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Abstract

We consider a set of users who are located along a pipeline with a single source. These users consume a good that is extracted from the source and flows downstream, with diminishing marginal returns for each user. In addition, flows along each edge in the pipeline create negative externalities, which are nondecreasing as a function of flow. The users cooperate toward obtaining group welfare maximization. In both the continuous and discrete cases, we obtain the group optimal solutions, and we then use cooperative game theory to determine how best to allocate the damages, using optimistic and pessimistic formulations for the characteristic function. Using core stability as our guiding principle, we provide a set of stable allocations that apportions the damages at a location among the set of downstream users, notably an average damage allocation and a marginal damage allocation. Given that the joint optimization forces agents to reduce (unequally) their consumption, we also examine the Shapley value of the optimistic game, also in the core, that allows to compensate agents who have sacrificed their consumption for the benefit of the group. Finally, we show that our pipeline externalities model generalizes some well-known problems from the literature, including the river sharing problem of Ambec and Sprumont 2002 and the joint production problem of Moulin and Shenker 1992.

Keywords: game theory, network flows, pipeline externalities; core; redistribution.

1 Introduction

While the optimization and applications of network flows have been well studied, there has been limited research on treating externalities that result from the flows. In the present work, we consider a pipeline where a good is extracted from the source and is transported along the pipeline to a series of users, who have decreasing benefits from consumption of the good. However, the flows distributed to the users subject them to increasing damages. We consider the novel problem whereby the users face the joint task of finding a set of quantities that maximizes their net benefit (taking the damages into account), and after solving this problem, we use cooperative game theory to equitably share the benefits (and damages) among the users.

Mitigating externalities is a significant cost component in the delivery of pipeline commodities. The consumption of gas, for example, by downstream agents creates externalities such as emissions, potential leaks, equipment breakdowns, or other serious events upstream. In the U.S. alone, there are over 2.7 million miles of pipelines that transmit natural gas, crude oil, and refined petroleum products, accounting for 5.7% of the total transportation value in the U.S. in 2017.[1] Regarding externality costs, in 2018, the total cost of oil and gas pipeline incidents in the U.S. was $2.26 billion.[2] In general, project planners need to balance the economic benefits of pipelines with their externalities, especially given that the distribution of benefits and externalities can vastly differ.

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[1] https://www.watershedcouncil.org/pipelines-101.html

[2] https://www.statista.com/statistics/1271679/us-pipeline-incidents-costs/

While typically externalities are modeled as being common to all users, pipeline externalities are localized and asymmetric, being created from local flows (degradation as a function of pipeline link lengths, increases in volume, and drops in supply capacity (Fan et al. 2022)). This provides a general framework to treat externalities across a wide array of problems that share this structure. A special case we explore in this work is the well-known river sharing problem (Ambec and Sprumont 2002), in which the consumption of water upstream reduces the flow downstream. Queueing, where agents at the front of a line impose externalities on those behind them, also fits the framework. Finally, our model can be used to represent joint production problems (Moulin and Shenker 1992), where agents cooperate to produce a commodity and share the cost of the units they wish to consume.

The joint creation of externalities can induce collaborative opportunities for their mitigation. To this end, we utilize cooperative game theory to propose how to share the joint benefits and costs generated along the pipeline. We begin by defining coalitional value functions, which is a challenging task given the externalities present. Following the approaches of Ambec and Sprumont 2002 for river sharing problems, Maniquet 2003 and Chun 2006 for queueing problems, and Atay and Trudeau 2024 for general frameworks, we construct pessimistic and optimistic value functions that provide lower and upper bounds, respectively, on what a coalition should receive. To determine the (hypothetical) value that a subgroup of users $S \subset N$ could obtain from the pipeline, various interpretations are possible. At one extreme, we could assume that non-members $N \backslash S$ can block the use of the pipeline at their locations. At the other extreme, we might assume that non-members in $N \backslash S$ have no say in the matter and can be made to bear externalities without compensation. We adopt a middle ground based on the polluter-pays principle, where non-members in $N \backslash S$ cannot block the pipeline as long as they are properly compensated for the externalities generated at their locations. This assumption yields the optimistic value function, as it assumes that non-members in $N \backslash S$ are not consuming. Conversely, the pessimistic value function allocates to a coalition S the additional value they can generate if they join after the agents in $N \backslash S$ have determined their consumption (optimistically). Because externalities increase with consumption, agents face higher marginal costs, resulting in reduced consumption and associated benefits.

Our approach is based around the concept of the core: no group should receive more than its optimistic upper bound, or less than its pessimistic lower bound. This line of analysis enables us to develop a series of contributions to the literature. We define a set of allocations that satisfy the core constraints by allocating the damage at each location among the downstream users, with the simplest of them charging users the average damage cost of their units consumed. In addition, when the users' consumption is measured in discrete units, we provide a greedy algorithm to optimize the total welfare as well as allocate costs to the users. This algorithm builds the optimal flow in the pipeline unit by unit while simultaneously allocating to each user the marginal damage their consumption generates.

The above allocations, while stable, ignore the sacrifices made by the agents, as group maximization necessarily implies a reduction in consumption compared to selfish decisions. The group-optimal consumption levels might not lead to uniform sacrifices. In particular, there can exist agents who are crowded out of consumption in the optimal solution. To address this, we consider the Shapley value with respect to the optimistic value function. We show that the optimistic function is always concave, thus ensuring that the Shapley allocation, which compensates agents for their cooperative roles, is an element of the core.

Moreover, with respect to generalizations from the literature, we demonstrate that river sharing problems are special cases of pipeline externalities problems, and we pinpoint how our results relate to that literature. In addition, we show how joint production models with a common technology (e.g., Moulin and Shenker 1992, Moulin 1996, and de Frutos 1998) fit within our formulation, allowing us to recover many of those results from our model.

1.1 Literature Review

The traditional focus in both the work on cooperative games that arise from general optimization problems (Owen 1975, Bird 1976, Granot and Huberman 1982, Faigle et al. 1998, Hamers et al. 1999, Dutta and Mishra 2012, Rosenthal 2013, Bahel and Trudeau 2019, Bergantiños et al. 2020 to name but a few) as well as the more focused literature on cooperative games arising from network optimization problems (Kalai and Zemel 1982, Rosenthal 1991, Granot et al. 2002, Perea et al. 2009, and Bertschinger et al. 2025) is restricted to sharing rewards (costs) from maximization (minimization) problems without regard to costly by-products of the main productive activities.

One application in network flows where (negative) externalities are central to the discussion is that of traffic networks (Carey and Srinivasan 1993, De Palma and Lindsey 2011, Vosough et al. 2022). Much of this literature has centered on models for vehicle flow and congestion pricing, and is far afield of the present work.

In the area of optimization-derived games, while the great majority of the literature does not consider externalities, there are other studies involving cooperative game theory where externalities are central to the analysis. Palestini and Poggio 2015 analyze welfare-maximizing players that implement optimal emissions strategies in deciding whether to join an international environmental agreement. Their model follows Finus 2001 for which the damage cost function is increasing and convex in aggregate emissions. Chander and Tulkens 1997 and Helm 2001 study an economy where agents' production generates environmental externalities that impact everyone. To achieve a stable way to reduce the externalities' impact, they develop a cooperative game and show that its core is nonempty. Central to their model is the assumption that players outside a particular coalition will adopt individual best reply strategies. The main result shows the non-emptiness of the resulting core. In addition to a different assumption on the behavior of external agents, in their model externalities are pooled, not locally generated and asymmetric as in our pipeline model.

Our model generalizes the line graph formulation of van den Brink et al. 2017. As in Myerson 1977, their work assumes a potential value for each coalition; to obtain this value, the graph is partitioned into connected components, such that each component is composed of adjacent agents on the line. The value of the coalition is then the sum of the potential values for each connected component. As opposed to this, our coalitional values are obtained from distinct maximization problems, and even if some users are not adjacent on the line, they still profit from joint maximization.

Other research in this setting employs games in partition function form (Thrall and Lucas 1963, Ray and Vohra 1999) to treat the issue of agent behavior external to a given coalition (Funaki and Yamoto 1999, Kóczy 2009), or else models the external player behavior probabilistically (Stamatopoulos 2021). As opposed to this modeling, we assume, as in Huang and Sjöström 2003, that the players external to a coalition S will adhere to the same optimization approach as S, namely, applying the same algorithm to the subgame under their control. Finally, Csercsik et al. 2019, extending previous work by Hubert and Cobanli 2015, study an optimization game where they transform a partition function formulation into a traditional characteristic function form game, but their application, to study transfer profits in a gas network as externalities, is far removed from ours.

Finally, our model is one of costly inclusion, in which we determine which users should be served (and with how much consumption). This property relates to the vast literature on the management of the commons with decreasing returns (Ostrom 1991). Usually, it is supposed that the manager of the commons does not know the willingness to pay of the agents, and there is a trade-off between efficiency, strategyproofness and budget balancedness (Moulin and Shenker 2001). As we assume full information and utilize cooperative game theory, our focus is on developing a fair allocation. To do this we exploit the pipeline structure to optimally distribute the goods, even though decreasing returns do not affect all users equally.

2 The model

Let $N = \{1, ..., n\}$ be a set of agents. We suppose that agents are on a line with agent 1 located at node 1 closest to the source 0, followed by 2, etc. Agents are interested in consuming a good provided by the source. The consumption by agent i implies that the desired quantity must be transported on the line from 0 to i. Let $\delta(i) = \{j \in N \,|\, j \geq i\}$ be the set of agents downstream of i. Let $\mu(i) = \{j \in N \,|\, j \leq i\}$ be the set of agents upstream of i. We use the notation S_{+i} as a shorthand for $S \cup \{i\}$ and S_{+ij} as a shorthand for $S \cup \{i, j\}$.

We first define a continuous version of the model, before considering a discrete version.

2.1 Continuous problems

Each agent $i \in N$ has a nondecreasing concave utility function U_i and a nondecreasing convex damage function D_i such that $U_i(0) = D_i(0) = 0$. We suppose that all functions are differentiable and u_i and d_i are the derivative functions of U_i and D_i, respectively. Let $\mathcal{U}$ and $\mathcal{D}$ be, respectively, the set of all such utility and damage functions. A problem is $P = (N, U, D)$, where $U = (U_i)_{i\in N}$ and $D = (D_i)_{i\in N}$. Let $\mathcal{P}^C$ be the set of all continuous problems.

We suppose that utility and damage functions are measured in monetary terms and that preferences are quasi-linear in money, so that we have transferable utility.

We assume that when cooperating, a planner will determine the optimal quantity consumed by each agent, in order to maximize the joint net surplus. Formally, let $x = (x_i)_{i\in N}$ be the vector of quantities consumed. Notice that since the good flows from the source to the agents, if x is the vector of consumption, the flow at node i is $\sum_{j\in\delta(i)} x_j$. For the grand coalition, the value generated is

$$V(N,P) = \max_{x\in\mathbb{R}^N_+} \sum_{i\in N} \left(U_i(x_i) - D_i\left(x_{\delta(i)}\right)\right) \tag{1}$$

where $x_S = \sum_{i\in S} x_i$. For all our maximization problems, we suppose that there is a unique solution yielding a finite value.

Maximand (1) sums the utilities for the consumption at each node i, minus the damages generated by its corresponding flow in the pipeline: at a given node the flow is the consumption of the agent located at this node plus the consumption of all agents downstream of that node.

A maximizer x is said to be a welfare-maximizing consumption for problem (N, U, D).

Example 1 *Consider a problem P with 3 agents, such that agent 1 is the closest to the source, and agent 3 the furthest. Suppose that agent i has the utility function: $U_i(x_i) = 60ix_i$ and that the damage functions are $D_i(f_i) = if_i^2$ where f_i is the incoming flow at node i. Thus, when the agents cooperate, our objective function is*

$$\max_{x_1,x_2,x_3} 60x_1 + 120x_2 + 180x_3 - (x_1 + x_2 + x_3)^2 - 2(x_2 + x_3)^2 - 3\,(x_3)^2$$

We can solve using the first-order conditions. The numbers are chosen so that even though the quantity can be real numbers, the optimal ones are integers. We obtain that it's optimal for agent 1 to consume 15 units, 5 units for agent 2 and 10 units for agent 3. The resulting net value created is $V(N,P) = 1650$.

2.2 Discrete problems

We now consider a discrete version of the problem. While we can simply define the functions U and D over $\mathbb{N}$ instead of $\mathbb{R}_+$, it is preferable to instead work with vectors of marginal utilities and marginal damages. Each agent $i \in N$ has a vector of marginal utilities $u_i = (u_{i1}, u_{i2}, ...)$, where u_{ik} is agent $i's$ utility for the k^{th} unit. We impose that if $k < l$ then $u_{ik} \geq u_{il} \geq 0$. We also have a vector of marginal damages $d_i = (d_{i1}, d_{i2}, ...)$ with $k < l$ implying that $0 \leq d_{ik} \leq d_{il}$. The damage is for each subsequent unit of flow arriving at node i. Let $\mathcal{U}^D$ and $\mathcal{D}^D$ be, respectively, the sets of all such marginal utility vectors and marginal damage vectors. A discrete problem is $P = (N, u, d)$, where $u = (u_i)_{i\in N}$ and $d = (d_i)_{i\in N}$. Let $\mathcal{P}^D$ be the set of all discrete problems.

For each $i \in N$, we suppose that there exists a finite k such that $u_{ik} < d_{ik}$, so that even if an agent was unconcerned with externalities, he would consume a finite amount.

One special case is a unitary utility vector, such that for all $i \in N$ and all $k > 1$, $u_{ik} = 0$. Let $\mathcal{U}^1$ be the set of all such utility vectors. Clearly $\mathcal{U}^1 \subset \mathcal{U}^D$. We define as $\mathcal{P}^1 \subset \mathcal{P}^D$ the set of problems with unitary utility vectors.

For discrete problems the planner's problem becomes

$$V(N,P) = \max_{x\in\mathbb{N}^N} \sum_{i\in N} \left(\sum_{l=0}^{x_i} u_{il} - \sum_{l=0}^{x_{\delta(i)}} d_{il} \right)$$

where we assume that $u_{i0} = d_{i0} = 0$ for all $i \in N$.

Starting from a continuous problem $P = (N, U, D) \in \mathcal{P}^C$ it is easy to see that we can build a discrete problem $P^D = (N, u, d) \in \mathcal{P}^D$: for each $i \in N$ and $k \in \mathbb{N}\backslash\{0\}$, let $u_{ik} = U_i(k) - U_i(k-1)$ and $d_{ik} = D_i(k) - D_i(k-1)$. Similarly, from a discrete problem (N, u, d) we can easily build the functions U and D (defined over $\mathbb{N}$). Given that, we sometimes talk of $(N, U, D) \in \mathcal{P} \equiv \mathcal{P}^C \cup \mathcal{P}^D$.

2.2.1 Optimization problem

While the continuous problem can be solved using first order conditions, the discretized version is not amenable to the same technique. Fortunately, we are able to provide an algorithm to determine a welfare-maximizing consumption vector for the discrete problem. The algorithm also provides a second benefit: later we are able to define cost sharing methods directly from it.

We need the following notation. For $i \in N$, let $e^i \in \mathbb{R}^N$ be such that $e^i_j = 1$ if $i = j$ and $e^i_j = 0$ otherwise. Let $\sigma : N \to N$ be a bijection, with $\sigma(i)$ interpreted as the priority rank of agent i. Let Σ be the set of all such bijections.

For each agent, we calculate the net benefit from adding a unit of flow to her node. If there is at least one agent for which the net benefit is strictly positive, we pick the agent creating the largest value (breaking ties with the priority ranking), send a unit of flow to that agent and update values of marginal flows. Otherwise we terminate.

Algorithm 1 *Input:* N, d, u, σ
Initialize: $v_i = u_{i1}$, $\theta_i = d_{i1}$, $f_i = 0$, $\forall i \in N$
Initialize: $\alpha = 0^N$, $W = 0$.
Initialize: $\pi_i = v_i - \sum_{k \leq i} \theta_k$, $\forall i \in N$
Initialize: $\gamma = \max_{i \in N} \pi_i$
While $\gamma > 0$ *pick* $j = \arg\min_{i \in \arg\max_{k \in N} \pi_k} \sigma(i)$
    Update $\alpha = \alpha + e^j$
    Update $f_i = f_i + 1$, $\forall i \leq j$
    Update $W = W + \gamma$
    Update: $v_j = u_{j,\alpha_j+1}$
    Update: $\theta_i = d_{i,f_i+1}, \forall i \leq j$
    Update: $\pi_i = v_i - \sum_{k \leq i} \theta_k$, $\forall i \leq j$
    Update: $\gamma = \max_{i \in N} \pi_i$
Return: $\alpha^* = \alpha$, $f^* = f$, $W^* = W$.

α^* is the vector of consumption at the optimal level, f^* the optimal flow and W^* is the net welfare at this optimal level.

We show that this greedy algorithm is optimal to solve our problem.

Theorem 1 *For any priority ordering $\sigma \in \Sigma$, Algorithm 1 provides a welfare-maximizing flow for all problems $(N, u, d) \in \mathcal{P}^D$.*

The proof, like all others that take more than a few lines, is in Appendix.

Example 2 *Suppose that $N = \{1, 2, 3\}$ and that we have unitary demands, with $u_{11} = 18$, $u_{21} = 56$, $u_{31} = 12$. Damages are $d_1 = (0, 6, 30, ...)$, $d_2 = (0, 30, ...)$ and $d_3 = (0, ...)$.*

In the first step of the algorithm we have $\pi_1 = 18$, $\pi_2 = 56$ and $\pi_3 = 12$. We thus serve agent 2.

In the second step we have $\pi_1 = 18 - 6 = 12$, $\pi_2 = 0 - 6 - 30 = -36$ and $\pi_3 = 12 - 6 - 30 - 0 = -24$. We thus serve agent 1.

In the third step we have $\pi_1 = 0 - 30 = -30$, $\pi_2 = 0 - 30 - 30 = -60$ and $\pi_3 = 12 - 30 - 30 - 0 = -48$. We do not serve anybody and terminate. The optimal flow is to send 2 units on arc (0,1) and 1 on (1,2). This creates a total value of 56+12=68.

Let $W(N, u, d)$ be the welfare associated to problem (N, u, d), as generated by Algorithm 1. We observe that while the welfare associated to N is independent of the priority ranking, the individual quantities are not. However, the results in this paper do not depend on the priority ranking chosen. To ease on the notation, for the rest of the paper we abstract away from them and implicitly assume that the algorithm has a single maximizer at each step. Alternatively, one can imagine the results below and the resulting allocations as being obtained by averaging over all priority rankings.

Let $\alpha(N, u, d)$ be the vector of demands generated by Algorithm 1. In other words, $W(N, u, d)$ and $\alpha(N, u, d)$ associates to inputs N, u, d the outputs W^* and α^* of Algorithm 1, respectively. For any α, α_S is the vector of demands for coalition S.

3 Defining coalitional games from the pipeline externalities problem

For a set of players N, a coalitional game V maps subsets of players to the real non-negative numbers: $V : 2^N \to \mathbb{R}_+$. A game V is said to be *convex* if $V(S_{+i}) - V(S) \leq V(T_{+i}) - V(T)$ for all $i \in N$ and $S \subset T \subset N \backslash \{i\}$. A game V is said to be *concave* if $V(S_{+i}) - V(S) \geq V(T_{+i}) - V(T)$ for all $i \in N$ and $S \subset T \subset N \backslash \{i\}$

For a game V, its Shapley value $Sh(V)$ is defined as follows: for all $i \in N$,

$$Sh_i(V) = \sum_{S \subseteq N \backslash i} \frac{|S|! (|N| - |S| - 1)!}{|N|!} (V(S_{+i}) - V(S)).$$

Developing a coalitional game from a pipeline externalities problem provides valuable insight into how damages can be equitably shared across the network. We consider two approaches to define the value associated to a coalition $S \subseteq N$.

3.1 The optimistic and pessimistic games

Given the externalities, there are many ways to define a coalitional game from the pipeline externalities problems. A simple and natural approach would be for $S \subseteq N$ to simply ignore agents in $N \backslash S$. A first version would suppose that we don't allow agents in $N \backslash S$ to consume and ignore the damage to these agents. Given that this game would not necessarily be monotonically increasing, we consider a less optimistic variant in which only agents in S consume, but must take into account the externalities they generate.

For any $P \in \mathcal{P}^C$, the problem for coalition S is thus

$$V^o(S, P) = \max_{x \in \mathbb{R}_+^S} \sum_{i \in S} U_i(x_i) - \sum_{i \in N} D_i\left(x_{\delta(i)}\right).$$

Stated otherwise, let $\bar{x}^S$ be the maximizer of the above problem. Then $V^o(S, P) = \sum_{i \in S} U_i(\bar{x}_i^S) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}^S\right)$. Throughout, we use $\bar{x}$ for $\bar{x}^N$. For the discrete problems, the approach is the same, except that we select $\bar{x}^S$ in $\mathbb{N}^S$.

This game is monotonically increasing: $V^o(S, d, u) \leq V^o(T, d, u)$ for all $S \subset T \subseteq N$, as T has access to the possible demands of agents in $T \backslash S$, to which coalition S does not have access. Stated otherwise, serving only agents in S is a possible solution for T, but it can possibly do better.

Given the optimistic assumptions taken, $V^o(S, P)$ is a best-case scenario, and it thus should be interpreted as an upper bound on what S should receive.

While the optimistic game supposes that coalition S has priority of access to consume, we can suppose the opposite, that is, coalition S comes in after $N \backslash S$ has maximized its welfare, consuming $\bar{x}^{N \backslash S}$.

Given its pessimistic nature, $V^p(S, P)$ is to be interpreted as a lower bound for the allocation to agents in S. We thus have that

$$V^p(S, P) = \max_{x \in \mathbb{R}_+^S} \sum_{i \in S} U_i(x_i) - \sum_{i \in N} \left(D_i\left(x_{\delta(i)} + \bar{x}_{\delta(i)}^{N \backslash S}\right) - D_i\left(\bar{x}_{\delta(i)}^{N \backslash S}\right)\right).$$

Since S chooses after $N \backslash S$, it must pay the higher marginal damage generated by its consumption, given the existing flow in the pipeline. Notice that $V^o(N, P) = V^p(N, P) = V(N, P)$.

Example 3 *Reconsider the discrete problem of Example 2. We find the pessimistic and optimistic values as follows:*

S	$V^p(S,P)$	$V^o(S,P)$
$\{1\}$	*12*	*18*
$\{2\}$	*0*	*56*
$\{3\}$	*0*	*12*
$\{1,2\}$	*20*	*68*
$\{1,3\}$	*12*	*24*
$\{2,3\}$	*50*	*56*
$\{1,2,3\}$	*68*	*68*

In particular, consider coalitions $\{3\}$ *and* $\{1,2\}$. *If agent 3 picks first, he consumes one unit, generating a benefit of* $V^o(\{3\},P) = 12$. *If* $\{1,2\}$ *pick after that and can't change the consumption of agent 3, they have agent 2 consume one unit, and agent 1 none, for a benefit of* $V^p(\{1,2\},P) = 20$. *Notice that this chain of event leads to a suboptimal outcome in which agent 3 consumes but not agent 1, as* $V^o(\{3\},P) + V^p(\{1,2\},P) = 32 < 68 = V(N,P)$.

If $\{1,2\}$ *picks first, each of them consumes one unit, and we obtain* $V^o(\{1,2\},P) = 68$. *If agent 3 comes afterwards, then he chooses not to consume, and* $V^o(\{3\},P) = 0$. *This particular chain of events leads to an efficient outcome.*

3.2 Core and anti-core

An allocation of the value $V(N,P)$ is a vector $y \in \mathbb{R}^N_+$ such that $\sum_{i\in N} y_i = V(N,P)$. We write $y(S)$ for $\sum_{i\in S} y_i$.

For the optimistic game V^o, since it provides an upper bound on the allocation for each coalition, we would like to provide an allocation that never exceeds these upper bounds. We call this set of allocations the anti-core of the optimistic game:

$$\mathcal{A}^o(P) = \left\{ y \in \mathbb{R}^N_+ \,|y(S) \leq V^o(S,P) \text{ for all } S \subset N \text{ and } y(N) = V^o(N,P) \right\}.$$

For the pessimistic game V^p, since it provides a lower bound on the allocation for each coalition, we would like to provide an allocation that never fails to provide these lower bounds. We call this set of allocations the core of the pessimistic game:

$$\mathcal{C}^p(P) = \left\{ y \in \mathbb{R}^N_+ \,|y(S) \geq V^p(S,P) \text{ for all } S \subset N \text{ and } y(N) = V^p(N,P) \right\}.$$

From the statements of the two objectives above (making sure that no coalition receive more benefits than their optimistic value, and not less than their pessimistic value), it is not clear if they are linked, or jointly feasible. We show, with a result close to that of Atay and Trudeau 2024, that making sure that no coalition receives more than their optimistic value is at least as difficult as making sure that they do not receive less than their pessimistic value.

Theorem 2 *For any problem* $P \in \mathcal{P}$, $\mathcal{A}^o(P) \subseteq \mathcal{C}^p(P)$.

Theorem 2 implies that i) if we verify that $y(S) \leq V^o(S,P)$ for all coalitions S, automatically we obtain that $y(S) \geq V^p(S,P)$, and ii) the set of allocations where both objectives are simultaneously satisfied is $\mathcal{A}^o(P)$.

4 A set of allocation methods that only reward served agents

An allocation method y assigns to any problem $P = (N,U,D)$ an allocation of the value it generates, such that $\sum_{i\in N} y_i(P) = V(N,P)$.

In the following sections, we explore methods that propose core allocations based on the optimal solution of flow assignments. Agents that consume units are said to be *served*. We start by characterizing a subset of the anti-core of the optimistic game, such that only agents that are served are awarded a share of the value created by the pipeline.

For all $i,j \in N$, let m_i^j be the damage at node j assigned to agent i. We write m_S^j for $\sum_{i\in S} m_i^j$. We impose the following restrictions:

1. $m_i^j = 0$ if $j > i$.

2. $D_j(\bar{x}_{S\cap\delta(j)}) \leq m_S^j \leq D_j(\bar{x}_{\delta(j)}) - D_j(\bar{x}_{\delta(j)\setminus S})$ for all $S \subseteq N$ and all $j \in N$.

3. $\sum_{i\in N} m_i^j = D_j(\bar{x}_{\delta(j)})$.

That is, condition 1 states that agents upstream of j do not share the damages at j. Condition 2 states that the share of any coalition for the damages at node j should be between a best-case scenario, in which they pay the first (lower) marginal damages for their units, and a worst-case scenario in which they pay the last (higher) marginal damages. Finally, condition 3 guarantees budget balance. Let $m = \left(m^j\right)_{j\in N}$ and let $M(P)$ be the set of all m that satisfy the conditions. Let $y^m(P) = U_i(\bar{x}_i) - \sum_{j\leq i} m_i^j$. Let $\mathcal{M}(P) = (y^m(P))_{m\in M(P)}$ be the set of allocations obtained from all eligible m.

Example 4 *Consider again Example 1. We describe the $m \in M$.*[3]

	$j=1$	$j=2$	$j=3$
$i=1$	$m_1^1 \in [225, 675]$	$m_1^2 = 0$	$m_1^3 = 0$
$i=2$	$m_2^1 \in [25, 275]$	$m_2^2 \in [50, 250]$	$m_2^3 = 0$
$i=3$	$m_3^1 \in [100, 500]$	$m_3^2 \in [200, 400]$	$m_3^3 = 300$
budget balance	$m_1^1 + m_2^1 + m_3^1 = 900$	$m_2^2 + m_3^2 = 450$	

Thus, for any m satisfying the constraints, we have $y_1^m = 900 - m_1^1$, $y_2^m = 600 - m_2^1 - m_2^2$ and $y_3^m = 1800 - m_3^1 - m_3^2 - m_3^3$.

We show that this set is always in the optimistic anti-core.

Theorem 3 *For all $P \in \mathcal{P}$, $\mathcal{M}(P) \subseteq \mathcal{A}^o(P)$.*

The set $\mathcal{M}(P)$ is large, and we define an allocation method that assigns particular allocations in it. Let $AD(P)$ be the *average damage allocation method.* It assigns to each agent the average damage on each segment needed to deliver his units consumed. Formally, for all $i \in N$,

$$AD_i(P) = U_i(\bar{x}_i) - \sum_{j\leq i} \frac{\bar{x}_i}{\bar{x}_{\delta(j)}} D_j\left(\bar{x}_{\delta(j)}\right)$$

.

Corollary 1 *For all $P \in \mathcal{P}$, $AD(P) \in \mathcal{A}^o(P) \subseteq \mathcal{C}^o(P)$.*

Notice that this implies that the optimistic anti-core and the pessimistic core are always non-empty.

Example 5 *Revisiting our continuous example, AD corresponds to the y^m such that $m_1^1 = 450$, $m_2^1 = 150$, $m_3^1 = 300$, $m_2^2 = 150$, $m_3^2 = 300$ and $m_3^3 = 300$. We thus obtain $AD_1 = 450$, $AD_2 = 300$ and $AD_3 = 900$.*

In the discrete case, we can also think in marginal terms and define a marginal allocation rule. Let $MD(P)$ be the marginal damage allocation method that is built from Algorithm 1: at each step it assigns the net value of the unit added to the agent consuming it. More precisely, if α^k, γ^k are respectively the vector of units consumed after step k and the net benefit obtained at step k and K^* is the last step in which we add a unit of consumption, then $MD(P) = \sum_{k=1}^{K^*}\left(\alpha^k - \alpha^{k-1}\right)\gamma^k$. Notice that $\sum_{i\in N} MD_i(P) = V(N,P)$.

Corollary 2 *For all $P = (N, U, D) \in \mathcal{P}^D$, $MD(P) \in \mathcal{A}^o(P) \subseteq \mathcal{C}^p(P)$.*

[3] When $n \leq 3$, it suffices to state condition 2 for singletons, as for other coalitions the condition is implied by combining the condition 2 for $N\setminus S$ and condition 3.

Example 6 *We revisit Example 2. Recall that we serve agents 1 and 2, and that the marginal damage of the first unit at node 1 is 0, and 6 for the second. The marginal damage of the first unit at node 2 is 0. Thus* $m_1^1 \in [0,6]$, $m_2^1 \in [0,6]$ *and* $m_2^2 = 0$. *The following table represents the allocations in, respectively,* $\mathcal{M}(P)$, $\mathcal{A}^o(P)$ *and* $\mathcal{C}^p(P)$.

$\mathcal{M}(P)$	$\mathcal{A}^o(P)$	$\mathcal{C}^p(P)$
$12 \le y_1 \le 18$	$12 \le y_1 \le 18$	$12 \le y_1 \le 18$
$50 \le y_2 \le 56$	$44 \le y_2 \le 56$	$0 \le y_2 \le 56$
$y_3 = 0$	$0 \le y_3 \le 12$	$0 \le y_3 \le 48$
$y_1 + y_2 = 68$	$56 \le y_1 + y_2 \le 68$	$20 \le y_1 + y_2 \le 68$
$12 \le y_1 + y_3 \le 18$	$12 \le y_1 + y_3 \le 24$	$12 \le y_1 + y_3 \le 68$
$50 \le y_2 + y_3 \le 56$	$50 \le y_2 + y_3 \le 56$	$50 \le y_2 + y_3 \le 56$
$y_1 + y_2 + y_3 = 68$	$y_1 + y_2 + y_3 = 68$	$y_1 + y_2 + y_3 = 68$

In particular, $MD(P) = (12, 56, 0)$ *and* $AD(P) = (15, 53, 0)$.

We conclude this section by showing a sufficient condition for the set $\mathcal{M}$ to be precisely the optimistic anti-core: when the optimal demand for each agent when they cooperate is the same as when they are alone on the pipeline. The demands are then essentially inelastic.

Theorem 4 *For any* $P \in \mathcal{P}$, *if* $\bar{x}_i^{\{i\}} = \bar{x}_i$ *for all* $i \in N$, *then* $\mathcal{A}^o(P) = \mathcal{M}(P)$.

Outside of this particular case, there are allocations in the anti-core outside of $\mathcal{M}$, and we examine some of them in the next section.

5 The optimistic anti-core and allocations that reward unserved agents

While the previous section defined an interesting subset of the optimistic anti-core, allocations in that subset do not allow for any reward to agents that were not served, like agent 3 in our discrete example, even though they might be priced out of consumption by the externalities generated by the consumption of those served. A way to achieve this goal would be to use either the optimistic or pessimistic games, for instance by applying the Shapley value. Going back to our example, however, we show that the pessimistic game might not be convex, a sufficient condition for its Shapley value to be in the pessimistic core.

Example 7 *Revisiting Example 1, we can verify that we have the following values for the pessimistic and optimistic games.*

S	$V^p(S)$	$V^o(S)$	S	$V^p(S)$	$V^o(S)$
$\{1\}$	*100*	*900*	$\{1,2\}$	*225*	*1350*
$\{2\}$	*12*	*1200*	$\{1,3\}$	*180*	*1620*
$\{3\}$	*150*	*1350*	$\{2,3\}$	*600*	*1500*
			$\{1,2,3\}$	*1650*	*1650*

Notice that V^p *is not convex:* $V^p(\{1,3\}) - V^p(\{3\}) = 30 < 100 = V^p(\{1\})$. *Here, it is optimal to limit the consumption of agent 2. When left to pick first, agent 2 consumes 20 units, imposing a large externality on agents 1 and 3, who can jointly only obtain 180 of surplus when picking last, by respectively consuming 6 and 4 units. If another agent selects first with agent 2, he is able to limit the consumption of agent 2, leaving a comparatively large surplus for the remaining agent who picks last. For instance, if agents 2 and 3 pick first, they consume 10 units each, allowing agent 1 to also consume 10 units when picking last, generating a surplus of 100.*

While the pessimistic game is not always convex, in the examples above the optimistic game is concave. We show that this is always true, allowing us to easily characterize all allocations in the optimistic anti-core, including allocations that reward unserved agents. In particular, this allows to conclude that the Shapley value of the optimistic game is in the optimistic anti-core, which provides stability to the entire user set. We show the result for continuous problems.

Theorem 5 *For all problems $P \in \mathcal{P}^C$, $V^o(P)$ is concave.*

Corollary 3 *For any problem $P \in \mathcal{P}^C$, $Sh(V^o(P)) \in \mathcal{A}^o(P) \subseteq \mathcal{C}^p(P)$.*

Proof. Fix a problem $P \in \mathcal{P}^C$. By the properties of the Shapley value and since V^o is concave, we have that $\sum_{i \in S} Sh_i(V^o(P)) \leq V^o(S, P)$ for all $S \subseteq N$. Thus, $Sh(V^o(P)) \in \mathcal{A}^o(P)$. By Theorem 2, $\mathcal{A}^o(P) \subseteq \mathcal{C}^p(P)$. ■

Example 8 *Revisiting Examples 1 and 2, we obtain that $Sh(V^o) = (420, 510, 720)$ in the continuous example, and $Sh(V^o) = (14, 49, 5)$ for the discrete problem.*

Compared to allocations in $\mathcal{M}$, the Shapley value offers more compensations to agents that severely restrict their consumption for the benefits of others. See that in our continuous example, we obtain an allocation that is much more advantageous to agent 2 than AD. Agent 2, who would consume 20 units if by himself, is restricted to 5 units when cooperating with others. In the discrete example, the results are even sharper: agent 3 never received any compensation in any allocation in $\mathcal{M}$ as he is abstaining from consuming any units for the benefits of others. The Shapley value does offer some compensation for the sacrifice.

Finally, we make a connection with the Aumann-Shapley 1974 approach to cost allocation. Applied to discrete problems (Sprumont 2005), it implies treating each unit of (potential) demand as a player in the cooperative game. The final allocation of a player is then the sum of the allocations made to his units of (potential) demands. Consider a discrete problem (N, u, d) and a vector $x \in \mathbb{N}^N$. We extend the set of players on the line as follows. Player 1_1 corresponds to the first unit of the agent closest to the source, player 1_2 to the second unit of that same agent, and so on, up to 1_{x_1}. Then, we have player 2_1, then 2_2 and so on. Call this set of players N^x. We also extend the utility vectors u_i to u_i^x as follows: For all $i \in N$ and all $k \leq x_i$, $u_{i_k,1}^x = u_{ik}$, and $u_{i_k,m}^x = 0$ for all $m > 1$. Similarly, the damage vectors are extended as follows: For all $i \in N$ let $d_{i_1}^x = d_i^x$ and $d_{i_m}^x = (0, 0, 0, ...)$ for all $m > 1$.[4] If we use $x = (\bar{x}_i)_{i \in N}$, i.e. the efficient consumption when cooperating, we obtain an allocation in $\mathcal{M}$. It would also be possible to use $x = \left(\bar{x}_i^{\{i\}}\right)_{i \in N}$, i.e. the quantities consumed by each agent if alone on the pipeline. In that case, the units in $(\bar{x}_i)_{i \in N}$ are consumed and receive benefits, but compensate the units in $\left(\bar{x}_i^{\{i\}} - \bar{x}_i\right)_{i \in N}$ that are sacrificed for the common good.

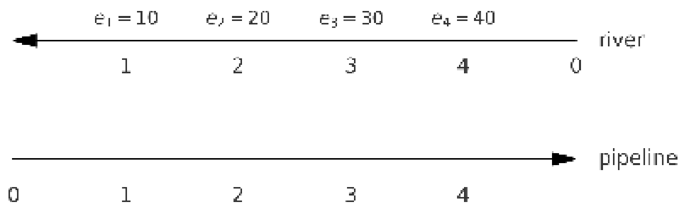
6 Special cases

6.1 River sharing problem

In river sharing problems (Ambec and Sprumont 2002), agents are located along a river, which is represented as a line, and consume its water. The problem is to identify the optimal consumption of water along the river, given the constraints imposed by the river (the direction of the flow, and water entries at various points). It is worth observing that in water sharing problems, the consumption of agents upstream imposes a negative externality on the agents downstream, as they have less water available for themselves. This is in opposition to our pipeline externalities problem, in which the consumption of agents downstream imposes a negative externality to agents upstream, as the transportation of this consumption from the source generates damages. Thus, to represent river sharing problems as pipeline problems, the ordering of the agents will need to be inverted. We suppose that agent n is the most upstream agent on the river, and agent 1 the most downstream. At each agent's location there is an entry of water $e_i > 0$. Each agent has an increasing and concave utility function $U_i(z_i)$, where z_i is the amount of water consumed. There is a constraint on how much water can be consumed: for all $i \in N$, $\sum_{j=i}^n z_j \leq \sum_{j=i}^n e_j$. Thus, a river sharing problem is (N, U, e), where $U = (U_i)_{i \in N}$. Let $\bar{z}$ be the optimal vector of water consumptions and let $W(N, U, e)$ be the maximum welfare obtained from the problem, i.e. $W(N, v, e) = \sum_{j=1}^n U_j(\bar{z}_j)$. Define the damage functions $D_i^e(z)$ such that $D_i^e(z) = 0$ if $z \leq \sum_{j=i}^n e_j$ and $D_i^e(z) = \infty$ otherwise.

Example 9 *Consider a 4-player example, with $e = (10, 20, 30, 40)$. The following schema represents the flow in the resulting river and pipeline problems.*

[4] The proof of Theorem 1 uses such a decomposition.



For all $i \in N$ we have $D_i^e(z) = 0$ if $z_i \leq \bar{e}_i$, and $D_i^e(z) = \infty$ otherwise, with $\bar{e} = (100, 90, 70, 40)$. Given these damage functions, it is optimal to maintain the flow below $\bar{e}$. Thus, optimal consumption in the pipeline externalities problem solves

$$\begin{aligned} &\max \sum_{i \in N} U_i(x_i) \\ s.t.\ x_4 &\leq 40 \\ x_3 + x_4 &\leq 70 \\ x_2 + x_3 + x_4 &\leq 90 \\ x_1 + x_2 + x_3 + x_4 &\leq 100 \end{aligned}$$

which is equivalent to the river sharing problem.

We formally show that this is true for all river sharing problems.

Theorem 6 *For any river sharing problem (N, U, e), the pipeline externalities problem (N, U, D^e) is such that $W(N, U, e) = V(N, U, D^e)$.*

Proof. Fix (N, U, e) and see that $W(N, U, e) = \max_z \sum_{j=1}^n U_j(z_j)$ such that $\sum_{j=i}^n z_j \leq \sum_{j=i}^n e_j$ for all $i \in N$.

Consider now $(N, U, D^e$. Since $D_i^e(x) = \infty$ if $x_i > \sum_{j=i}^n e_j$, it is optimal to have $x_i \leq \sum_{j=i}^n e_j$ for all $i \in N$. In that case, the damages are null at each node, and thus, the maximization problem can be rewritten as $V(N, U, D^e) = \max_x \sum_{j=1}^n U_j(x_j)$ such that $\sum_{j=i}^n x_j \leq \sum_{j=i}^n e_j$ for all $i \in N$, and thus $W(N, U, e) = V(N, U, D^e)$. ■

Thus, we can interpret the pipeline externalities problem as an extension of river sharing problems in which each agent's consumption increases the cost of consumption for agents downstream, while the river sharing problem has these hard constraints where the cost becomes infinite when the water is exhausted. Thus, in pipeline externalities problems, the water is never fully exhausted - it just becomes more and more expensive to extract more, perhaps because the flow reduces, making it technically more difficult to extract. It is worth noting that $(N, U, D^e) \notin \mathcal{P}$ given the shape of the damage functions. Nonetheless, our results, excepted Theorem 5, still hold.

Ambec and Sprumont 2002 define a game using the Unlimited Territorial Integrity (UTI) doctrine, which allows a group to consume all the water that enters their territory, i.e. $W^{UTI}(S, U, e) = \max_{(z_i)_{i \in S}} \sum_{i \in S} U_i(z_i)$ under the constraints that $\sum_{\substack{j \geq i \\ j \in S}} z_j \leq \sum_{j=i}^n e_j$ for all $i \in S$.

Theorem 7 *For any river sharing problem (N, U, e), $W^{UTI}(\cdot, U, e) = V^o(\cdot, P^e)$, where $P^e = (N, U, D^e)$.*

Proof. Fix (N, U, e) and consider $V^o(S, P^e)$ for some $S \subset N$. Since $D_i^e(x) = \infty$ if $x_i > \sum_{j=i}^n e_j$, it is optimal to have $x_i \leq \sum_{j=i}^n e_j$ for all $i \in S$. In that case, the damages are null at each node, including at nodes in $N \backslash S$, and thus, the maximization problem can be rewritten as $V^o(S, U, D^e) = \max_{(z_i)_{i \in S}} \sum_{j \in S} U_j(x_j)$ under the constraints that $\sum_{\substack{j \geq i \\ j \in S}} z_j \leq \sum_{j=i}^n e_j$ for all $i \in S$, and thus $W(S, U, e) = V^o(S, P^e)$. ■

Ambec and Sprumont 2002 also define a game under the Absolute Territorial Sovereignty doctrine, which supposes that any water that flows in a territory will be consumed. This means that if $i, j \in S$ but $i < l < j$, with $l \notin S$, coalition S assumes that if it attempts to send water from j to i, that water will be intercepted and consumed by l. To properly define the constraints under this doctrine, we say that a coalition is consecutive if for any pair of agents in that coalition, adjoining agents are also in the coalition. Let $\Gamma(S)$ be the coarsest partition of S into consecutive coalitions. Then, $W^{ATS}(S) = \sum_{T\in\Gamma(S)} \max_{(z_i)_{i\in T}} \sum_{i\in T} U_i(z_i)$ under the constraints that $\sum_{\substack{j\geq i\\ j\in T}} z_i \leq \sum_{\substack{j\geq i\\ j\in T}} e_i$ for all $i \in T$ and all $T \in \Gamma(S)$.

This is thus a pessimistic approach, but it does not correspond to our pessimistic game. In fact, we have the following result:

Theorem 8 *For any river sharing problem (N, v, e), $V^p(S, P^e) \leq W^{ATS}(S, U, e) \leq W^{UTI}(S, U, e)$ for all $S \subset N$, where $P^e = (N, U, D^e)$.*

We illustrate the differences with our example.

Example 10 *We consider the different coalitional values for coalition $\{1, 3, 4\}$. As mentioned, the UTI approach is equivalent to our optimistic approach, and lets the coalition be the first to consume. We obtain:*

$$\begin{aligned} V^{UTI}(\{1,3,4\}) &= \max_{x_1,x_3,x_4} U_1(x_1) + U_3(x_3) + U_4(x_4) \\ s.t.\ x_4 &\leq 40 \\ x_3 + x_4 &\leq 70 \\ x_1 + x_3 + x_4 &\leq 100 \end{aligned}$$

Here agents 3 and 4 are constrained by the maximum flow of the river at their locations. Notice that they can decide to consume less to let some water flow to agent 1 downstream.

By opposition, under the ATS doctrine any attempt to let water flow downstream will be intercepted by a non-member of the coalition. We thus obtain:

$$\begin{aligned} V^{ATS}(\{1,3,4\}) &= \max_{x_1,x_3,x_4} U_1(x_1) + U_3(x_3) + U_4(x_4) \\ s.t.\ x_4 &\leq 40 \\ x_3 + x_4 &\leq 70 \\ x_1 &\leq 10 \end{aligned}$$

Agent 4 can still let water flow to agent 3 - there is no non-members between them. But if agent 3 attempts to let water flow to agent 1, we assume that agent 2 will intercept it, and thus agent 1 can consume at most the entry of water at his location.

In our pessimistic approach we assume that agent 2 has first made its "optimistic" choice, and then coalition $\{1, 3, 4\}$ make their decisions. Given the assumption that our utility functions are strictly increasing, when going first agent 2 has consumed all it could, which here is 90 units. Thus we are left with:

$$\begin{aligned} V^{p}(\{1,3,4\}) &= \max_{x_1,x_3,x_4} U_1(x_1) + U_3(x_3) + U_4(x_4) \\ s.t.\ x_4 &\leq 0 \\ x_3 + x_4 &\leq 0 \\ x_1 + x_3 + x_4 &\leq 10 \end{aligned}$$

and agents 3 and 4 cannot consume anything. Agent 1 can consume 10 units.[5] *See that $V^p(\{1,3,4\}) \leq V^{ATS}(\{1,3,4\})$.*

Using our results, we can link the optimistic anti-core and pessimistic core with their equivalent using the UTI and ATS doctrine.

[5] Observe that in general, the pessimistic approach does not yield results as extreme. If the damage function is increasing, convex and differentiable, then agent 2 choosing first increases the marginal cost of consuming for agents 3 and 4, but does not push that cost to infinity as in the river sharing problem. It would be possible to use a more pessimistic approach in which a non-cooperating agent could veto the flow at his location; here agent 2 could block any quantity from flowing to agents 3 and 4. Our approach is that agent 2 can only petition for being fully compensated for the damage sustained at its location.

Corollary 4 *For any river sharing problem (N, U, e), let $P^e = (N, U, D^e)$ be the corresponding pipeline externalities problem. We have:*

i) $\mathcal{A}^o(P^e) = \mathcal{A}\left(W^{UTI}(\cdot, U, e)\right) \subseteq \mathcal{C}^p(P^e)$;

ii) $\mathcal{C}\left(W^{ATS}(\cdot, U, e)\right) \subseteq \mathcal{C}^p(P^e)$.

Ambec and Sprumont 2002 propose the downstream incremental rule:

$$\begin{aligned} y_i^{DI} &= W^{UTI}(\{i, ..., n\}, U, e) - W^{UTI}(\{i+1, ..., n\}, U, e) \\ &= W^{ATS}(\{i, ..., n\}, U, e) - W^{ATS}(\{i+1, ..., n\}, U, e) \end{aligned}$$

for all $i = 1, ..., n-1$ and $y_n^{DI} = W^{UTI}(\{n\}, U, e) = W^{ATS}(\{n\}, U, e)$.

We obtain the following result:

Theorem 9 *For any river sharing problem (N, U, e), we have:*

i) (Ambec and Sprumont, 2002) $y^{DI}(N, U, e) = \mathcal{A}\left(W^{UTI}(\cdot, U, e)\right) \cap \mathcal{C}\left(W^{ATS}(\cdot, U, e)\right)$;

ii) $y^{DI}(N, U, e) \in \mathcal{C}^p(\cdot, P^e)$, *with* $P^e = (N, U, D^e)$.

In the context of river sharing problems, our set $\mathcal{M}$ reduces to a single allocation, as the damage is zero at each node. Thus, $y_i^{\mathcal{M}}(P^e) = U_i(\bar{z}_i)$ for all $i \in N$, where $\bar{z}$ is the optimal water consumption vector for N. While simpler because it does not involve any monetary transfer, $y^{\mathcal{M}}$ is not necessarily less interesting than y^{DI}. The agent that is closest to the source of water receives its full optimistic value in y^{DI}, as if it could ignore the presence of others. By opposition, in $y^{\mathcal{M}}$ the agent closest to the source of water receives the value of its consumption when we maximize the joint value over all agents. That value cannot be higher for the agent than in its optimistic game. Thus, $y^{\mathcal{M}}$ redistributes utility (money) from upstream to downstream agents compared to y^{DI}.

Our allocation $Sh(V^o(P^e))$ also compares to y^{DI}. By our results above we have that $y_i^{DI} = V^o(\{i, ..., n\}, P^e) - V^o(\{i+1, ..., n\}, P^e)$ for all $i = 1, ...n-1$ and $y_n^{DI} = V^o(\{n\}, P^e)$. Thus, y^{DI} takes the vector of marginal contributions for one specific ordering, from upstream to downstream (compared to the source of water), while $Sh(V^o(P^e))$ takes the average over all orderings. Once again, this will redistribute utility (money) from upstream to downstream agents compared to y^{DI}.

It is worth noting that these allocations have received little attention in the river sharing literature because the nice properties that they have in the pipeline externalities problems do not carry over to river sharing problems. This is because our proof that V^o is concave uses the assumption that the damage functions are differentiable, which is not verified for water sharing problems. Ambec and Sprumont 2002 show that $W^{UTI}(S_{+i}) - W^{UTI}(S) \geq W^{UTI}(T_{+i}) - W^{UTI}(T)$ if $S \subset T \subseteq N \backslash i$ and i downstream of all members in T according to the river, which is a weaker result than concavity. Interestingly, W^{ATS} has been shown to be convex by Ambec and Sprumont 2002. van den Brink et al. 2007 study its Shapley value.

Among other methods proposed, Wang 2011 proposes a market-based allocation that has no evident counterpart in our setting, while van den Brink et al. 2007 propose methods based on the fact that W^{ATS} is a line graph game. Since it does not correspond to our pessimistic or optimistic games, there is no direct counterpart in our setting.

Khmelnitskaya 2010 extends the model to cases where the river has a delta or multiple springs. For simplicity of exposition, we have focused on pipelines represented as lines, although other topologies are possible. Given that river sharing problems reverse the ordering of the agents compared to pipeline externalities problems, extensions to a more complicated river structure would require a reverse structure for the corresponding pipeline.

6.2 Joint production

Many have considered the problem of agents cooperating to produce and share the cost of the units they want to consume.

A first stream of the literature (Moulin and Shenker, 1992; Moulin, 1996; de Frutos, 1998) supposes that the quantities demanded are exogenous. We consider here the version in which all agents are demanding a single, homogenous good. Thus, we have demands $q = (q_i)_{i \in N} \in \mathbb{R}_+^N$ and a cost function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the cost of producing k units of the good is $\omega(k)$. Thus, the joint production problem (with exogenous demand) is (N, q, ω). There is no optimization problem, we simply must share $\omega\left(\sum_{i \in N} q_i\right)$.

The case where ω is a convex function fits within our pipeline externalities framework. As the utility functions do not matter other than to impose that each agent i consumes exactly q_i, suppose that $U_i^q(x) = \beta x$ for all $0 \leq x \leq q_i$ and $U_i^q(x) = \beta q_i$ otherwise, where β is larger than the marginal cost of the last unit produced, so that it's optimal to consume q. We then set $D_1^\omega(x) = \omega(x)$ and $D_i^\omega(x) = 0$ for all $i \neq 1$ and all $x \in \mathbb{R}_+$. The pipeline externalities problem (N, U^q, D^ω) is thus

$$\begin{aligned} V(N, U^q, D^\omega) &= \max_{(x_i)_{i \in N}} \sum_{i \in N} \left(U_i^q(x_i) - D_i^\omega\left(x_{d(i)}\right)\right) \\ &= \max_{(x_i)_{i \in N}} \sum_{i \in N} U_i^q(x_i) - D_1^\omega\left(x_{d(i)}\right) \\ &= \max_{(x_i)_{i \in N}} \sum_{i \in N} U_i^q(x_i) - \omega\left(\sum_{i \in N} x_i\right) \\ &= \sum_{i \in N} \beta q_i - \omega\left(\sum_{i \in N} q_i\right) \end{aligned}$$

Notice that because there is no optimization, our optimistic and pessimistic games are dual, and $\mathcal{A}^o(P) = \mathcal{C}^p(P)$ (see Atay and Trudeau, 2024).

Thus, by sharing this value, we obtain a cost sharing value (i.e., all interesting value allocations will assign to agent i the value βq_i and subtract some cost shares). Our allocations in $\mathcal{M}$ are simple and natural, with each agent paying a cost share in between $\omega(q_i)$ and $\omega\left(\sum_{j \in N} q_j\right) - \omega\left(\sum_{j \in N \setminus i} q_j\right)$. In fact, given Theorem 4, the cost shares associated to $\mathcal{M}$ are precisely those in the core of the corresponding cost sharing problem. The Shapley value is a well-known solution for this problem, as we proposed, and in the optimistic anti-core if ω is convex. As all units are symmetric (as the damage is only at the first node), the Aumann-Shapley solution yields the proportional solution (Moulin 1987), which is nothing other than our allocation AD.

Another stream of the literature (Moulin, 1990; Roemer and Silvestre, 1993; Fleurbaey and Maniquet, 1996) supposes that the quantity consumed is endogenous. While we keep the common cost function as above, we suppose now that agents have a non-decreasing, concave and differentiable utility function U_i. We call this problem (N, U, ω) an endogenous consumption with joint production problem. To transform into a pipeline externalities problem, we do as above for the damage functions, and directly keep the utility functions. While the above papers suppose ordinal utilities, if we suppose that we have comparable valuations by the agents, as in Mackenzie and Trudeau 2023, we can use the pipeline externalities framework. The resulting pipeline externalities problem (N, U, D^ω) is thus

$$\begin{aligned} V(N, U, D^\omega) &= \max_{(x_i)_{i \in N}} \sum_{i \in N} \left(U_i(x_i) - D_i^\omega\left(x_{\delta(i)}\right)\right) \\ &= \max_{(x_i)_{i \in N}} \sum_{i \in N} U_i(x_i) - D_i^\omega\left(x_{\delta(i)}\right) \\ &= \max_{(x_i)_{i \in N}} \sum_{i \in N} U_i(x_i) - \omega\left(\sum_{i \in N} x_i\right). \end{aligned}$$

If $\bar{x}$ is the optimal vector of consumption, our allocations in $\mathcal{M}$ starts by allocating each agent with $U_i(\bar{x}_i)$, and then allocates to each agent a cost share in between $\omega(\bar{x}_i)$ and $\omega\left(\sum_{j \in N} \bar{x}_j\right) - \omega\left(\sum_{j \in N \setminus i} \bar{x}_j\right)$. Thus, allocations in $\mathcal{M}$, once the consumption has been chosen, treats the sharing of its production cost as if these quantities were exogenous. While simple, our results show that those allocations are in the optimistic anti-core and the pessimistic core.

Instead of only allocating costs, the Shapley value takes into account the consumption sacrifices made by agents. Notice that here the externalities are fully symmetric, while they are (weakly) larger for downstream agents in our model. Our concavity result on V^o applies, as long as ω is convex and differentiable, and thus its Shapley value is in the optimistic anti-core and the pessimistic core.

7 Conclusion

Aside from transportation applications, it is uncommon to see a treatment of externalities in the literature on network flows and applications. The pipeline externalities problem we introduce optimizes flows in a pipeline shared by different users, all of which are attempting to extract value while subject to local damages that increase with flow through their locations. We first establish pessimistic and optimistic conditions on what coalitions of users are able to achieve. With these constraints in place, we find the group optimal solution (net benefits minus damages) and then use cooperative game theory to develop fair allocations to the users. We first propose solutions that simply allocate damages among users, including an average damage-based allocation and a marginal damage-based allocation. While this yields core allocations, these allocations ignore the (varying) sacrifices made by agents to reach the group-optimal usage of the pipeline. We turn to the Shapley value, which takes into account these sacrifices; further, we prove that the optimistic value function is concave, ensuring that the Shapley value lies in the core and is thus a stable solution.

In addition, although our pipeline externalities model is at first sight rather specialized, we show that it is surprisingly general: a number of well-known problems in the literature can be subsumed within this framework. These problems include Ambec and Sprumont's river sharing problem, as well as joint production problems as studied by Moulin and Shenker 1992. Additionally, we observe that while in general we assume asymmetric damage costs to serve the various users, when damages are limited to the first node we recover symmetric costs. If in addition we suppose that users demand at most one unit, the problem can be interpreted either as providing an excludable public good (Deb and Razzolini 1999) or as allocating objects when facing an increasing marginal cost function (Mackenzie and Trudeau 2023). If the marginal damage on the first edge is zero up to some flow value k, and infinite after that (as in river sharing problems), we recover the problem of distributing k free objects among $n > k$ agents (Varian 1974, Ohseto 2006).

We have supposed throughout that the pipeline is a line. As seen in the river sharing literature (see Beal et al. 2013 for review), a next step is to extend to more general topologies.

Finally, we discuss the allocations in $\mathcal{C}^p(P)\backslash\mathcal{A}^o(P)$. As seen in Example 6, those allocations typically allow agents that make larger sacrifices in terms of consumption for the collective benefits to extract more compensations, by taking into account that without their sacrifices, they would force others to consume much less. How much we want to reward some agents for this possibility is up for debate. As the pessimistic game is not convex, the description of these allocations is not as easy as for the optimistic anti-core.

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A Appendix

A.1 Proof of Theorem 1

To prove Theorem 1, we proceed with several lemmas.

First, we show that if we have a unitary utility vector ($u \in \mathcal{U}^1$), a modified version of the algorithm that serves K agents maximizes the surplus among flows that serve K agents.

Algorithm 2 *Input:* N, d, u, σ, K
Initialize $v_i = u_{i1}$, $\theta_i = d_{i1}$, $f_i = 0$, $\forall i \in N$
Initialize $\alpha = 0^N$, $W = 0$.
Initialize: $\pi_i = v_i - \sum_{k \leq i} \theta_k$, $\forall i \in N$
Initialize: $\gamma = \max_{i \in N} \pi_i$
While $\sum \alpha_i \leq K$ *pick* $j \in \arg\min_{i \in \arg\max_{k \in N} \pi_k} \sigma(i)$
  Update $\alpha = \alpha + e^j$
  Update $f_i = f_i + 1$, $\forall i \leq j$
  Update $W = W + \gamma$
  Update: $v_j = u_{j,\alpha_j+1}$
  Update: $\theta_i = d_{i,f_i+1}, \forall i \leq j$
  Update: $\pi_i = v_i - \sum_{k \leq i} \theta_k$, $\forall i \in N$
  Update: $\gamma = \max_{i \in N} \pi_i$

Return $\alpha^* = \alpha$, $f^* = f$, $W^* = W$

In this algorithm, we use $\delta_i(T)$ and $\mu_i(T)$ to be respectively, the agents downstream and upstream of i in $T \subseteq N$, including i if i is in T. Let ϕ_i be the position of agent i on the line.

Lemma 1 *For any problem* $(N, u, d) \in \mathcal{P}^1$ *and any* $K \leq |N|$, *Algorithm 2 provides a welfare-maximizing flow among those serving* K *agents.*

Proof. Let j_k be the agent picked at step k of the algorithm. Let $T^k = \{j_1, ..., j_k\}$. Let S be a coalition of size K. We build a pairing between T^K and S.

Step 1: Start with j_1. If there are members of S downstream of (or at) j_1, let S_1 be the closest to j_1 among them. Otherwise, let S_1 be the closest to j_1 among those upstream of j_1. Let $\tilde{S}^1 = S \backslash S_1$ and $S^1 = \{S_1\}$.

Step m : Consider j_m : If there are members of $\tilde{S}^{m-1}$ downstream of (or at) j_m, let S_m be the closest to j_m among them. Otherwise, let S_m be the closest to j_m among those upstream of j_m. Let $\tilde{S}^m = \tilde{S}^{m-1} \backslash S_m$ and $S^m = S^{m-1} \cup \{S_m\}$.

Thus, $\tilde{S}^k$ is the set of agents not assigned a position after step k, and S^k is the set of agents assigned a position at or before step k.

The proof consists of first proving 2 claims, then using the inequalities in the algorithm to conclude.

Claim 1: for all $k = 1, ..., |T|$, $\left|\delta_{S_k}(S^{k-1})\right| = \left|\delta_{S_k}(T^{k-1})\right|$.

At each step l, we place S_l downstream of (or at) j_l or as close to j_l as possible. If in all steps before k we place S_l downstream of (or at) j_l, then it's immediate that $\left|\delta_{S_k}(S^{k-1})\right| = \left|\delta_{S_k}(T^{k-1})\right|$.

Otherwise, we could have $\left|\delta_{S_k}(S^{k-1})\right| < \left|\delta_{S_k}(T^{k-1})\right|$ if some S_l, paired with j_l, could not be placed downstream of (or at) j_l but was placed upstream of the (eventual) location of S_k. But this is impossible, as S_l should have been placed as close as possible to j_l, and location S_k was still available and is closer.

It's also impossible to have $\left|\delta_{S_k}(S^{k-1})\right| > \left|\delta_{S_k}(T^{k-1})\right|$: this would require that at some step $l < k$, S_l is placed downstream of j_l, with the eventual location of S_k being in between j_l and S_l. But this is impossible, as S_l should have been placed at a location that's downstream of j_l, but as close to j_l as possible: location S_k was then available and closer to j_l.

Thus, Claim 1 is verified.

Claim 2: For all $k = 1, ..., |T|$, and all $l < k$, if $j_l \in \mu_{S_k}(T^{k-1})$, then $\phi(j_l) < \phi(S_l) < \phi(S_k)$. First, from Claim 1, $\left|\delta_{S_k}(S^{k-1})\right| = \left|\delta_{S_k}(T^{k-1})\right|$, and from its proof we know that $j_l \in \delta_{S_k}(T^{k-1}) \Leftrightarrow S_l \in \delta_{S_k}(S^{k-1})$. Since $|S| = \left|T^K\right|$, we then have that $\left|\mu_{S_k}(S^{k-1})\right| = \left|\mu_{S_k}(T^{k-1})\right|$, and $j_l \in \mu_{S_k}(T^{k-1}) \Leftrightarrow S_l \in \mu_{S_k}(S^{k-1})$. Thus, $\phi(S_l) < \phi(S_k)$.

It remains to show that $\phi(j_l) < \phi(S_l)$. Suppose it's not true. Then, by the pairing algorithm, it has to be because there is no spot available downstream of (or at) j_l. But S_k is downstream of j_l and was available when S_l was chosen. We thus have a contradiction and Claim 2 is satisfied.

We are now ready for the main part of the proof.

We write as $\rho_i(R)$ the marginal cost of serving agent i when agents in R have already been served.

$$\begin{aligned}
\rho_i(R) &= \sum_{j=1}^{i} d_{j,|\delta_j(R)|+1} - d_{j,|\delta_j(R)|} \\
&= \sum_{j=1}^{i} d_{j,|\delta_i(R)|+|\delta_j(\mu_i(R))|+1} - d_{j,|\delta_i(R)|+|\delta_j(\mu_i(R))|}
\end{aligned}$$

that is we add the demand of agent i to all his upstream edges. The quantity demanded at node j is the number of agents downstream of i plus the number of agents upstream of i that are also downstream of node j.

At step k of the algorithm, we have that

$$v_{j_k} - \rho_{j_k}\left(T^{k-1}\right) \geq v_j - \rho_j\left(T^{k-1}\right)$$

for all $j \in N \backslash T^{k-1}$. In particular, we have

$$v_{j_k} - \theta_{j_k}\left(T^{k-1}\right) \geq v_{S_k} - \theta_{S_k}\left(T^{k-1}\right)$$

We show that $\rho_{S_k}\left(T^{k-1}\right) \leq \rho_{S_k}\left(S^{k-1}\right)$, or

$$\sum_{j=1}^{S_k}\left[\begin{array}{c} d_{j,\left|\delta_{S_k}(T^{k-1})\right|+\left|\delta_j\left(\mu_{S_k}(T^{k-1})\right)\right|+1} \\ -d_{j,\left|\delta_{S_k}(T^{k-1})\right|+\left|\delta_j\left(\mu_{S_k}(T^{k-1})\right)\right|} \end{array}\right] \leq \sum_{j=1}^{S_k}\left[\begin{array}{c} d_{j,\left|\delta_{S_k}(S^{k-1})\right|+\left|\delta_j\left(\mu_{S_k}(S^{k-1})\right)\right|+1} \\ -d_{j,\left|\delta_{S_k}(S^{k-1})\right|+\left|\delta_j\left(\mu_{S_k}(S^{k-1})\right)\right|} \end{array}\right]$$

By Claim 1, $\left|\delta_{S_k}\left(T^{k-1}\right)\right| = \left|\delta_{S_k}\left(S^{k-1}\right)\right|$.

By Claim 2, $\left|\delta_j\left(\mu_{S_k}(T^{k-1})\right)\right| \leq \left|\delta_j\left(\mu_{S_k}(S^{k-1})\right)\right|$ for all $j = 1, ..., \phi(S_k)$

Thus, for all $j = 1, ..., \phi(S_k)$, $\left|\delta_{S_k}\left(T^{k-1}\right)\right| + \left|\delta_j\left(\mu_{S_k}(T^{k-1})\right)\right| \leq \left|\delta_{S_k}\left(S^{k-1}\right)\right| + \left|\delta_j\left(\mu_{S_k}(S^{k-1})\right)\right|$. By the convexity of the cost functions, it is thus no cheaper to add i to S^{k-1} than to T^{k-1}, i.e. $\rho_{S_k}\left(T^{k-1}\right) \leq \rho_{S_k}\left(S^{k-1}\right)$.

We thus have, for all $k = 1, ..., |T|$, that

$$v_{j_k} - \rho_{j_k}\left(T^{k-1}\right) \geq v_{S_k} - \rho_{S_k}\left(T^{k-1}\right) \geq v_{S_k} - \rho_{S_k}\left(S^{k-1}\right)$$

Summing up, we obtain

$$\begin{aligned} \sum_{k=1}^{|T|}\left(v_{j_k} - \rho_{j_k}\left(T^{k-1}\right)\right) &\geq \sum_{k=1}^{|T|}\left(v_{S_k} - \rho_{S_k}\left(S^{k-1}\right)\right) \\ \sum_{i\in T} u_i - \Theta(T) &\geq \sum_{i\in S} u_i - \Theta(S) \end{aligned}$$

where $\Theta(R)$ is the total cost to serve R. Thus, we have that the surplus when serving T is no smaller than when serving S. ■

Lemma 2 *Let K^* be the number of iterations in Algorithm 1. Let W^K be the net surplus generated by Algorithm 2 with K as input. Then $W^{K^*} \geq W^K$ for all K.*

Proof. First, notice that Algorithm 2 picks a flow that is equivalent, in terms of surplus, to Algorithm 1 if K^* is inputed in Algorithm 2.

Let γ^k be the net surplus of step k. By definition of Algorithm 1, we have $\gamma^{K^*} > 0 \geq \gamma^{K^*+1}$. By the assumptions on u and d, $k < l \Rightarrow \gamma^k \geq \gamma^l$, and thus $\gamma^l \leq 0$ for all $l > K^*$. Therefore, it is immediate that $W^{K^*} \geq W^k$ for all $k > K^*$.

In the same manner, we have $\gamma^k \geq \gamma^{K^*} > 0$ for all $k < K^*$, and thus $W^{K^*} \geq W^k$ for all $k < K^*$.

Combining, we obtain the desired result. ■

Lemma 3 *For any problem $(N, d, u) \in \mathcal{P}^1$, Algorithm 1 provides a welfare-maximizing flow.*

Proof. Let $W(S)$ be the net surplus of serving coalition S. By Lemma 2, $W(\{1, .., j_{K^*}\}) \geq W(\{1, ...j_K\})$ for all K. By Lemma 1, for all K we have $W(\{1, ...j_K\}) \geq W(S)$ for all S such that $|S| = K$. Combining, we have $W(\{1, .., j_{K^*}\}) \geq W(S)$ for all $S \subseteq N$, as desired. ■

Lemma 4 *Suppose that Algorithm 1 provides a welfare-maximizing flow for all problems $(N, u, d) \in \mathcal{P}^1$. Then Algorithm 1 provides a welfare-maximizing flow for all problems $(N, d, u) \in \mathcal{P}^D$.*

Proof. For each agent i, let q_i denote the largest k for which $u_{ik} > d_{ik}$.

For each agent i, construct a series of agents $(i_1, ..., i_{q_i})$. Let $\bar{N}$ be that new set of agents.

We suppose that agents are ordered on the line with i_k being closer to the source than $j_l \Leftrightarrow \{i < j\}$ or $\{i = j \text{ and } k < l\}$.

Construct $\bar{d}$ in the following way

For all $i \in N$, $\bar{d}_{i_1} = d_i$, and $\bar{d}_{i_k} = (0, 0, ...)$ for all $k > 1$.

Construct $\bar{u}$ in the following way:

$\bar{u}_{i_k} = (u_{ik}, 0, 0...)$ for all $i \in N$, all $k = 1, ..., q_i$.

It is obvious that $\bar{u} \in \mathcal{U}^1$, so by assumption, Algorithm 1 is optimal for the problem $(\bar{N}, \bar{u}, \bar{d})$

It is obvious that we can always serve agent i_k before i_l for $k < l$ (there might be ties). Thus, agent i_k in the problem $(\bar{N}, \bar{u}, \bar{d})$ has the same utility as agent i has for his k^{th} unit. The damage is also the same, given that damage is null between new agents built from the same original agent, and that a given agent in the original tree is replaced by a line of his new agents, all with zero damage to go from one to the other.

Thus, the problem $(\bar{N}, \bar{u}, \bar{d})$ has the same net utility as (N, u, d). If $\bar{\alpha}$ is the optimal consumption vector for $(\bar{N}, \bar{u}, \bar{d})$, then an optimal consumption vector for (N, u, d) is α such that $\alpha_i = \sum_{l=1}^{q_i} \bar{a}_{i_l}$. It is easy to see that the consumption vector picked by Algorithm 1 on (N, u, d) is surplus-equivalent to that consumption vector. ■

Proof of Theorem 1. Combining the 4 above Lemmas, we obtain that Algorithm 1 yields a welfare-maximizing flow for all problems $(N, u, d) \in \mathcal{P}^D$ as desired. Notice in particular that for σ and σ', the resulting flows might not be identical, in which case both are welfare-maximizing. ■

A.2 Proof of Theorem 2

Proof. Fix $P = (N, U, D) \in \mathcal{P}$. Recall that $V^o(N, P) = V^p(N, P)$.

Next, fix $S \subset N$. If $y \in \mathcal{A}^o(P)$, then $y(S) \leq V^o(S, P)$ and $y(N \backslash S) \leq V^o(N \backslash S, P)$. Together this implies that $V^o(N, P) - V^o(N \backslash S, P) \leq y(S) \leq V^o(S, P)$, or equivalently, that

$$\sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right) - \left(\sum_{i \in N \backslash S} U_i(\bar{x}_i^{N \backslash S}) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}^{N \backslash S}\right)\right) \leq y(S) \leq \sum_{i \in S} U_i(\bar{x}_i^S) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}^S\right)$$

If $y \in \mathcal{C}^p(P)$, then $y(S) \geq V^p(S, P)$ and $y(N \backslash S) \geq V^p(N \backslash S, P)$. Together this implies that $V^p(S, P) \leq y(S) \leq V^p(N, P) - V^p(N \backslash S, P)$, or equivalently, that

$$\begin{array}{c} \sum_{i \in S} U_i(\hat{x}_i^S) \\ -\sum_{i \in N}\left(D_i\left(\hat{x}_{\delta(i)}^S + \bar{x}_{\delta(i)}^{N \backslash S}\right) - D_i\left(\bar{x}_{\delta(i)}^{N \backslash S}\right)\right) \end{array} \leq y(S) \leq \begin{array}{c} \sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right) \\ -\sum_{i \in S} U_i(\hat{x}_i^{N \backslash S}) \\ +\sum_{i \in N}\left(D_i\left(\hat{x}_{\delta(i)}^{N \backslash S} + \bar{x}_{\delta(i)}^S\right) - D_i\left(\bar{x}_{\delta(i)}^S\right)\right) \end{array}$$

where $\hat{x}^T$ is the optimal consumption of T when $N \backslash T$ has chosen first.

To obtain the desired result, we need to show that

$$\begin{array}{c} \sum_{i \in S} U_i(\hat{x}_i^S) \\ -\sum_{i \in N}\left(D_i\left(\hat{x}_{\delta(i)}^S + \bar{x}_{\delta(i)}^{N \backslash S}\right) - D_i\left(\bar{x}_{\delta(i)}^{N \backslash S}\right)\right) \end{array} \leq \begin{array}{c} \sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right) \\ -\left(\sum_{i \in N \backslash S} U_i(\bar{x}_i^{N \backslash S}) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}^{N \backslash S}\right)\right) \end{array}$$

and

$$\sum_{i \in S} U_i(\bar{x}_i^S) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}^S\right) \leq \begin{array}{c} \sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right) \\ -\left(\sum_{i \in S} U_i(\hat{x}_i^{N \backslash S}) - \sum_{i \in N}\left(D_i\left(\hat{x}_{\delta(i)}^{N \backslash S} + \bar{x}_{\delta(i)}^S\right) - D_i\left(\bar{x}_{\delta(i)}^S\right)\right)\right) \end{array}.$$

We start with the first inequality, which can be rewritten as

$$\sum_{i \in S} U_i(\hat{x}_i^S) + \sum_{i \in N \backslash S} U_i(\bar{x}_i^{N \backslash S}) - \sum_{i \in N} D_i\left(\hat{x}_{\delta(i)}^S + \bar{x}_{\delta(i)}^{N \backslash S}\right) \leq \sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right)$$

Recall that $\sum_{i \in N} U_i(\bar{x}_i) - \sum_{i \in N} D_i\left(\bar{x}_{\delta(i)}\right) = \max_{x \in \mathbb{R}_+^N} \sum_{i \in N}\left(U_i(x_i) - D_i\left(x_{\delta(i)}\right)\right)$. Clearly $\left(\hat{x}^S, \bar{x}^{N \backslash S}\right) \in \mathbb{R}_+^N$, and the net benefit of this consumption vector is precisely the LHS of our inequality. Thus, $\left(\hat{x}^S, \bar{x}^{N \backslash S}\right)$ is a feasible consumption vector but not necessarily an optimal one, and the result follows.

The second inequality follows using the same argument as the first one.

Thus, any allocation in $\mathcal{A}^o(P)$ is also in $\mathcal{C}^p(P)$. ■

A.3 Proof of Theorem 3

Proof. Fix $P \in \mathcal{P}$ and $m \in M(P)$. It is easy to see that by condition 3 for an eligible m we have that y^m is budget-balanced. Let $S \subset N$. We show that $y_S^m \leq V^o(S, P)$.

We have that

$$
\begin{aligned}
y_S^c &= \sum_{i\in S}\left(U_i(\bar{x}_i) - \sum_{j\in N} m_i^j\right) \\
&= \sum_{i\in S} U_i(\bar{x}_i) - \sum_{j\in N} m_S^j \\
&\leq \sum_{i\in S} U_i(\bar{x}_i) - \sum_{j\in N} D_j\left(\bar{x}_{S\cap\delta(i)}\right) \\
&\leq \max_{x\in\mathbb{R}_+^S} \sum_{i\in S} U_i(x_i) - \sum_{i\in N} D_i\left(x_{\delta(i)}\right) \\
&= V^o(S,P)
\end{aligned}
$$

where the first inequality comes from condition 2 of an eligible m. The second inequality comes from the definition of $V^o(S,P)$, and the fact that the consumption $x_i = \bar{x}_i$ for all $i \in S$ and $x_i = 0$ otherwise is feasible for S when choosing first, but not necessarily optimal, with the third line providing exactly the value obtained by S if it chooses that consumption. ■

A.4 Proof of Theorem 4

Proof. Fix $P \in \mathcal{P}$, and suppose that $\bar{x}_i^{\{i\}} = \bar{x}_i$ for all $i \in N$. By Theorem 3, we have $\mathcal{M}(P) \subseteq \mathcal{A}^o(P)$. We show that $\mathcal{A}^o(P) \subseteq \mathcal{M}(P)$.

By Lemma 6, $\bar{x}_i^S = \bar{x}_i$ for all $S \ni i$. Thus, the agents' consumption stay constant throughout, and

$$
V^o(S,P) = \sum_{i\in S} U_i(\bar{x}_i) - \sum_{i\in N} D_i\left(\sum_{\substack{j\in S\\ j\in\delta(i)}} \bar{x}_j\right)
$$

for all $S \subseteq N$.

Notice that since consumption is fixed, we can write $V^o(\cdot,P)$ in the following way: for all $S \subseteq N$, let $U(S,P) = \sum_{i\in S} U_i(\bar{x}_i)$, and let $\psi(S,P) = \sum_{i\in N} D_i\left(\sum_{\substack{j\in S\\ j\in\delta(i)}} \bar{x}_j\right)$. We then have $V^o(S,P) = U(S,P) - \psi(S,P)$. It is easy to see that U is a linear game. $z_i = U_i(\bar{x}_i)$ for all $i \in N$ is its only anti-core allocation. Thus, we have that $y \in \mathcal{A}^o(P)$ is such that $y_i = U_i(\bar{x}_i) - z_i$. Thus, we can focus, on ψ.

Let z be such that $z(N) = \psi(N,P)$. Start with agent 1. By the anti-core constraints, we must have

$$
D_1(\bar{x}_1) \leq z_1 \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus 1})
$$

Take agent 2. By the anti-core constraints, we must have

$$
\begin{aligned}
D_1(\bar{x}_2) + D_2(\bar{x}_2) &\leq z_2 \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus 2}) + D_2(\bar{x}_{N\setminus 1}) - D_2(\bar{x}_{N\setminus\{1,2\}}). \\
D_1(\bar{x}_{\{1,2\}}) + D_2(\bar{x}_2) &\leq z(\{1,2\}) \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus\{1,2\}}) + D_2(\bar{x}_{N\setminus 1}) - D_2(\bar{x}_{N\setminus\{1,2\}})
\end{aligned}
$$

We can see that this implies that agent 1 is not responsible for the cost in D_2. We can rewrite $z_1 = z_1^1$ and $z_2 = z_2^1 + z_2^2$ with

$$
\begin{aligned}
D_1(\bar{x}_1) &\leq z_1^1 \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus 1}) \\
D_1(\bar{x}_2) &\leq z_2^1 \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus 2}) \\
D_1(\bar{x}_{\{1,2\}}) &\leq z_2^1 + z_2^1 \leq D_1(\bar{x}_N) - D_1(\bar{x}_{N\setminus\{1,2\}}) \\
D_2(\bar{x}_2) &\leq z_1^2 \leq D_2(\bar{x}_{N\setminus 1}) - D_2(\bar{x}_{N\setminus\{1,2\}})
\end{aligned}
$$

Continue adding agents one by one, to obtain that $z_i = \sum_{j\in N} z_i^j$ with $z_i^j = 0$ if $j > i$, $D_j(\bar{x}_{S\cap\delta(j)}) \leq z_S^j \leq D_j(\bar{x}_{\delta(j)}) - D_j(\bar{x}_{\delta(j)\setminus S})$ for all $S \subseteq N$ and all $j \in N$, and $\sum_{i\in N} z_i^j = D_j(\bar{x}_{\delta(j)})$. Thus, $z \in M(P)$ and $y \in \mathcal{M}(P)$. ■

A.5 Proof of Theorem 5

To show the result for continuous problems, we need the following lemmas.

The first lemma just formally states that if an agent consumes a strictly positive quantity, the first order condition with respect to his consumption is binding, and thus the marginal utility of his consumption is equal to the sum of marginal damages on all segments on his path from the source.

Lemma 5 *For any problem $P \in \mathcal{P}^C$, let $S \subseteq N$ and $i \in S$ such that $\bar{x}_i^S > 0$. Then, $u_i(\bar{x}_i^S) = \sum_{j\in\mu(i)} d_j(\bar{x}_{\delta(j)}^S)$. If $\bar{x}_i^S = 0$ then $u_i(0) \leq \sum_{j\in\mu(i)} d_j(\bar{x}_{\delta(j)}^S)$.*

Proof. Comes from First Order Conditions for the problem for coalition S. ■

The next lemma states that adding more agents to the maximization cannot lead to our original agents consuming more, but the overall consumption cannot decrease.

Lemma 6 *For any problem $P \in \mathcal{P}^C$, if $S \subset T$, then for all $j \in S$, $\bar{x}_j^S \geq \bar{x}_j^T$. Moreover, for all $S' \subseteq S$, $\sum_{j\in S'} \bar{x}_j^S \leq \sum_{j\in S'\cup(T\setminus S)} \bar{x}_j^T$.*

Proof. We first show that $\bar{x}_j^S \geq \bar{x}_j^T$, for $T = S_{+i}$ and $j \in S$. Suppose that $\bar{x}_S^S = 0$. It is easy to see that agent i joining S cannot lead to $\bar{x}_S^{S_{+i}} > 0$ given the increasing marginal damages. Thus, suppose that $\bar{x}_S^S > 0$.

Let s_1 be the most upstream agent in S, and suppose first that $S = \{s_1\}$. If $s_1 < i$, by Lemma 5 we have $u_{s_1}(\bar{x}_{s_1}^R) = \sum_{k\leq s_1} d_k(\bar{x}_R^R)$, for $R \in \{S, S_{+i}\}$, as $\bar{x}_{\delta(j)}^S = R$ for all $j \leq s_1$. By the assumptions on the utility and damage functions, $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$ implies that $\bar{x}_{s_1}^{S_{+i}} + \bar{x}_i^{S_{+i}} < \bar{x}_{s_1}^S$, which is impossible since $\bar{x}_i^{S_{+i}} \geq 0$. If $s_1 > i$, by Lemma 5 we have $u_{s_1}(\bar{x}_{s_1}^S) = \sum_{k\leq i} d_k(\bar{x}_S^S) + \sum_{i<k\leq s_1} d_k(\bar{x}_S^S)$ and $u_{s_1}(\bar{x}_{s_1}^{S_{+i}}) = \sum_{k\leq i} d_k(\bar{x}_{S_{+i}}^{S_{+i}}) + \sum_{i<k\leq s_1} d_k(\bar{x}_S^{S_{+i}})$. By the assumptions on the utility and damage functions, $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$ implies that $\bar{x}_{s_1}^{S_{+i}} + \bar{x}_i^{S_{+i}} < \bar{x}_{s_1}^S$, which is also impossible. Thus, in all cases, $\bar{x}_{s_1}^{S_{+i}} \leq \bar{x}_{s_1}^S$.

Thus, suppose that $|S| > 1$. By Lemma 5, if $s_1 < i$, we have $u_{s_1}(\bar{x}_{s_1}^R) = \sum_{k\leq s_1} d_k(\bar{x}_R^R)$, for $R \in \{S, S_{+i}\}$. Suppose that $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$. By the convexity of the utility function and the concavity of the damage functions, this is only possible if $\bar{x}_S^S > \bar{x}_{S_{+i}}^{S_{+i}}$, which we rewrite as $\bar{x}_{S\setminus s_1}^S + \bar{x}_{s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}} + \bar{x}_{s_1}^{S_{+i}}$. We thus need $\bar{x}_{s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}} + \bar{x}_{s_1}^{S_{+i}} - \bar{x}_{S\setminus s_1}^S$. Since by assumption we have $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$, a necessary condition is $\bar{x}_{S\setminus s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}}$. If $s_1 < i$ we have $u_{s_1}(\bar{x}_{s_1}^S) = \sum_{k\leq i} d_k(\bar{x}_S^S) + \sum_{i<k\leq s_1} d_k(\bar{x}_S^S)$ and $u_{s_1}(\bar{x}_{s_1}^{S_{+i}}) = \sum_{k\leq i} d_k(\bar{x}_{S_{+i}}^{S_{+i}}) + \sum_{i<k\leq s_1} d_k(\bar{x}_S^{S_{+i}})$. A necessary condition for $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$ is that $\bar{x}_{S\setminus s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}}$.

Move now to s_2, the second most upstream agent in S. If $s_2 < i$, we have that $u_{s_2}(\bar{x}_{s_1}^R) = \sum_{k\leq s_1} d_k(\bar{x}_R^R) + \sum_{s_1<k\leq s_2} d_k(\bar{x}_{R\setminus s_1}^R)$ for $R \in \{S, S_{+i}\}$. By the above result, $\bar{x}_S^S > \bar{x}_{S_{+i}}^{S_{+i}}$ and $\bar{x}_{S\setminus s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}}$ and by the assumptions on utility and damages functions, we must have $\bar{x}_{s_2}^{S_{+i}} > \bar{x}_{s_2}^S$. If $s_2 > i$, we have $u_{s_2}(\bar{x}_{s_1}^S) = \sum_{k\leq s_1} d_k(\bar{x}_S^S) + \sum_{s_1<k\leq s_2} d_k(\bar{x}_{S\setminus s_1}^S)$ and $\sum_{k\leq s_1} d_k(\bar{x}_{S_{+i}}^{S_{+i}}) + \sum_{s_1<k\leq i} d_k(\bar{x}_{S_{+i}\setminus s_1}^{S_{+i}}) + \sum_{i<k\leq s_2} d_k(\bar{x}_{S\setminus s_1}^{S_{+i}})$. By the above result, $\bar{x}_S^S > \bar{x}_{S_{+i}}^{S_{+i}}$ and $\bar{x}_{S\setminus s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}} > \bar{x}_{S\setminus s_1}^{S_{+i}}$ and by the assumptions on utility and damages functions, we must have $\bar{x}_{s_2}^{S_{+i}} > \bar{x}_{s_2}^S$. Thus, in all cases $\bar{x}_{s_2}^{S_{+i}} > \bar{x}_{s_2}^S$

Repeat the argument for each agent in S, going from upstream to downstream, to obtain that $\bar{x}_j^{S_{+i}} > \bar{x}_j^S$ for all $j \in S$, contradicting $\bar{x}_{S\setminus s_1}^S > \bar{x}_{S_{+i}\setminus s_1}^{S_{+i}}$, the necessary condition for $\bar{x}_{s_1}^{S_{+i}} > \bar{x}_{s_1}^S$. Thus, $\bar{x}_{s_1}^{S_{+i}} \leq \bar{x}_{s_1}^S$.

Suppose that we have shown that $\bar{x}_{s_l}^{S_{+i}} \leq \bar{x}_{s_l}^S$ for $l = 1, ..., m-1$. We show that $\bar{x}_{s_m}^{S_{+i}} \leq \bar{x}_{s_m}^S$. By Lemma 5, we have $u_{s_m}(\bar{x}_{s_m}^R) = \sum_{k\leq s_m} d_k(\bar{x}_{\delta(k)}^R)$, for $R \in \{S, S_{+i}\}$. Because $\bar{x}_{s_l}^{S_{+i}} \leq \bar{x}_{s_l}^S$ for $l < m$, we must have that $\sum_{k\leq s_l} d_k(\bar{x}_{\delta(k)}^{S_{+i}}) \geq \sum_{k\leq s_l} d_k(\bar{x}_{\delta(k)}^S)$ for all $l < m$. Thus, a necessary condition for $\bar{x}_{s_m}^{S_{+i}} > \bar{x}_{s_m}^S$ is $\sum_{s_{m-1}<k\leq s_m} d_k(\bar{x}_{\delta(k)}^S) > \sum_{s_{m-1}<k\leq s_m} d_k(\bar{x}_{\delta(k)}^{S_{+i}})$. Given our assumptions on the damage functions, this can be rewritten as $\bar{x}_{\delta(s_m)}^S > \bar{x}_{\delta(s_m)}^{S_{+i}} \geq \bar{x}_{\delta(s_m)\setminus i}^{S_{+i}}$. Thus, a necessary condition is $\bar{x}_{\delta(s_m)}^S > \bar{x}_{\delta(s_m)\setminus i}^{S_{+i}}$. Repeat the argument for each agent in S downstream of s_m, going from upstream to downstream, to obtain that $\bar{x}_j^{S_{+i}} > \bar{x}_j^S$ for all $j \in S$ such that $j > s_m$, contradicting $\bar{x}_{\delta(s_m)}^S > \bar{x}_{\delta(s_m)\setminus i}^{S_{+i}}$, the necessary condition for $\bar{x}_{s_m}^{S_{+i}} > \bar{x}_{s_m}^S$. Thus, $\bar{x}_{s_m}^{S_{+i}} \leq \bar{x}_{s_m}^S$.

All together, $\bar{x}_j^{S_{+i}} \leq \bar{x}_j^S$ for all $j \in S$, as desired. The result can be extended agent by agent to a general T.

For the second part of the lemma, we first show the result for $S' = S$. Notice that if $\sum_{i\in S} \bar{x}_i^S > \sum_{i\in T} \bar{x}_i^T$ then there is less flow between the source and s_1 in T than in S. By Lemma 5, it would be optimal for agent s_1 to increase its consumption, contradicting the first part of the lemma. Thus, $\sum_{i\in S} \bar{x}_i^S \leq \sum_{i\in T} \bar{x}_i^T$, or $\sum_{i\in T\setminus S} \bar{x}_i^T \geq \sum_{i\in S} \left(\bar{x}_i^S - \bar{x}_i^T\right)$. By the first part of the lemma, $\sum_{i\in S} \left(\bar{x}_i^S - \bar{x}_i^T\right) \geq \sum_{i\in S'} \left(\bar{x}_i^S - \bar{x}_i^T\right)$ for all $S' \subseteq S$, and thus $\sum_{i\in S'} \bar{x}_i^S \leq \sum_{i\in S'\cup(T\setminus S)} \bar{x}_i^T$ for all $S' \subseteq S$. ■

For $i \in N \cup \{0\}$, $j \in N\backslash i$ and a flow x such that $x_i > 0$, let $\gamma_{ij}(x)$ be the marginal cost of sending a marginal quantity from i to j, which not only includes the marginal damage along the way, but also the marginal utlility lost by reducing the consumption of i to increase the consumption of j. We have that if $0 < i < j$, $\gamma_{ij}(x) = u_i(x_i) + \sum_{i<k\leq j} d_k(x_{\delta(k)})$, if $i = 0$, $\gamma_{ij}(x) = \sum_{k\leq j} d_k(x_{\delta(k)})$ and if $i < j$, $\gamma_{ij}(x) = u_i(x_i) - \sum_{j<k\leq i} d_k(x_{\delta(k)})$.

Lemma 7 *For any problem $P \in \mathcal{P}^C$, all $S \subseteq N$ and all $i, j \in S \cup \{0\}$ such that $\bar{x}_l^S > 0$ for $l = i, j$, then $\gamma_{ik}(\bar{x}^S) = \gamma_{jk}(\bar{x}^S) = \gamma_{0k}(\bar{x}^S)$ for all $k \in N\backslash \{i, j\}$.*

Proof. Take $i \in S$ such that $\bar{x}_i^S > 0$. By definition, $\gamma_{ik}(\bar{x}^S) = u_i(\bar{x}_i^S) + \sum_{i<l\leq k} d_l(\bar{x}_{\delta(l)}^S)$ if $i < k$ and $\gamma_{ik}(x) = u_i(\bar{x}_i^S) - \sum_{k<l\leq i} d_l(\bar{x}_{\delta(l)}^S)$ otherwise. By Lemma 5, $u_i(\bar{x}_i^S) = \sum_{j\in\mu(i)} d_j(\bar{x}_{\delta(j)}^S)$, and thus, in both cases, $\gamma_{ik}(\bar{x}^S) = \sum_{l\leq k} d_l(x_{\delta(l)}) = \gamma_{0k}(\bar{x}^S)$. ■

We are now ready to show the concavity of V^o.

Proof of Theorem 5. We show that for $i, j \in N\backslash S$, $V^o(S_{+ij}) - V^o(S_{+j}) \leq V^o(S_{+i}) - V^o(S)$. Notice first that if $\bar{x}_j^{S_{+j}} = 0$, then adding j has no impact, so the result is trivial. If $\bar{x}_i^{S_{+i}} = 0$, then by Lemma 6 $\bar{x}_i^{S_{+ij}} = 0$ and $V^o(S_{+ij}) - V^o(S_{+j}) = V^o(S_{+i}) - V^o(S) = 0$. Thus, suppose $\bar{x}_j^{S_{+j}} > 0$ and $\bar{x}_i^{S_{+i}} > 0$.

Start from $\bar{x}^S$, and consider the marginal cost of supplying agent i. If that supply comes from the source, the marginal cost is $\sum_{j\leq i} d_j(\bar{x}_{\delta(j)}^S)$. If the supply comes from agent $k < i$, the marginal cost is $u_k(\bar{x}_k^S) + \sum_{k<l\leq i} d_l(\bar{x}_{\delta(l)}^S)$. If the supply comes from agent $k > i$, the marginal cost is $u_k(\bar{x}_k^S) - \sum_{l=i+1}^k d_l(\bar{x}_{\delta(l)}^S)$. By Lemma 7 all these marginal costs have the same value. Let $\Theta_i^S(0)$ be that value. We compare this marginal cost to $\Theta_i^{S_{+j}}(0)$, i.e. starting from $\bar{x}^{S_{+j}}$.

If $i < j$, then by the second part of Lemma 6, the marginal cost to supply from the source, and thus from all agents, is $\sum_{k\leq i} d_k(\bar{x}_{\delta(k)}^{S_{+j}}) \geq \sum_{k\leq i} d_k(\bar{x}_{\delta(k)}^S)$. Thus, $\Theta_i^{S_{+j}}(0) \geq \Theta_i^S(0)$.

If $j < i < n$, there exists $k > i$ and consider the marginal cost to supply from agent k : $u_k(\bar{x}_k^{S_{+j}}) - \sum_{i<l\leq k} d_l(\bar{x}_{\delta(l)}^{S_{+j}}) \geq u_k(\bar{x}_k^S) - \sum_{i<l\leq k} d_l(\bar{x}_{\delta(l)}^S)$ by the first part of Lemma 6. Thus, $\Theta_i^{S_{+j}}(0) \geq \Theta_i^S(0)$.

If $i = n$ and $j < i - 1$, there exists $j < k < i = n$ and consider the marginal cost to supply from k : $u_k(\bar{x}_k^{S_{+j}}) + \sum_{k<l\leq i} d_l(\bar{x}_{\delta(l)}^{S_{+j}}) \geq u_k(\bar{x}_k^S) + \sum_{k<l\leq i} d_l(\bar{x}_{\delta(l)}^S)$ by Lemma 6. Thus, $\Theta_i^{S\cup j}(0) \geq \Theta_i^S(0)$.

It remains the case where $j = n - 1$ and $i = n$. Consider the marginal cost to supply i from j : $u_j(\bar{x}_j^{S_{+j}}) + d_i(0)$. By Lemma 5, we have that $u_j(\bar{x}_j^{S_{+j}}) = \sum_{k\in\mu(j)} d_k(\bar{x}_{\delta(k)}^{S_{+j}})$. Thus, the marginal cost to supply agent i is $\sum_{k\in\mu(j)} d_k(\bar{x}_{\delta(k)}^{S_{+j}}) + d_i(0) \geq \sum_{k\in\mu(j)} d_k(\bar{x}_{\delta(k)}^S) + d_i(0)$ by Lemma 6. Thus, $\Theta_i^{S\cup j}(0) \geq \Theta_i^S(0)$.

Therefore, in all cases, $\Theta_i^{S_{+j}}(0) \geq \Theta_i^S(0)$.

Now consider $x \leq \bar{x}_i^{S_{+ij}}$. We argue that $\Theta_i^{S_{+j}}(x) \geq \Theta_i^S(x)$. Suppose it's not true, i.e. that $\Theta_i^{S_{+j}}(x) < \Theta_i^S(x)$, and change U_i such that $u_i(x) = \Theta_i^S(x)$. We would then have that $\bar{x}_i^{S_{+i}} = x$, and by assumption on the utility function, $\bar{x}_i^{S_{+ij}} > x = \bar{x}_i^{S_{+i}}$, contradicting Lemma 6. Thus, $\Theta_i^{S_{+j}}(x) \geq \Theta_i^S(x)$ for all $x \in \left[0, \bar{x}_i^{S_{+ij}}\right]$.

Thus, the result above shows that when i starts consuming, its marginal cost of consumption is no smaller when j is present, and the same is true for each (marginal) unit consumed until $\bar{x}_i^{S_{+j}}$.

Since $V^o(S_{+ij}) - V^o(S_{+j})$ is the sum of marginal utility minus marginal cost for each (marginal) unit from 0 to $\bar{x}_i^{S_{+ij}}$, and $V^o(S_{+i}) - V^o(S)$ is the sum of marginal utility minus marginal cost of each (marginal) unit from 0 to $x_i^{S_{+i}}$, with $x_i^{S_{+ij}} \leq x_i^{S_{+i}}$, we have that

$$\begin{aligned} V^o(S_{+ij}) - V^o(S_{+j}) &= \int_0^{\bar{x}_i^{S_{+ij}}} \left(u_i(x) - \Theta_i^{S_{+j}}(x)\right) dx \\ &\leq \int_0^{\bar{x}_i^{S_{+ij}}} \left(u_i(x) - \Theta_i^{S}(x)\right) dx \\ &\leq \int_0^{\bar{x}_i^{S_{+i}}} \left(u_i(x) - \Theta_i^{S}(x)\right) dx \\ &= V^o(S_{+i}) - V^o(S) \end{aligned}$$

where the first inequality comes from $\Theta_i^{S_{+j}}(x) \geq \Theta_i^S(x)$ for all $x \in \left[0, \bar{x}_i^{S_{+ij}}\right]$ and the second one from $u_i(x) - \Theta_i^S(x) \geq 0$ for all $x \in \left[\bar{x}_i^{S_{+ij}}, \bar{x}_i^{S_{+i}}\right]$ (by definition of $\bar{x}_i^{S_{+i}}$), and thus V^o is concave. ■

A.6 Proof of Theorem 8

Proof. Fix a river sharing problem (N, v, e) and let P^e be the corresponding pipeline externalities problem.

$W^{ATS}(S, U, e) \leq W^{UTI}(S, U, e)$ follows from Ambec and Sprumont 2002. We show $V^p(S, P^e) \leq W^{ATS}(S, U, e)$.

Fix (N, U, e), $S \subset N$ and examine $V^p(S, P^e)$. To define $V^p(S, P^e)$, we need to examine $V^o(N \backslash S, P^e)$. Given the assumptions of the game, notably the fact that the utility function is non decreasing, we have that $\sum_{j \in N \backslash S} z_j^* = \sum_{j=m_{-S}}^n e_j$, where m_{-S} is the agent in $N \backslash S$ that is furthest from the source of the river (lowest index). Thus, if $1 \notin S$, then $m_{-S} = 1$ and there is nothing to consume for coalition S, and $V^p(S, P^e) = 0 \leq W^{ATS}(S, U, e)$.

If $1 \in S$, then $m_{-S} > 1$ and let $\hat{T}$ be the largest consecutive coalition containing 1 that is a subset of S. Then, $V^p(S, P^e) = \max_{(z_i)_{i \in \hat{T}}} \sum_{i \in \hat{T}} v_i(z_i)$ under the constraints that $\sum_{\substack{j \geq i \\ j \in \hat{T}}} z_i \leq \sum_{\substack{j \geq i \\ j \in \hat{T}}} e_i$ for all $i \in \hat{T}$. It is easy to see that $V^p(S, P^e)$ is the problem $W^{ATS}(S, U, e)$ with the additional constraint that $z_i = 0$ for all $i \in S \backslash \hat{T}$. Thus, $V^p(S, P^e) \leq W^{ATS}(S, U, e)$. ■