2. But the job of agent 3 can now be seen as 3 jobs of length 1. We complete one of them in period 2 (on the server used by agent 1), and the 2 servers are used to process the 2 remaining jobs in period 3. Thus agent 3 is now served after 3 periods (with agents 2 still served after 2 periods, agent 1 after 1) for a total cost of 6. We obtain the following values for the accountable function in the indivisible and divisible cases

| $S$ | $c^a_{IND}(S)$ | $c^a_{DIV}(S)$ |
|---|---|---|
| $\{1\}$ | 2 | 2 |
| $\{2\}$ | 4 | 3 |
| $\{3\}$ | 6 | 4 |
| $\{1,2\}$ | 4 | 4 |
| $\{1,3\}$ | 5 | 5 |
| $\{2,3\}$ | 7 | 6 |
| $\{1,2,3\}$ | 7 | 6 |

[2]We suppose that jobs cannot be divided in smaller pieces than the integer.

The accountable Shapley value gives $(0.5, 2.5, 4)$ in the indivisible case and $(1, 2, 3)$ in the divisible case. We compare with the reverse serial method. If lengths were $(3, 3, 3)$, in the indivisible case, the cost would be $3 + 3 + 6 = 12$, so we assign 4 to all agents. In the divisible case, the cost is $2 + 3 + 5 = 10$ (as we process a job as soon as possible, processing on both servers in the same period if possible), assigning $\frac{10}{3}$ to all agents. When we move to $(2, 2, 3)$, in the indivisible case, the cost is $2 + 2 + 5 = 9$, so the saving of 3 is shared by agents 1 and 2, giving a final share of $4 - 1.5 = 2.5$ to agent 2. In the divisible case, the total cost changes to $1 + 2 + 4 = 7$, also for a saving of 3 shares by agents 1 and 2. The final share of agent 2 is thus $\frac{10}{3} - \frac{3}{2} = \frac{11}{6}$. We know the cost for $(1, 2, 3)$, yielding the allocation $(0.5, 2.5, 4)$ for the indivisible case and $\left(\frac{5}{6}, \frac{11}{6}, \frac{10}{3}\right)$ for the divisible case. We see that this corresponds to the accountable Shapley value in the indivisible case, but not in the divisible case.

## 4.2 Queueing problems and general sequencing problems

It is worth observing that the accountable Shapley value suffers from the same type of problem in the queueing problems as the pessimistic value function does in scheduling problems: it exhibits order reversal. In queueing problems, in which agents are differentiated by their unit-waiting cost, if agent $i$ has a lower unit waiting cost than $j$, agent $i$ will pay at least as much (see below).

When moving to general sequencing problems, the accountable function and its Shapley value are well-defined, as are the optimistic and pessimistic functions/Shapley values.

However, since general sequencing include as special cases queueing and scheduling problems, it is obvious that pessimistic and accountable functions/Shapley values will not always behave well. By contrast, the optimistic function/Shapley value, which behaves well in both problems, is expected to have nice properties in sequencing problems.

Suppose that $N = \{1, 2, 3\}$, $l = (1, 1, 2)$ and $w = (6, 3, 3)$. We can verify that it is optimal to serve agents in the order $1, 2, 3$. In addition, notice that the jobs of agents 1 and 2 have the same length, but that agent 1 has a higher waiting cost; thus agent 1 should pay at least as much as agent 2 (Condition 1). Notice also that agents 2 and 3 have the same waiting costs, but agent 3 has a longer job than agent 2; thus agent 3 should pay at least as much as agent 2 (Condition 2).

We obtain the following value functions:

| $S$ | $c^o(S)$ | $c^p(S)$ | $c^a(S)$ |
|---|---|---|---|
| $\{1\}$ | 6 | 24 | 12 |
| $\{2\}$ | 3 | 12 | 12 |
| $\{3\}$ | 6 | 12 | 24 |
| $\{1,2\}$ | 12 | 30 | 18 |
| $\{1,3\}$ | 15 | 24 | 24 |
| $\{2,3\}$ | 12 | 18 | 30 |
| $\{1,2,3\}$ | 30 | 30 | 30 |

We obtain $Sh^o = (11, 8, 11)$, $Sh^p = (17, 8, 5)$ and $Sh^a = (5, 8, 17)$. See that $Sh^a$ fails Condition 1 while $Sh^p$ fails Condition 2.

An open question is if there are ways to take into account both pessimistic and accountable costs in a way that prevents order reversal.

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