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Abstract: This paper considers the river sharing problem first studied in Ambec and Sprumont (2002). We use the Equal Share Proportional Solution (ESPS) for the permit sharing problem introduced in Suh and Wang (2023) to define a solution, also called the ESPS, for the river sharing problem. We first show that a river sharing problem can be divided into a list of subproblems, each of which can be considered as a permit sharing problem (Decomposition Lemma). Then, we apply the ESPS solution to each of the subproblems. The ESPS for the river sharing problem is the aggregation of the ESPS for all the subproblems. We also compare the ESPS with the well-known Downstream Incremental Distribution solution by Ambec and Sprumont (2002). We show that for a *dummy* agent whose optimal consumption coincides with his initial endowment, the agent obtains his stand-alone benefit in the ESPS. In contrast, the Downstream Incremental Distribution solution may assign welfare levels to dummy agents that are higher than their stand-alone benefits. On the other hand, the ESPS violates the aspiration upper bounds.

Keywords: River, efficiency, welfare distribution, fair allocation.

1 Introduction

There has been a growing body of literature on the river sharing problem inspired by Ambec and Sprumont (2002). The Downstream Incremental Distribution (DID) solution proposed in the latter has played an important role in this literature (see Beal et al. (2013) for a survey). Recently, in Gudmundsson et al. (2019) it is shown that there is no implementable mechanism such that for a dummy agent his equilibrium payoff is his stand-alone benefit. An agent is a dummy agent if at the efficient allocation of water, his optimal consumption is his initial endowment. This result implies that the Downstream Incremental Distribution solution would usually assign a welfare level to a dummy agent that is higher than his stand-alone level of welfare if the dummy agent is not the first agent along the river.

In this paper. we consider the river sharing problem from a different perspective. We first observe that there exists a connection between the river sharing problem and the permit sharing problem studied in Suh and Wang (2023).[1] We show that a river sharing problem can be divided into a list of subproblems, each of which is equivalent to a permit sharing problem individually. Based on this observation, we make use of the Equal Share Proportional Solution (ESPS) for the permit sharing problem introduced in Suh and Wang (2023) to define a solution, also called the ESPS, for the river sharing problem. Specifically, the ESPS for the river sharing problem is an aggregation of all the individual ESPS of the subproblems. By imposing a consistency axiom called the Aggregation Principle, we show that the ESPS is well-defined for the river sharing problem.

Then, we compare the ESPS with the Downstream Incremental Distribution solution. We show that for a dummy agent whose optimal consumption coincides with his initial endowment the agent obtains his stand-alone benefit in the ESPS. An agent's stand-alone benfit is defined by the benfit he obtains when he consumes his own endowment and receives no monetary transfer. In contrast, the Downstream Incremental Distribution solution often assigns to a dummy agent a welfare level that is different from the agent's stand-alone

[1]For a related earlier study on the pollution permits allocation problem, see Montgomery (1972).

benfit.[2]

Specifically, we take a two-step approach. In Section 2, we first establish the Decomposition Lemma (Lemma 1). It states that a river sharing problem can be uniquely divided into a list of subproblems. These subproblems arise from a partition of the set of agents, in which each subproblem is associated with a consecutive coalition. Based on this partition, each subproblem turns out to be simply a permit sharing problem. Essentially, the unidirectional constraints on the water allocation within the coalition become irrelevant as long as the total amount of water owned by the agents in the coalition is fully allocated among the agents in the coalition. The Decomposition Lemma is simply a corollary of the characterization of the optimal allocation of water for the river sharing problem (see Kilgour and Dinar (2001), Ambec and Sprumont (2002)).

The decomposition of the river sharing problem into a list of subproblems each of which can be treated as a permit sharing problem (Suh and Wang, 2023) opens to us more choice of solutions for the river sharing problem. In this paper, we consider the ESPS for each subproblem in a river sharing problem. As we pointed out earlier, the ESPS for the subproblems are independent of the unidirectional nature of the river flows in the river problem since each subproblem can be treated as a permit sharing problem individually with no restrictions on how agents reallocating their water between themselves. In Section 3, we define the ESPS for the river sharing problem as the aggregation of the ESPS for all its subproblems.

In Section 4, we introduce a new axiom along with the existing axioms for the permit sharing problem (Suh and Wang (2023)) to characterize the ESPS for the river sharing problem. This axiom, called the Aggregation Principle, requires a consistent application of the ESPS to each of the subproblems in a river sharing problem. With the Aggregation Principle, the characterization of the ESPS for the river sharing problem is straightforward and essentially identical to the characterization of the ESPS for the permit sharing problem in Suh and Wang (2023).

In Section 5, we compare the ESPS with the Downstream Incremental Distribution solution. On the one hand, the ESPS violates the aspiration upper bounds (Ambec and Sprumont (2002)). On the other hand, it satisfies the stand-alone benefit for dummy agent property. In contrast, the Downstream Incremental Distribution satisfies the aspiration upper bounds by definition. But, it violates the stand-alone benefit for dummy agent property. Between the aspiration upper bounds and the stand-alone benefit for dummy agent property, we argue for the latter for the river sharing problem. We consider the aspiration upper bounds property is too restrictive for upstream agents, particularly for the first agent along the river. Imagine a two-agent river sharing problem. Suppose that in the optimal allocation agent 1 (the upstream agent) needs to transfer certain amount of water to agent 2. According to the Downstream Incremental Distribution solution, in return for the transfer of water agent 1 receives a monetary payment from agent 2 in an amount that just makes him indifferent between either consuming his endowment or transferring some water to agent 2 and receiving a payment from the latter. In this solution, agent 1 does not share any portion of the total increased benefit because of the aspiration upper

[2] We assume that the first agent is not a dummy agent. Indeed, Gudmundsson et al. (2019) showed that no implementable allocation mechanisms including the Downstream Incremental Distribution would assign all dummy agents their stand-alone benefits.

bounds restriction. In contrast, the ESPS allows agent 1 to share a part of the increase in the total benefit from the optimal allocation of water.

We conclude this introduction by pointing out one possible direction for future research. After a connection between the river sharing problem and the permit sharing problem is established, it would be worthwhile to consider other solution concepts beyond the ESPS for the permit sharing problem that might be useful for the river sharing problem.

2 Model

The following is a diagrammatic representation of the river sharing problem.

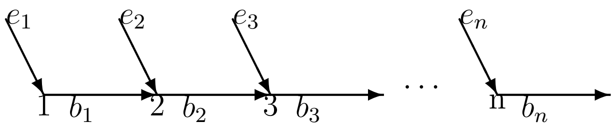

Fig. 1. $e_i(i = 1, ..., n)$ is agent i's endowment of water, $b_i(i = 1, ..., n)$ is agent i's benefit function.

Following Ambec and Sprumont (2002), the problem is described as follows. A river flows through a finite number of agents, $N = \{1, 2, ..., n\}$, from upstream to downstream. Here, $i < j$ indicates that i is upstream from j. Assume that each agent i, $i = 1, ..., n$, initially picks up $e_i(\geq 0)$ volume of water along the river. Agents consume water from their parts of the river basin and may trade water with their downstream neighbors and in return may receive a payment. For each agent, money is available to make payments. Agents have valuations on water consumption. Specifically, each agent i has a quasilinear utility function, which is linear in money but nonlinear in the amount of water. Assume further that agents have utility functions represented by

$$u_i(x_i, t_i) = b_i(x_i) + t_i, \ i \in N \tag{1}$$

where $b_i : R_+ \to R_+$ is agent i's benefit function, x_i is the amount of water consumed by agent i, and t_i is the monetary transfer to agent i. The function b_i is assumed to be strictly increasing, concave, and continuously differentiable at every x_i and its derivative $b_i'(x_i)$ goes to infinity as x_i goes to zero.

The following notations are useful. A coalition S is a subset of N, i.e., $S \subseteq N$. Given two coalitions S and T, we write $S < T$ if $i < j$ for all $i \in S$ and $j \in T$. Call a coalition T consecutive if $k \in T$ whenever $i, j \in T$ and $i < k < j$. Every coalition S has a unique coarsest partition $\mathcal{P}$ into consecutive components. Denote the set of predecessors of agent i by $P_i = \{j \in N | j \leq i\}$.

For any given initial endowments $e = (e_1, ..., e_n)$, an allocation

$$(x, t) = (x_1, ..., x_n, t_1, ..., t_n) \in R_+^N \times R^N$$

is feasible if it satisfies the following feasibility constraints

$$\sum_{i=1}^{n} t_i \leq 0, \tag{2}$$

$$\sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} e_i \text{ for all } j = 1, ..., n-1, \text{ and} \tag{3}$$

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i. \tag{4}$$

An allocation (x^*, t^*) is *efficient* if and only if, x^* maximizes the total benefits,

$$\begin{cases} B(N) = \max \sum_{i=1}^{n} b(x_i) \\ \text{subject to feasibility constraints (3) and (4).} \end{cases} \tag{5}$$

and

$$\sum_{i=1}^{n} t_i^* = 0, \tag{6}$$

and we call x^* an efficient allocation of water.

A welfare distribution is a vector $z = (z_1, ..., z_n) \in R^N$ which is the utility image of some allocation (x, t) in the sense that $z_i = b_i(x_i) + t_i, i \in N$.

Our main question is: How should we distribute the total optimal benefits $B(N)$?

Ambec and Sprumont (2002) take a game-theoretical approach that is based on the two well-known conflicting doctrines advocated in international disputes on the use of river water. The theory of *absolute territorial sovereignty* (ATS) says that a country has full property rights over the water originating on its territory. The theory of *unlimited territorial integrity* (UTI) says that a country has property rights to all the water originating its upstream. Treating the river sharing problem as a cooperative game, the ATS defines a core lower bound and the UTI defines an aspiration upper bound for any coalition of agents. They show that only one welfare allocation that satisfies the core lower bounds and the aspiration upper bounds: it is the marginal contribution vector corresponding to the ordering of the agents along the river. This allocation is called the Downstream Incremental Distribution solution mentioned earlier.

As already pointed out by the authors, the Downstream Incremental Distribution solution is special to their model. In particular, the particular nature of the feasibility constraints that water can only be transferred downstream essentially determines that the Downstream Incremental Distribution solution is the only solution that satisfies the core lower bounds and the aspiration upper bounds.

In order to find alternative solutions, first we turn to the feasibility constraints on the efficient allocation. Note that in the efficient allocation of water, many feasibility constraints are in fact not binding. The only binding constraints are on certain coalitions of agents. Also, these coalitions are consecutively ordered along the river and form a partition of all agents. This observation allows us to treat each such coalition as a separate problem. From this perspective, the river sharing problem is technically equivalent to the permit sharing problem. Therefore, we can treat a river sharing problem as a permit sharing problem. In this paper, we focus on the Equal Share Proportional solution first defined in

Suh and Wang (2023) for the permit sharing problem and define the corresponding solution for the river sharing problem.

Specifically, we take a two-step approach to the problem of how to share the total optimal benefits $B(N)$. We first show that the problem can be decomposed into a list of subproblems. These subproblems are generated from a partition of N in which each subproblem is associated with a consecutive coalition. Each subproblem turns out to be simpler than the original problem in the sense that the constraints on water allocation within the coalition are irrelevant (not binding) except for the whole coalition. Based on this observation, then we can apply solution concepts for resource allocation problems that are free from the restrictions imposed by the linear unidirectional structure of the river sharing problem.

We begin with the following lemma.

Lemma 1 *(Decomposition Lemma) The problem (5) can be decomposed into a list of subproblems* $B(N_k), k = 1, ..., K$ *such that*

$$B(N) = \sum_{k=1}^{K} B(N_k), \tag{7}$$

where

$$\begin{aligned} B(N_k) &= \max \sum_{i \in N_k} b_i(x_i) \\ \sum_{i \in N_k} x_i &= \sum_{i \in N_k} e_i, \ k = 1, ..., K. \end{aligned}$$

and $\{N_k\}_{k=1,...,K}$ *is a partition of* N*, each* $N_k, k = 1, ..., K$ *is a consecutive coalition, and* $N_k < N_{k+1}, k = 1, ..., K - 1$.

Proof. Consider the problem

$$\begin{aligned} B(N) &= \max \sum_{i=1}^{n} b(x_i) \\ \sum_{i=1}^{j} x_i &\leq \sum_{i=1}^{j} e_i, \ j = 1, ..., n - 1, \\ \sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} e_i. \end{aligned}$$

If x^* is an efficient allocation, then there exists a partition $\{N_k\}_{k=1,...,K}$ of N and a list $\{\beta_k\}_{k=1,...,K}$ of numbers such that $N_k < N_{k'}$ and $\beta_k > \beta_{k'}$ whenever $k < k'$,

$$b'_i(x^*_i) = \beta_k \text{ for every } i \in N_k \text{ and every } k = 1, ..., K, \text{ and}$$

$$\sum_{i \in N_k} (x^*_i - e_i) = 0 \text{ for every } k = 1, ..., K. \tag{8}$$

For each $N_k, k = 1, ..., K$, define the subproblem:

$$\begin{aligned} B(N_k) &= \max \sum_{k \in N_k} b_i(x_i) \\ \sum_{i \in N_k} x_i &= \sum_{i \in N_k} e_i. \end{aligned}$$

Then,

$$B(N) = \sum_{k=1}^{K} B(N_k).$$

This completes the proof of the lemma. □

Based on Lemma 1, our two-step approach to the problem of distributing the optimal total benefits $B(N)$ in (5) can be described as follows. First, we distribute the optimal benefits $B(N_k)$ among the agents in N_k for each $k = 1, ..., K$. Then we combine these distributions to obtain the distribution of $B(N)$ for all the agents in N.

To make the distributions across $N_k, k = 1, ..., K$ consistent so that we apply the same solution for all (sub)problems, we need to impose a consistency requirement on the solution with regard to all the possible subproblems that might be generated in a problem. This is given by the Aggregation Principle in Section 4. Now we turn to our solution for the subproblems in the next section.

3 The Equal Share Proportional Solution

In Suh and Wang (2023), a solution called the Equal Share Proportional Solution (ESPS) was proposed to solve a *permit sharing problem*. A permit sharing problem is described by a list of countries, each of which owns a certain amount of emission permits and has a unique technology that requires permits to produce output. The question is how to share the optimal global surplus generated beyond the autarky economy output. To answer the question, the ESPS is defined based on a number of axioms that are pertinent to the permit sharing problem.

On the surface, the river sharing problem and the permit sharing problem are different. Despite the differences between their feasibility constraints on allocation, they involve essentially the same question about how to share the joint optimal benefits that are generated from reallocating among the agents their given resources from less productive agents to more productive ones to increase the total benefits or output. Therefore, solutions for the permit sharing problem can be useful to construct solutions for the river sharing problem. What we attempt to do in this paper is to show that the Equal Share Proportional Solution for the permit sharing problem can be used to define a solution for the river sharing problem.

To define a solution for the river sharing problem using the ESPS, we need the following notations.

Given a problem (N, e, b). By Lemma 1, we only need to restrict ourselves to the consecutive coalitions $S \subseteq N$ where the following two problems are equivalent:

$$\begin{aligned} B(S) &= \max \sum_{i \in S} b_i(x_i) \\ \sum_{i \in S \cap P_j} x_i &\leq \sum_{i \in S \cap P_j} e_i, j \in S, &(9) \\ \sum_{i \in S} x_i &= \sum_{i \in S} e_i &(10) \end{aligned}$$

and

$$\begin{aligned} B(S) &= \max \sum_{i \in S} b_i(x_i) \\ \sum_{i \in S} x_i &= \sum_{i \in S} e_i. \end{aligned}$$

In other words, in the first problem only constraint (10) is relevant. Constraints in (9) are redundant and can be removed.

To define the ESPS for problem (N, e, b), by Lemma 1 we can apply the ESPS to each of its subproblems and the aggregation of these solutions will define a solution for (N, e, b). For simplicity, in the following we will use the same notation (N, e, b) for a subproblem instead of (S, e_S, b_S).

Let the optimal water allocation be x^* for (N, e, b). Denote $S^+ = \{i \in N | x_i^* \geq e_i\}$ and $S^- = \{i \in N | x_i^* < e_i\}$. Define

$$\Delta z_i = b_i(x_i^*) - b_i(e_i), i = 1, ..., n, \tag{11}$$

$$\Delta x_i = x_i^* - e_i. \tag{12}$$

The total positive surplus of benefits is obtained by aggregating the positive individual surpluses, i.e,

$$\Delta O^+ \equiv \sum_{i \in S^+} \Delta z_i. \tag{13}$$

To generate the positive surplus, the agents in S^- reduce their consumptions of water below their stand-alone levels and the unused portion of water are supplied to the agents in S^+ and in doing so, generating negative surpluses of benefits for themselves. Thus, we call agent $i(\in S^+)$ a positive benefit contributor but at the same time a negative water contributor, and agent $i(\in S^-)$ a negative benefit contributor but at the same time a positive water contributor. (Obviously, agents in S^- are upstream from agents in S^+, i.e., $S^- < S^+$.)

The total negative surplus of benefits is obtained by aggregating the negative individual surpluses, i.e,

$$\Delta O^- \equiv \sum_{i \in S^-} \Delta z_i. \tag{14}$$

And the total surplus $\Delta O \equiv \sum_{i\in N}(b_i(x_i^*) - b_i(e_i))$ is obtained by combining the total positive and negative surpluses, i.e., $\Delta O = \Delta O^+ + \Delta O^-$.

The following diagraphs show a two-agent example in which agent i is both a positive water contributor and a negative benefit contributor, and the downstream agent j is both a negative water contributor and a positive benefit contributor in the optimal allocation.

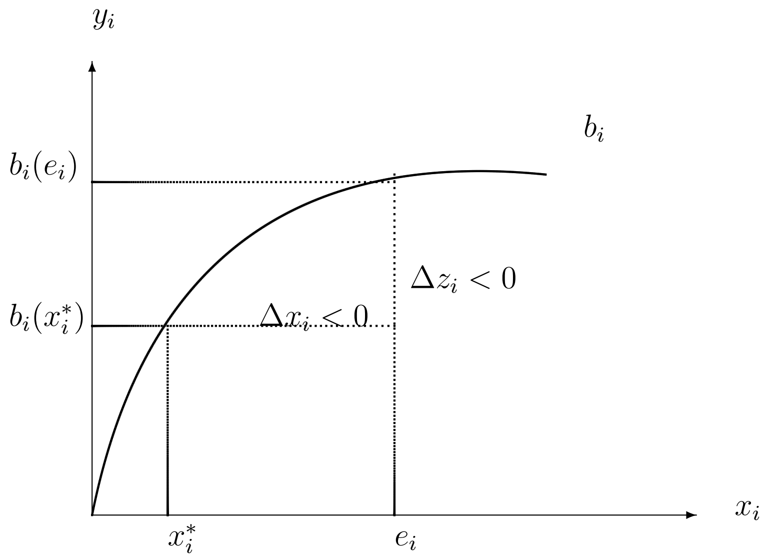

Fig. 2. Agent $i \in S^-$

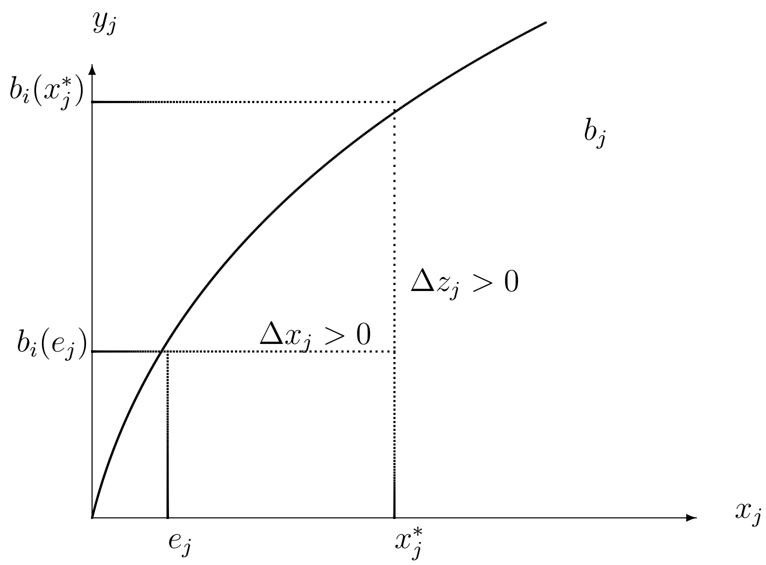

Fig. 3. Agent $j \in S^+$

In this case, the total increase in welfare is $\Delta z_j + \Delta z_i$. Agent i shares half of the increase Δz_j and half the decrease Δz_i in addition to his stand-alone welfare $b_i(e_i)$. Similarly, agent j shares half of the increase Δz_j and half the decrease Δz_i in addition to his stand-alone welfare $b_j(e_j)$. The Equal Share Proportional Solution is a generalization of the above two-agent solution to any finite number of agents.

Now we introduce the Equal Share Proportional Solution for a (sub)problem (N, e, b) in which

$$B(N) = \max \sum_{i=1}^{n} b_i(x_i), \text{ s.t. } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} e_i.$$

Definition 1 *The Equal Share Proportional Solution φ is defined as follows. For each (N, e, b), let x^* be the optimal allocation of water for (N, e, b). Denote $S^+ = \{i \in N | x_i^* \geq e_i\}$, $S^- = \{i \in N | x_i^* < e_i\}$. If $x^* \neq e$, then for each $i \in S^+$, define*

$$\varphi_i(N, e, b) = b_i(e_i) + \frac{1}{2}(b_i(x_i^*) - b_i(e_i)) + \frac{1}{2} \sum_{j \in S^-} (b_j(x_j^*) - b_j(e_j)) \frac{x_i^* - e_i}{\sum_{l \in S^+}(x_l^* - e_l)}, \tag{15}$$

and for each $i \in S^-$, define

$$\varphi_i(N, e, b) = b_i(e_i) + \frac{1}{2}(b_i(x_i^*) - b_i(e_i)) + \frac{1}{2} \sum_{j \in S^+} (b_j(x_j^*) - b_j(e_j)) \frac{x_i^* - e_i}{\sum_{l \in S^-}(x_l^* - e_l)}. \tag{16}$$

If $x^* = e$, then define

$$\varphi_i(N, e, b) = b_i(e_i), i = 1, ..., n. \tag{17}$$

Now we are ready to define the Equal Share Proportional Solution for the river sharing problem (N, e, b).

Definition 2 *The Equal Share Proportional Solution (ESPS) φ is defined as follows. Let φ be the Equal Sharing Proportional Solution defined in Definition 1. For a given problem (N, e, b), let the collection $\{(N_k, e_{N_k}, b_{N_k})\}_{k=1}^{K}$ be the (unique) decomposition of (N, e, b) in Lemma 1, define the Equal Share Proportional Solution $\varphi(N, e, b)$ as follows:*

$$\varphi_i(N, e, b) = \varphi_i(N_k, e_{N_k}, b_{N_k}), \text{ for } i \in N_k, k = 1, ..., K. \tag{18}$$

4 Characterization of the ESPS

In this section, we provide a characterization of the ESPS. The characterization is essentially identical to the characterization of the ESPS for the permit sharing problem in Suh and Wang (2023).

Formally, let $N = \{1, ..., n\}$ be the set of all agents on a linear river. Let $\mathcal{P}$ be the set of all non-empty consecutive subsets of N. Each $S = \{i_1, ..., i_s\} \in \mathcal{P}$ denotes a consecutive coalition of agents located along a river. For a given $S \in \mathcal{P}$, we denote the set of problems on S by $\{(S, e_S, b_S)\}$

A *solution* φ is a mapping that assigns for each $S \in \mathcal{P}$ and each (S, e_S, b_S) an allocation,

$$\varphi(S, e_S, b_S) = (\varphi^I(S, e_S, b_S), \varphi^O(S, e_S, b_S))$$

$$= ((\varphi^I_{i_1}(S, e_S, b_S), ..., \varphi^I_{i_s}(S, e_S, b_S)), (\varphi^O_{i_1}(S, e_S, b_S), ..., \varphi^O_{i_s}(S, e_S, b_S)))$$

such that

$$\sum_{j=1}^{l} \varphi^I_{i_j}(S, e_S, b_S) \leq \sum_{j=1}^{l} e_{i_j}, l = 1, ..., s-1, \tag{19}$$

$$\sum_{j=1}^{s} \varphi^I_{i_j}(S, e_S, b_S) = \sum_{j=1}^{s} e_{i_j}, \tag{20}$$

and

$$\sum_{j=1}^{s} \varphi^O_{i_j}(S, e_S, b_S) = \sum_{j=1}^{s} b_{i_j}(\varphi^I_{i_j}(S, e_S, b_S)), \tag{21}$$

where $\varphi^I_{i_j}(S, e_S, b_S)$ is the water allocation and $\varphi^O_{i_j}(S, e_S, b_S)$ is the welfare allocation of agent $i_j \in S$.

For notational simplicity, denote problem (N, e_N, b_N) as (N, e, b). Now we introduce the axioms that we will use in the characterization of the ESPS.

Optimality. For each (N, e, b),

$$\sum_{i=1}^{j} \varphi^I_i(N, e, b) \leq \sum_{i=1}^{j} e_i, j = 1, ..., n-1, \tag{22}$$

$$\sum_{i=1}^{n} \varphi^I_i(N, e, b) = \sum_{i=1}^{n} e_i, \tag{23}$$

and

$$\sum_{i \in N} b_i(\varphi^I_i(N, e, b)) \geq \sum_{i \in N} b_i(x'_i), \tag{24}$$

$$\forall x' \in I\!R^n_+ \text{ s.t. } \sum_{i=1}^{j} x'_i \leq \sum_{i=1}^{j} e_i, j = 1, ..., n-1,$$
$$\sum_{i=1}^{n} x'_i = \sum_{i=1}^{n} e_i.$$

Since we will focus on the distribution of the optimal total benefits, in the rest of paper we use $\varphi(S, e_S, b_S)$ instead of $\varphi^O(S, e_S, b_S)$.

Voluntary Participation (VP). For each (N, e, b), each $i \in N$,

$$\varphi_i(N, e, b) \geq b_i(e_i). \tag{25}$$

For the purpose of this paper, we now introduce a key axiom called the Aggregation Principle that will play an important role in the characterization. This axiom is closely related to the Decomposition Lemma (**Lemma 1**). To better understand this principle, it is helpful to recall and compare it with a similar type of axiom, usually called *Consistency*, in the cost (or surplus) sharing literature (Moulin (2002), Thomson (2003)). In this literature, Consistency axiom requires that reallocation of costs or surplus among a subcoalition of agents should result in cost (surplus) shares identical to those resulting from the original allocation problem. In other words, for agents of *any* coalition, cost (surplus) shares should be the same regardless of whether we apply the allocation rule on the original problem or on a *reduced problem* where agents outside the coalition have taken their assigned shares by the original allocation and left.

While both are related to subproblems, the Aggregation Principle differs from the Consistency axiom substantially. In the Aggregation Principle, the subcoalitions are *not* arbitrary coalitions but are associated with separate subproblems, each of which itself is induced by the original problem according to the Decomposition Lemma. The Aggregation Principle requires that the same allocation rule applies to each of these subproblems. Unlike Consistency, these subproblems in the Aggregation Principle *do not* depend on the allocation rule. Simply speaking, the Aggregation Principle states that the allocation for the problem is an aggregation of the allocations from all the subproblems that the problem is decomposed into (**Lemma 1**).

Now we are ready to state the principle.

Aggregation Principle (AP). If problem (N, e, b) is decomposed into a list of subproblems $B(N_k), k = 1, ..., n$, such that

$$B(N) = \sum_{k=1}^{K} B(N_k). \tag{26}$$

then

$$\varphi(N, e, b) = (\varphi(N_1, e_{N_1}, b_{N_1}), ..., \varphi(N_K, e_{N_K}, b_{N_K})). \tag{27}$$

Separation Principle of Parameter α on Subproblems (SP).[3]

(1) The total positive and negative surpluses that are defined in (13) and (14) respectively are divided between the benefit contributors and water contributors with weight $\alpha \in [0, 1]$:

$$\Delta O^+ = \Delta B^+ + \Delta W^+, \Delta O^- = \Delta B^- + \Delta W^-,$$
$$\Delta B^+ = \alpha \Delta O^+, \Delta B^- = \alpha \Delta O^-.$$

(2) There are two lists of share functions $y_i : I\!R^{n+1} \to I\!R, y_i \geq 0$ if $i \in S^+$ and $y_i \leq 0$ if $i \in S^-$, and $w_i : I\!R^{n+1} \to I\!R, w_i \leq 0$ if $i \in S^+$ and $w_i \geq 0$ if $i \in S^-$, that are defined on

[3]The idea of the Separation Principle may be explained simply as follows.

For each subproblem (N_k, e_{N_k}, b_{N_k}), for notational simplicity we ignore the subscript k and use the notation (N, e, b).

(1) The total surplus is divided into the water input share and the benefit share in a fixed proportion α.

(2) For example, an agent's water input share, w_i, is assumed to be a certain function of the total water share and the water contributions. There is no restriction imposed on the function, w_i except that the sum of water input shares must be equal to ΔW. It is yet to be shown in Theorem 1 that the function w_i is a proportional solution.

(3) An agent's total share is the sum of its water input share and benefit share.

The precise condition becomes more complicated as expressed in the paper because we consider the negative and positive contributions of the water input and the benefit contributions.

the benefit contributions and water contributions beyond their stand-alone benefits. They satisfy the following budget balance conditions:

$$\sum_{j\in S^+} y_j(\Delta z;\Delta B^+) = \Delta B^+,$$

$$\sum_{j\in S^-} y_j(\Delta z;\Delta B^-) = \Delta B^-,$$

$$\sum_{j\in S^+} w_j(\Delta x;\Delta W^-) = \Delta W^-,$$

$$\sum_{j\in S^-} w_j(\Delta x;\Delta W^+) = \Delta W^+,$$

where $\Delta x, \Delta z$ are defined in (11) and (12).

(3) The benefit share for each $i \in N$ is given by the sum of the agent's stand-alone benefits and his shares of the total surplus given by y_i and w_i:
If $i \in S^+$, then

$$\varphi_i(N,e,b) = b_i(e_i) + y_i(\Delta z;\Delta B^+) + w_i(\Delta x;\Delta W^-), \tag{28}$$

if $i \in S^-$, then

$$\varphi_i(N,e,b) = b_i(e_i) + y_i(\Delta z;\Delta B^-) + w_i(\Delta x;\Delta W^+). \tag{29}$$

Water Reallocation-Proofness on Subproblems (WRPS).

(1) For each $\Delta x \in I\!R^n$ and each $\Delta W^- \in I\!R_-$, each $S \subseteq S^+$ and each $\Delta x' \in I\!R^n$, if $\sum_{i\in S}\Delta x_i = \sum_{i\in S}\Delta x'_i$, then

$$\sum_{i\in S} w_i(\Delta x;\Delta W^-) = \sum_{i\in S} w_i(\Delta x'_S, \Delta x_{N\setminus S};\Delta W^-).$$

(2) For each $\Delta x \in I\!R^n$ and each $\Delta W^+ \in I\!R_+$, each $S \subseteq S^-$ and each $\Delta x' \in I\!R^n$, if $\sum_{i\in S}\Delta x_i = \sum_{i\in S}\Delta x'_i$, then

$$\sum_{i\in S} w_i(\Delta x;\Delta W^+) = \sum_{i\in S} w_i(\Delta x'_S, \Delta x_{N\setminus S};\Delta W^+).$$

Note that if $\sum_{i\in S} p_i(\Delta x;\Delta W^+) \neq \sum_{i\in S} w_i(\Delta x';\Delta W^+)$, for example

$$\sum_{i\in S} w_i(\Delta x;\Delta W^+) < \sum_{i\in S} w_i(\Delta x';\Delta W^+),$$

then coalition S could increase its share by reallocating their water contributions from Δx to $\Delta x'$.

If a solution satisfies WRPS, then no group of agents in S^- or S^+ can strictly increase their combined surplus share received from their water contributions by redistributing their water contributions among themselves.

Surplus Reallocation-Proofness on Subproblems (SRPS).

(1) For each $\Delta z \in I\!R^n$ and each $\Delta B^+ \in I\!R_+$, each $S \subseteq S^+$ and each $\Delta\tilde{z} \in I\!R^n$, if $\sum_{i\in S} \Delta z_i = \sum_{i\in S} \Delta\tilde{z}_i$, then

$$\sum_{i\in S} y_i(\Delta z; \Delta B^+) = \sum_{i\in S} y_i(\Delta\tilde{z}_S, \Delta z_{N\setminus S}; \Delta B^+).$$

(2) For each $\Delta z \in I\!R^n$ and each $\Delta B^- \in I\!R_-$, each $S \subseteq S^-$ and each $\Delta\tilde{z} \in I\!R^n$, if $\sum_{i\in S} \Delta z_i = \sum_{i\in S} \Delta\tilde{z}_i$, then

$$\sum_{i\in S} y_i(\Delta z; \Delta B^-) = \sum_{i\in S} y_i(\Delta\tilde{z}_S, \Delta z_{N\setminus S}; \Delta B^-).$$

The SRPS condition says that the aggregate benefit share of the surplus for the agents in S^- or S^+ should remain the same regardless of the rearrangement of their benefit contributions. Note that the condition SRPS does not require a solution to be immune to manipulation via benefit functions that may occur when the information about the benefit functions is private.[4]

Null Consistency on Subproblems (NCS).[5] For each $\Delta x \in I\!R^n$ and each $\Delta z \in I\!R^n$, each $\Delta B \in I\!R$ and each $\Delta W \in I\!R$, if $\Delta z_i = 0$, then for each $j \in N \setminus \{i\}$,

$$y_j(\Delta z; \Delta B) = y_j(\Delta z_{N\setminus\{i\}}; \Delta B),$$

if $\Delta x_i = 0$, then for each $j \in N \setminus \{i\}$,

$$w_j(\Delta x; \Delta W) = w_j(\Delta x_{N\setminus\{i\}}; \Delta W).$$

The Null Consistency on Subproblem axiom says that if an agent's benefit (or water) contribution is zero, then the agent's presence does not affect the shares of other agents.

Definition 3 *The generalized proportional solution φ^α of parameter $\alpha \in [0,1]$ is defined as follows. For each (N,e,b), let x^* be the optimal water allocation for (N,e,b) (i.e., $\varphi^I(e,b) = x^*$). Denote $S^+ = \{i \in N | x_i^* \geq e_i\}$, $S^- = \{i \in N | x_i^* < e_i\}$. If $x^* \neq e$, then for each $i \in S^+$, define*

$$\varphi_i^\alpha(N,e,b) = b_i(e_i) + \alpha(b_i(x_i^*) - b_i(e_i)) + (1-\alpha)\sum_{j\in S^-}(b_j(x_j^*) - b_j(e_j))\frac{x_i^* - e_i}{\sum_{l\in S^+}(x_l^* - e_l)}, \quad (30)$$

and for each $i \in S^-$, define

$$\varphi_i^\alpha(N,e,b) = b_i(e_i) + \alpha(b_i(x_i^*) - b_i(e_i)) + (1-\alpha)\sum_{j\in S^+}(b_j(x_j^*) - b_j(e_j))\frac{x_i^* - e_i}{\sum_{l\in S^-}(x_l^* - e_l)}, \quad (31)$$

If $x^* = e$, then define

$$\varphi_i^\alpha(N,e,b) = b_i(e_i), i = 1, ..., n. \quad (32)$$

In particular, the Equal Share Proportional Solution is obtained by setting $\alpha = 1/2$, and is denoted as φ.

[4]For the work related to the issue of manipulation via benefit functions in this model, refer to Suh (2001) and Shin and Suh (2007).

[5]Null Consistency is similar to *null claims consistency* in claims problems. See Thomson (2003).

Now we state our first theorem.

Theorem 1 *There exists a unique solution satisfying Optimality, the Aggregation Principle, the Separation Principle of Parameter $\alpha \in [0,1]$ on Subproblems, Null Consistency on Subproblems, Water Reallocation-Proofness on Subproblems and Surplus Reallocation-Proofness on Subproblems. It is the generalized proportional solution of parameter α:*

$$\varphi^{\alpha}(N,e,b) = (\varphi^{\alpha}(N_1, e_{N_1}, b_{N_1}), ..., \varphi^{\alpha}(N_K, e_{N_K}, b_{N_K})). \tag{33}$$

Proof. The proof mimics the proof of **Theorem 1** in Suh and Wang (2023). We only need to show that if a solution φ satisfies the axioms stated above, we must have $\varphi = \varphi^{\alpha}$.

Given a problem (N,e,b), by Optimality and the Aggregation Principle, we only need to consider $\varphi(N_k, e_K, b_k), k = 1, ..., K$, where $\{(N_k, e_{N_k}, b_{N_k})\}_{k=1}^{K}$ is the unique decomposition of (N,e,b) by Lemma 1. But by the characterization theorem (Theorem 1) in Suh and Wang (2023),

$$\varphi(N_k, e_K, b_k) = \varphi^{\alpha}(N_k, e_{N_k}, b_{N_k}), k = 1, ..., K,$$

This proves that φ is the generalized proportional solution of parameter α for problem (N,e,b).

This completes the proof of the theorem. □

Corollary 1 *In the family of the generalized proportional solution of parameter $\alpha \in [0,1]$, only the Equal Share Proportional Solution φ $(\alpha = 1/2)$ satisfies Voluntary Participation.*

The proof of the corollary can be found in Suh and Wang (2023).

Consider the following example.

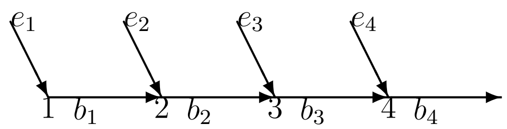

Fig. 4. $e_i (i = 1, ..., 4)$ is agent i's endowment of water, $b_i (i = 1, ..., 4)$ is agent i's benefit function.

Example 1. Let $b_1(x_1) = x_1^{2/3}$, $b_2(x_2) = x_2^{2/3}$, $b_3(x_3) = x_3^{1/3}$, $b_4(x_4) = x_3^{1/3}$. Let $e_1 = 8, e_2 = 4, e_3 = 10, e_4 = 6$.

$$\begin{aligned} B(N) \quad = \quad & \max x_1^{2/3} + x_2^{2/3} + x_3^{1/3} + x_4^{1/3} \\ & x_1 \leq 8, \\ & x_1 + x_2 \leq 12, \\ & x_1 + x_2 + x_3 \leq 22, \\ & x_1 + x_2 + x_3 + x_4 = 28. \end{aligned}$$

The problem can be decomposed into two subproblems:

$$\begin{aligned} B(\{1,2\}) &= \max x_1^{2/3} + x_2^{2/3} \\ & \quad\; x_1 + x_2 = 12, \end{aligned}$$

thus

$$B(\{1,2\}) = 2 \times (6)^{2/3}.$$

And

$$\begin{aligned} B(\{3,4\}) &= \max x_3^{1/3} + x_4^{1/3} \\ & \quad\; x_3 + x_4 = 16, \end{aligned}$$

thus

$$B(\{3,4\}) = 4.$$

That is,

$$B(\{1,2,3,4\}) = B(\{1,2\}) + B(\{3,4\}).$$

Denote $N_1 = \{1,2\}, N_2 = \{3,4\}$. By the ESPS, we have

$$\varphi_1(N_1, e_{N_1}, b_{N_1}) = 8^{2/3} + \frac{1}{2}(6^{2/3} - 8^{2/3}) + \frac{1}{2}(6^{2/3} - 4^{2/3}) = 6^{2/3} + \frac{1}{2}(8^{2/3} - 4^{2/3}) = 4.042,$$

$$\varphi_2(N_1, e_{N_1}, b_{N_1}) = 4^{2/3} + \frac{1}{2}(6^{2/3} - 4^{2/3}) + \frac{1}{2}(6^{2/3} - 8^{2/3}) = 6^{2/3} + \frac{1}{2}(4^{2/3} - 8^{2/3}) = 2.561,$$

$$\varphi_3(N_2, e_{N_2}, b_{N_2}) = 10^{1/3} + \frac{1}{2}(8^{1/3} - 10^{1/3}) + \frac{1}{2}(8^{1/3} - 6^{1/3}) = 6^{2/3} + \frac{1}{2}(8^{2/3} - 4^{2/3}) = 2.1687,$$

$$\varphi_3(N_2, e_{N_2}, b_{N_2}) = 6^{1/3} + \frac{1}{2}(8^{1/3} - 6^{1/3}) + \frac{1}{2}(8^{1/3} - 10^{1/3}) = 6^{2/3} + \frac{1}{2}(4^{2/3} - 8^{2/3}) = 1.8313.$$

5 Comparison with the Downstream Incremental Distribution Solution

Recall the Downstream Incremental Distribution (DID) solution for the river sharing problem (Ambec and Sprumont, 2002).

Consider a coalition $S \subseteq N$. Let $\mathcal{P}$ be the coarsest partition of S into consecutive components. Let $x^*(S)$ be the unique consumption plan for S that maximizes $\sum_{i \in S} b_i(x_i)$ subject to the constraints

$$\sum_{i \in P_j \cap T} (x_i - e_i) \leq 0, \text{ for all } j \in T \text{ and } T \in \mathcal{P},$$

and call

$$v(S) = \sum_{i \in S} b_i(x_i^*(S))$$

the secure benefit of S.

We say that a welfare distribution $z = (z_1, ..., z_n)$ satisfies the *core lower bounds* if

$$\sum_{i \in S} z_i \geq v(S) \text{ for every } S \subseteq N.$$

The core lower bounds property draws upon the theory of *absolute territorial sovereignty* (Barret, 1994) which states that a country has absolute sovereignty over the area of any river basin on its territory.

On the other hand, the theory of *unlimited territorial integrity* (Barret, 1994) forbids a country to alter the natural conditions on its own territory to the disadvantage of a neighboring country. This theory generates an upper bound on the welfare distribution as follows.

Consider the aspiration welfare of an arbitrary coalition S which is defined as the highest welfare it cound achieve in the absence of $N \setminus S$. It is obtained by choosing a consumption plan x_S maximizing $\sum_{i \in S} b_i(x_i)$ subject to the constraints

$$\sum_{i \in P_j \cap S} x_i \leq \sum_{i \in P_j} e_i \text{ for every } j \in S.$$

This problem has a unique solution, which is denoted by $x^{**}(S)$. The aspiration welfare of S is

$$w(S) = \sum_{i \in S} b_i(x_i^{**}(S)).$$

We say that a welfare distribution z satisfies the aspiration upper bounds if

$$\sum_{i \in S} z_i \leq w(S) \text{ for every } S \subseteq N.$$

An important observation is the relation between the core lower bounds and the aspiration upper bounds: $v(P_i) = w(P_i)$ for every $i \in N$. Thus there is only one welfare distribution satisfying these two bounds. It is called the Downstream Incremental Distribution (DID) z^* defined by Ambec and Sprumont (2002):

$$z_i^* = v(P_i) - v(P_{i-1}), \ i \in N. \tag{34}$$

Now we can compare the ESPS with the DID solution. Consider the following example.

Example 2. Let $b_1(x_1) = x_1^{2/3}$, $b_2(x_2) = x_2^{2/3}$. Let $e_1 = 8, e_2 = 4$

$$\begin{aligned} B(\{1,2\}) &= \max x_1^{2/3} + x_2^{2/3} \\ & \quad\ x_1 \leq 8, \\ & \quad\ x_1 + x_2 = 12. \end{aligned}$$

$$B(\{1,2\}) = 2 \times (6)^{2/3}.$$

The ESPS gives the following

$$\varphi_1 = 8^{2/3} + \frac{1}{2}(6^{2/3} - 8^{2/3}) + \frac{1}{2}(6^{2/3} - 4^{2/3}) = 6^{2/3} + \frac{1}{2}(8^{2/3} - 4^{2/3}),$$

$$\varphi_2 = 4^{2/3} + \frac{1}{2}(6^{2/3} - 4^{2/3}) + \frac{1}{2}(6^{2/3} - 8^{2/3}) = 6^{2/3} + \frac{1}{2}(4^{2/3} - 8^{2/3}).$$

If we compute the Downstream Incremental Distribution solution (DID), we have

$$\pi_1^{DID} = 8^{2/3},$$

$$\pi_2^{DID} = 2 \times 6^{2/3} - 8^{2/3}.$$

It is easy to see that

$$\varphi_1 = 4.042 > \pi_1^{DID} = 4,$$

$$\varphi_2 = 2.561 < \pi_2^{DID} = 2.603.$$

Thus, the ESPS violates the aspiration upper bounds.

We argue that the aspiration upper bounds constraints for the upstream agents (i.e., agent 1 in this example) is too restrictive because agent 1 should be compensated by a fair share of the surplus increase from transferring some water to the downstream agent 2, instead of just making him indifferent between transferring water and receiving his stand-alone benefit as if he is consuming his entire endowment of water. If we insist that agent 1 can not have the welfare that is above his stand-alone benefit level, he would have no incentive to share his water to increase the total welfare if he is ruled out to share any part of the increased total welfare.

Example 3. Let $b_1(x_1) = \sqrt{x_1}$, $b_2(x_2) = \sqrt{2}\sqrt{x_2}$, $b_3(x_3) = \sqrt{2}\sqrt{x_3}$, and $e_1 = 2, e_2 = 2, e_3 = 1$. Then, by the DID solution, we have

$$\pi_1^{DID} = \sqrt{2}, \pi_2^{DID} = \frac{6}{\sqrt{3}} - \sqrt{2} \approx 2.05, \pi_3^{DID} = 5 - \frac{6}{\sqrt{3}} \approx 1.54.$$

On the other hand, by the ESPS, we have

$$\varphi_1 = \sqrt{2} + \frac{1}{2}(1 - \sqrt{2}) + \frac{1}{2}(2 - \sqrt{2}) = \frac{3}{2},$$

$$\varphi_2 = 2,$$

$$\varphi_3 = \sqrt{2} + \frac{1}{2}(2 - \sqrt{2}) + \frac{1}{2}(1 - \sqrt{2}) = \frac{3}{2}.$$

In **Theorem 6** in Gudmundsson et al. (2019), it is shown that there is no implementable mechanism that for a dummy agent, his equilibrium payoff is $b_i(e_i)$. An agent is a dummy agent if at the efficient allocation x^*, $x_i^* = e_i$. According to the ESPS, if agent i is dummy, $\varphi_i = b_i(e_i)$. This means that the ESPS does not belong to the set of solutions that can be implemented by a mechanism from the class of mechanisms defined in Gudmundsson et al. (2019).

Houba (2008) provided a bargaining interpretation of the Downstream Incremental Distribution solution of Ambec and Sprumont (2002) (denoted as the AS solution in Fig 5 below). The following Fig. 5 shows a three-agent problem (the third agent is not shown in the figure):

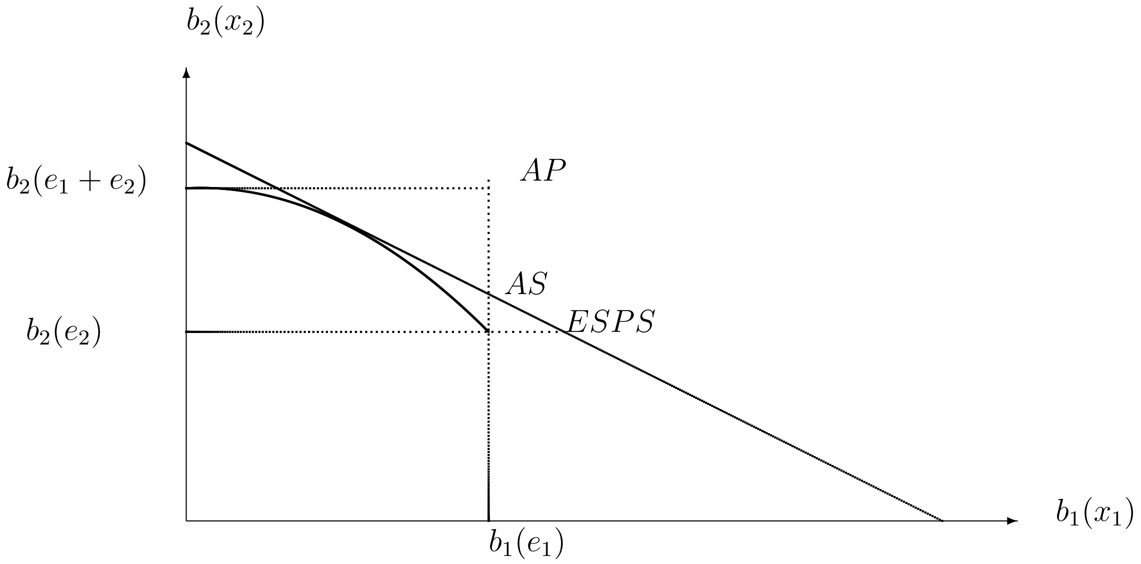

Fig. 5. $\pi_1^{AS} = b_1(e_1), \pi_2^{AS} > b_2(e_2), \varphi_1^{ESPS} > b_1(e_1), \varphi_2^{ESPS} = b_2(e_2)$.

In Fig.5., AP is the aspiration upper bounds. It can be seen that when agent 2's optimal allocation is $x_2^* = e_2$, agent 1 obtains $b_1(e_1)$ and agent 2 obtains more than $b_2(e_2)$ in AS. But in the ESPS, agent 1 obtains more than $b_1(e_1)$ and agent 2 obtains just $b_2(e_2)$. Agent 3 shares the rest.

6 Concluding Remarks

In this paper, we took a two-step approach to the river sharing problem. We divided the problem into a list of subproblems so that each of these subproblems can be treated as a permit sharing problem. In these permit sharing problems, the unidirectional constraints on the water allocations become irrelevant. This approach mitigates the structural restrictions of the river sharing problem and provides us opportunities to take advantage of other solution concepts in the permit sharing literature.

In particular, we considered the Equal Sharing Proportional Solution for the permit sharing problem (Suh and Wang (2023)) as the solution concept for the subproblems in the river sharing problem. The Aggregation Principle we proposed in the paper guarantees that the aggregation of the solutions for all the subproblems is a solution for the river sharing problem.

We compared the Equal Sharing Proportional Solution with the Downstream Incremental Distribution solution. On the one hand, the Equal Sharing Proportional Solution violates the aspiration upper bounds. On the other hand, it satisfies the stand-alone welfare property for dummy agent. In contrast, the Downstream Incremental Distribution solution meets the aspiration upper bounds by definition. But it violates the property of the stand-alone welfare for dummy agent.

The decomposition approach suggested in this paper opens us to other solution concepts for the permit sharing problem such as the price solution (e.g., Wang (2011)) and the equal division of surplus solution that we can apply to the river sharing problem. But, as shown in Suh and Wang (2023), each of these solutions also has drawbacks from either violating Separation Principle (see Suh and Wang (2023)) or being manipulable via utility functions (Suh (2001), Shin and Suh (2007)). Which solution concepts shall be used in practice ultimately is an empirical question. Further discussion in this direction is beyond the scope of this paper and it is left for future study.

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