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Abstract

We introduce the notion of preconvexity, which extends the familiar concept of convexity found in cooperative games with transferable utility. In a convex game, the larger the group joined by an agent, the larger the marginal value brought to the group by that agent. By contrast, in strictly preconvex games, an agent's marginal contribution is initially decreasing (when joining small groups), and it eventually becomes increasing at (and above) some critical group size. As a consequence, the core of a preconvex game may be empty. Defining the property of semicohesiveness (related to marginal contributions at this critical group size), we prove that it is sufficient to guarantee a nonempty core. We also propose a new solution for the set of preconvex games; and we characterize this solution by combining three axioms which are natural in our framework. A stronger cohesiveness property (guaranteeing that our solution falls in the core) is also studied. Some additional results are provided for the special case of anticonvex games, for which marginal contributions are always non-increasing.

JEL classification numbers: C71; D63.

Keywords: cooperation, allocation, core, preconvexity, cohesiveness.

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1 Introduction

Convex games (Shapley, 1971) play a special role in cooperative game theory. These games, in which the marginal contribution of every agent i weakly increases as the group of players joined by i grows larger, always have a non-empty core; and every marginal contribution vector (obtained by selecting an arbitrary order and assigning to each player his marginal contribution to the group preceding him) is in that core. Shapley (1953)'s value, which is the average of these marginal contribution vectors is thus a core allocation. Moreover, a game is convex if and only if the convex hull of these marginal contribution vectors equals the core (Ichiishi, 1981).

The present paper studies an extension of the class of convex games: the members of this wider family are called *preconvex games*, based on the behavior of the marginal contribution vectors. In a preconvex game with n players, there is a critical size k such that the marginal contribution of any agent i to a group is a decreasing (an increasing) function when i joins a coalition containing at most (least) $n-k$ agents. We refer to the integer k as the *degree* of preconvexity of the game. An interesting application of preconvex games is the problem where (i) a large group of n producers decide to form a cooperative and (ii) the net revenue generated when a number m of producers work together is given by the function $f(m)$, which first exhibits decreasing returns to scale (below some threshold $\bar{m}$) and then increasing returns to scale (above $\bar{m}$). The situation can be modeled by means of a preconvex game with degree $k = n - \bar{m}$.[1]

Our preconvex games do not always have a non-empty core; and this emphasizes the need to find verifiable conditions guaranteeing the existence of core allocations. We offer a condition (called k-*semicohesiveness*) guaranteeing the non-vacuity of the core for preconvex games —see Theorem 1. Specifically, k-semicohesiveness requires that the value of at least one coalition of size $n-k+1$ be no larger than the sum of the marginal contributions of its members.

Moreover, we adjust the Shapley value to propose a solution for preconvex games. Fix an ordering of the agents. Since the last $k-1$ marginal contributions behave as in a convex game, we keep these contributions unchanged in the allocation associated to the ordering. The remaining agents are each assigned their respective marginal contribution to the group of size $n-k$, except the last agent, whose allocation is defined to maintain budget balance. In the spirit of the Shapley value, the average of the resulting payoff vectors over all orderings is then

[1] In the same vein, it can be shown that some of the Production Allocation Games studied in Bahel and Trudeau (2018) are preconvex.

taken. We call this allocation the k-adjusted average marginal value. Of course, if $k = n$, we recover convex games, and the n-adjusted average marginal value then coincides with the Shapley value. A stronger cohesiveness condition (called k-cohesiveness) is shown to guarantee the stability of our solution for preconvex games (see Theorem 2).[2]

We also propose an axiomatization of the adjusted average marginal value (defined over the set of preconvex games) by using Anonymity and Cone Additivity (two well-known axioms), and Sequentially Marginal Equivalence (or SM-Equivalence), an adaptation of Young (1985)'s Marginality axiom correcting the players' marginal contributions if they decrease as the group joined becomes larger. Specifically, SM-Equivalence says that, if a player i's corrected vector of marginal contributions does not change from a preconvex game to a convex game, then i's payoff should be the same in these two games. Theorem 3 then states that the adjusted average marginal value is the unique solution meeting Anonymity, Cone Additivity and SM-Equivalence.

Convex games, which have the highest possible degree of preconvexity $k = n$ (*i.e.*, marginal contributions are always non-decreasing), are an extreme case within our family of preconvex games. The other extreme members of the family are the so-called *anticonvex games*, which have the lowest possible degree of preconvexity ($k = 1$) —their marginal contributions are always non-increasing. Interestingly, for anticonvex games, we prove that k-cohesiveness is both necessary and sufficient for the existence of stable allocation; and our adjusted average marginal value falls in the core whenever it is non-empty (see Theorems 5 and 6).

The paper is divided as follows. After introducing our notation and recalling some basic concepts of cooperative game theory in Section 2, we define and illustrate preconvex games in Section 3. Section 4 defines our solution for preconvex games, and it offers conditions for the non-vacuity of the core. In Section 5, we axiomatize the adjusted average marginal value. Section 6 offers further results specific to anticonvex games. All proofs are in the appendix.

2 Preliminaries

Let $n \geq 3$ and fix the set of players $N = \{1, 2, \ldots, n\}$. Given that N is fixed, a *cooperative game with transferable utility* (or *TU game*, for short) is defined by a characteristic function $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. For any coalition $S \in 2^N$, the real number $v(S)$ stands for the *worth* of S.

[2] In the context of cost problems, a property similar to k-cohesiveness was studied by Mackenzie and Trudeau (2023); and their property is called Inclusion Cost Coverage.

We use the symbol $\subseteq$ for weak set inclusion; and $\subsetneq$ stands for strict set inclusion. A TU game v is *monotonic* if $v(S) \leq v(T)$, for any coalitions $S, T \in 2^N$ s.t. $S \subseteq T$. In other words, in a monotonic TU game, larger coalitions are worth more. Note from what precedes that a monotonic TU game is necessarily *nonnegative* [that is, $v(S) \geq 0$ for all $S \in 2^N$] since all coalitions contain the empty set. Throughout this paper we will restrict attention to monotonic TU games. Let then $\mathcal{G}$ denote the set of monotonic TU games (with player set N).

For all $x \in \mathbb{R}^N$ and $S \in 2^N$, define the shorthand $x_S := \sum\limits_{i \in S} x_i$. Given a TU game (N, v), an *allocation* is a tuple $x \in \mathbb{R}^N$ satisfying $x_N = v(N)$ [efficiency]. An allocation x is called *stable* if $x_S \geq v(S)$ [rationality of the coalition S], for any $\emptyset \neq S \subsetneq N$. The *core* of (N, v) is the set of stable allocations, that is to say,

$$Core(v) \equiv \{x \in \mathbb{R}^N \mid x_N = v(N) \text{ and } x_S \geq v(S), \text{ for all } \emptyset \neq S \subsetneq N\}.$$

A TU game is called *balanced* if its core is nonempty. The notion of the core was formally introduced by Gillies (1953) as a set-valued solution concept for TU games.

We introduce the following notational conventions. Given $i \in N$, we will often omit curly brackets by writing i instead of $\{i\}$. For all $t \in \{0, 1, \ldots, n\}$, we write $\mathbf{C}_t = \{S \in 2^N : |S| = t\}$ to denote the collection of coalitions containing exactly t players. Moreover, given any $i \in N$, let $\mathbf{C}_t^{-i} = \{S \in \mathbf{C}_t : i \notin S\}$. Finally, we will need the shorthands $\overline{\mathbf{C}}_t = \bigcup\limits_{k=t}^{n} \mathbf{C}_k$, $\overline{\mathbf{C}}_t^{-i} = \bigcup\limits_{k=t}^{n} \mathbf{C}_k^{-i}$, and $\underline{\mathbf{C}}_t = \bigcup\limits_{k=0}^{t} \mathbf{C}_k$. For any TU game (N, v), any player $i \in N$ and any $S \subseteq N \setminus i$, we use

$$mc_i^v(S) = v(S \cup i) - v(S) \tag{1}$$

to denote the marginal contribution of player i to player set S.

3 Preconvex games: definition and examples

Recall that a TU game v is *convex* if $mc_i^v(S) \leq mc_i^v(T)$, for all $i \in N$ and all $S \subseteq T \subseteq N \setminus i$. Put differently, in a convex game, the marginal contribution of a player i is an increasing (with respect to set inclusion) function of the coalition joined by i. We define below a new family of TU games for which the players' marginal contributions do not necessarily satisfy this property.

Definition 1. *Let $k \in \{1, \ldots, n\}$ and consider $v \in \mathcal{G}$. We say that the TU game v is k-**preconvex** if, for every $i \in N$, we have:*

(a) $mc_i^v(S) \leq mc_i^v(T), \forall T \in \overline{\mathbf{C}}_{n-k+1}^{-i}, \forall S \in 2^T$;

(b) $mc_i^v(S) \geq mc_i^v(T), \forall T \in \underline{\mathbf{C}}_{n-k}^{-i}, \forall S \in 2^T \setminus \{\emptyset\}$.

Condition (a) in Definition 1 says that, in a k-preconvex TU game, the marginal contribution of a player i to a coalition T containing at least $n-k+1$ players is weakly higher than i's marginal contribution to a subset of T; and by Condition (b) the reverse inequality must hold for coalitions T that contain at most $n-k$ players (and their nonempty subsets S). We will often refer to k as the *degree* of preconvexity of the TU game v.

Let us denote by $\mathcal{S}_k$ the set containing all k-preconvex TU games. We also define $\mathcal{S} := \bigcup_{k=1}^{n} \mathcal{S}_k$; and we call a TU game v *preconvex* if $v \in \mathcal{S}$.

The following remark follows directly from the combination of the respective definitions of convexity and preconvexity.

Remark 1. *For all $v \in \mathcal{G}$, we have the following equivalence*

$$[v \text{ is } n - \text{preconvex}] \Leftrightarrow [v \text{ is convex}].$$

It is well known from Shapley (1971) that convex games are balanced. One of our objectives in this paper is to find conditions guaranteeing a nonempty core in the larger class of preconvex TU games. We provide below an example of a preconvex TU game.

Example 1. *Suppose $N = \{1,2,3,4\}$ and consider the TU game v defined by*

$$v(S) = \begin{cases} 200 & \text{if } S = N; \\ 100 & \text{if } |S| = 3; \\ 60 & \text{if } |S| = 2; \\ 0 & \text{if } |S| \leq 1. \end{cases}$$

Taking $k = 2$ and picking any distinct $i,j,l,t \in N$, note that $v(ijl) - v(jl) = 40 < 100 = v(ijlt) - v(ijl) = 100$; and hence the condition (a) of Definition 1 is satisfied.[3] *Moreover, one can write $v(ij) - v(j) = 60 > 40 = v(ijl) - v(jl)$, thus showing condition (b). We therefore conclude that v is 2-preconvex (and* ***not*** *4-preconvex, 3-preconvex or 1-preconvex). Note that the TU game v is balanced: for instance, the tuple $x = (100, 40, 40, 20)$ is stable since it is efficient and satisfies coalitional rationality (for all coalitions $S \in 2^N$).*

[3] To ease on notation, we often omit curly brackets by writing for instance $v(ijk)$ instead of $v(\{i,j,k\})$.

Definition 2. *We say that a game* $v \in \mathcal{G}$ *is anticonvex if* $v \in \mathcal{S}_1$*, that is to say, if* v *is* 1*-preconvex.*

Noting that $\mathbf{C}_n = \{N\}$ (and hence $\mathbf{C}_n^{-i} = \emptyset$ for any $i \in N$), observe that condition (a) in Definition 1 becomes mute when $k = 1$. In other words, a game v is anticonvex if and only if it satisfies condition (b) in Definition 1 for every player $i \in N$. The following example illustrates the notion of anticonvexity.

Example 2. *Suppose* $N = \{1, 2, 3, 4\}$ *and consider the TU game* w *defined by*

$$w(S) = \begin{cases} 200 & \textit{if } S = N; \\ 149 & \textit{if } |S| = 3 \textit{ and } 4 \in S; \\ 151 & \textit{if } |S| = 3 \textit{ and } 4 \notin S; \\ 80 & \textit{if } |S| = 2; \\ 0 & \textit{otherwise.} \end{cases}$$

Letting $k = 1$*, it is not difficult to check that condition (b) is satisfied. Indeed, for any distinct* $i, j, l, t \in N$*, observe that the following inequalities hold:* $\underbrace{w(ijlt) - v(jlt)}_{\in\{49,51\}} \leq 60 \leq \underbrace{w(ijl) - w(jl)}_{\in\{69,71\}} \leq w(ij) - w(j) = 80$*. We thus conclude that* w *is anticonvex.*
Also notice that the TU game w *is balanced: in particular, the tuple* $x = (49, 51, 51, 49)$ *is in the core because it is efficient and satisfies coalitional rationality for all coalitions* $S \in 2^N$*.*

We have emphasized for the reader that the TU games v and w described respectively in Example 1 and Example 2 are balanced (even though neither one of these two games is convex). The following section will provide an explanation for this common feature of the two games. We conclude this section with the remark below.

Remark 2. *Note that the respective classes of convex games and anticonvex games are diametrically opposed within the family of preconvex games: the former (latter) class has the highest (lowest) possible degree of preconvexity.*

4 Some new values for TU games

Let us denote by $\Pi(N)$ the set of permutations of our player set $N = \{1, \ldots, n\}$. Given a permutation $\pi \in \Pi(N)$ and a player $i \in N$, we write $\pi(i)$ to refer to i's position (or rank) under π. Conversely, letting π^{-1} be the inverse of π, we write

$\pi^{-1}(\ell)$ to refer to the player ranked in ℓth position under π (for any possible rank $\ell = 1, \ldots, n$).

For any rank $\ell = 0, 1, \ldots, n-1$, one can then define

$$F_\ell^\pi := \{\pi^{-1}(\ell+1), \ldots, \pi^{-1}(n)\}.$$

as the set of players with rank greater than ℓ. Moreover, for all $i \in N$, we will write $\bar{F}^\pi(i) = F^\pi_{\pi(i)} = \{\pi^{-1}(\pi(i)+1), \ldots, \pi^{-1}(n)\}$ to denote the set of players with rank greater than player i's. Finally, for any rank $\ell = 0, 1, \ldots, n-1$, any player $i \in N$ and any TU game $v \in \mathcal{G}$, let us use the shorthand

$$m_i^{\pi,\ell}(v) = mc_i^v(F^\pi_{\ell-1} \setminus i). \tag{2}$$

Recall that an *efficient value* for TU games is a map $\tilde{x} : \mathcal{G} \to \mathbb{R}^N$ such that $\tilde{x}_N(v) = v(N)$, for all $v \in \mathcal{G}$. As an example, given a permutation π, the π-descending *marginal worth value* x^π is defined by $x_i^\pi(v) = mc_i^v(\bar{F}^\pi(i))$. The well-known Shapley value, x^{Sh}, can then be computed as the average of all descending marginal worth values: for all $v \in \mathcal{G}$,

$$x^{Sh}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} x^\pi(v). \tag{3}$$

The following definition introduces some new values for TU games.

Definition 3. ***(i)***. *Consider a permutation $\pi \in \Pi(N)$. We call k-**adjusted** π-**marginal value** the efficient value ${}^k\tilde{x}^\pi$ defined as follows: for all $v \in \mathcal{G}$ and $i \in N \setminus \{\pi^{-1}(n)\}$,*

$${}^k\tilde{x}_i^\pi(v) = \begin{cases} mc_i^v(\bar{F}^\pi(i)) & \text{if } \pi(i) < k; \\ m_i^{\pi,k}(v) & \text{if } k \leq n-1 \text{ and } k \leq \pi(i) \leq n-1. \end{cases} \tag{4}$$

(ii). *We define the k-**adjusted average marginal value** the efficient value ${}^k\tilde{x}$ defined as follows: for all $v \in \mathcal{G}$, ${}^k\tilde{x}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} {}^k\tilde{x}^\pi(v)$.*

Note that Equation (4) defines the payoff assigned to the first $n-1$ players served under π, with the payoff of the last player determined by efficiency, that is, ${}^k\tilde{x}_{\pi^{-1}(n)} = v(N) - {}^k\tilde{x}_{N \setminus \{\pi^{-1}(n)\}}$. With a few manipulations, we obtain ${}^k\tilde{x}_i^\pi(v) = v(F^\pi_{k-1}) - \sum_{j=k}^{n-1} mc^v_{\pi^{-1}(j)}\left(F^\pi_{k-1} \setminus \pi^{-1}(j)\right)$ if $\pi(i) = n$. The following observation obtains from the combination of Definition 3 and the definition of the Shapley value.

Remark 3. *For any convex game v (i.e., $k = n$ and $v \in \mathcal{S}_k$), it holds that* ${}^k\tilde{x}(v) = x^{Sh}(v)$.

In other words, the n-adjusted marginal value coincides with the Shapley value on the set of convex games. For classes of k-preconvex game such that $k \neq n$, this coincidence does not hold. The following Example illustrates Definition 3.

Example 3. *Recall the TU game v introduced in Example 1 (which is 2-preconvex); and consider the permutation $\pi = \overline{1234} \in \Pi(N)$. First, it is easy to see that the π-descending marginal value assigns to v the allocation $x_i^\pi(v) = (100, 40, 60, 0)$. Second, note that our 2-adjusted marginal value associated with π gives*

$$\begin{aligned} {}^2\tilde{x}^\pi(v) &= (v(N) - v(234), \underbrace{v(234) - v(34)}_{m_2^{\pi,2}(v)}, \underbrace{v(234) - v(24)}_{m_3^{\pi,2}(v)}, v(24) + v(34) - v(234)) \\ &= (100, 40, 40, 20). \end{aligned}$$

As explained in Example 1, the allocation ${}^2\tilde{x}^\pi(v) = (100, 40, 40, 20)$ is in the core of v. On the other hand, the π-descending marginal worth allocation $x^\pi(v) = (100, 40, 60, 0)$ is not stable (since players 2 and 4 jointly receive less than their worth 60). Finally, taking the average for all permutations, it is not difficult to see that the Shapley value and the k-adjusted marginal value coincide for v: ${}^2\tilde{x}(v) = x^{Sh}(v) = (50, 50, 50, 50)$.

Considering now the game w of Example 2 (which is 1-preconvex) and the permutation $\pi' = \overline{4321} \in \Pi(N)$, we get $x_i^{\pi'}(v) = (0, 80, 71, 49)$ and

$$\begin{aligned} {}^1\tilde{x}_i^{\pi'}(w) &= (w(N) - m_4^{\pi',1}(w) - m_3^{\pi',1}(w) - m_2^{\pi',1}(w), m_2^{\pi',1}(w), m_3^{\pi',1}(w), m_4^{\pi',1}(w)) \\ &= (49, 51, 51, 49). \end{aligned}$$

Note from Table 1 that the Shapley value and the 1-adjusted average marginal value pick different allocations for the game w: The Shapley value falls out of the core ($50\frac{1}{6} + 50\frac{1}{6} + 50\frac{1}{6} < 151$), while the 1-adjusted average marginal value falls in the core. The game w in Example 2 thus illustrates how this new value can correct shortcomings of the Shapley value of preconvex games.

We examine circumstances under which a k-preconvex game is balanced. Furthermore, we want to determine conditions guaranteeing that our k-adjusted marginal values fall in $Core(v)$. As already pointed out, it is well known that every convex game ($k = n$) is balanced.

Order π	${}^{k}\widetilde{x}^{\pi}(w)$	${}^{Sh}x^{\pi}(w)$
$\overline{1234}$	$(51, 51, 51, 47)$	$(51, 69, 80, 0)$
$\overline{1243}$	$(51, 51, 49, 49)$	$(51, 69, 0, 80)$
$\overline{1324}$	$51, 51, 51, 47)$	$(51, 80, 69, 0)$
$\overline{1342}$	$(51, 49, 51, 49)$	$(51, 0, 69, 80)$
$\overline{1423}$	$(51, 51, 49, 49)$	$(51, 80, 0, 69)$
$\overline{1432}$	$(51, 49, 51, 49)$	$(51, 0, 80, 69)$
$\overline{2134}$	$(51, 51, 51, 47)$	$(69, 51, 80, 0)$
$\overline{2143}$	$(51, 51, 49, 49)$	$(69, 51, 0, 80)$
$\overline{2314}$	$(51, 51, 51, 47)$	$(80, 51, 69, 0)$
$\overline{2341}$	$(49, 51, 51, 49)$	$(0, 51, 69, 80)$
$\overline{2413}$	$(51, 51, 49, 49)$	$(80, 51, 0, 69)$
$\overline{2431}$	$(49, 51, 51, 49)$	$(0, 51, 80, 69)$
$\overline{3124}$	$(51, 51, 51, 47)$	$(69, 80, 51, 0)$
$\overline{3142}$	$(51, 49, 51, 49)$	$(69, 0, 51, 80)$
$\overline{3214}$	$(51, 51, 51, 47)$	$(80, 69, 51, 0)$
$\overline{3241}$	$(49, 51, 51, 49)$	$(0, 69, 51, 80)$
$\overline{3412}$	$(51, 49, 51, 49)$	$(80, 0, 51, 69)$
$\overline{3421}$	$(49, 51, 51, 49)$	$(0, 80, 51, 69)$
$\overline{4123}$	$(51, 51, 49, 49)$	$(71, 80, 0, 49)$
$\overline{4132}$	$(51, 49, 51, 49)$	$(71, 0, 80, 49)$
$\overline{4213}$	$(51, 51, 49, 49)$	$(80, 71, 0, 49)$
$\overline{4231}$	$(49, 51, 51, 49)$	$(0, 71, 80, 49)$
$\overline{4312}$	$(51, 49, 51, 49)$	$(80, 0, 71, 49)$
$\overline{4321}$	$(49, 51, 51, 49)$	$(0, 80, 71, 49)$
average	$(50\frac{1}{2}, 50\frac{1}{2}, 50\frac{1}{2}, 48\frac{1}{2})$	$(50\frac{1}{6}, 50\frac{1}{6}, 50\frac{1}{6}, 49\frac{1}{2})$

Table 1: Shapley value and the 1-adjusted average marginal value of Example 2

Given $k \in \{1, \dots n-1\}$, $v \in \mathcal{S}_k$, and $T \in \mathbf{C}_{n-k+1}$, we say that v is *cohesive at* T if it holds that

$$\sum_{i \in T} v(T \setminus i) \leq (n-k)\ v(T). \tag{5}$$

Furthermore, a TU game $v \in \mathcal{S}_k$ will be called:

(a) *k-semicohesive* if there exists $T \in \mathbf{C}_{n-k+1}$ s.t. v is cohesive at T;

(b) *k-cohesive* if, for all $T \in \mathbf{C}_{n-k+1}$, v is cohesive at T.

Obviously, the definitions above mean that k-semicohesiveness is a weaker property than k-cohesiveness. Intuitively, note from k-cohesiveness at T that the worth of any $(n-k+1)$-player coalition T [in a k-cohesive game] should be relatively high (compared to a specific linear combination of the respective worths of its subsets of size $n-k$). Moreover, condition (5) for cohesiveness at T can be rewritten as

$$\begin{aligned} v(T) &\leq \sum_{i \in T} [v(T) - v(T \setminus i)] \\ &= \sum_{i \in T} mc_i^v(T \setminus i). \end{aligned}$$

For all permutations π that serve the players in T last (i.e., $T = F^{\pi}_{k-1}$), we get

$$\begin{aligned} v(F^{\pi}_{k-1}) &\leq \sum_{i \in F^{\pi}_{k-1}} mc^v_i(F^{\pi}_{k-1} \setminus i) \\ &= \sum_{j=k}^{n-1} mc^v_{\pi^{-1}(j)} \left(F^{\pi}_{k-1} \setminus \pi^{-1}(j)\right) + mc^v_{\pi^{-1}(n)} \left(F^{\pi}_{k-1} \setminus \pi^{-1}(n)\right) \end{aligned}$$

Using the fact that ${}^k\tilde{x}^{\pi}_i(v) = v(F^{\pi}_{k-1}) - \sum_{j=k}^{n-1} mc^v_{\pi^{-1}(j)} \left(F^{\pi}_{k-1} \setminus \pi^{-1}(j)\right)$ if $\pi(i) = n$, we can restate the above equation as

$$ {}^k\tilde{x}^{\pi}_i(v) \leq mc^v_{\pi^{-1}(n)} \left(F^{\pi}_{k-1} \setminus \pi^{-1}(n)\right) \tag{6}$$

if $\pi(i) = n$. Recalling (4) and (6), note that cohesiveness at T guarantees that the payoff of any member of T [under our allocations ${}^k\tilde{x}^{\pi}$] worsens when he is relegated to the last position in the permutation.

We provide two examples to illustrate k-semicohesiveness and k-cohesiveness.

Example 4. *Suppose $N = \{1, 2, 3, 4\}$ and consider the TU game u defined by*

$$u(S) = \begin{cases} 200 & \textit{if } S = N; \\ 100 & \textit{if } |S| = 3; \\ 60 & \textit{if } |S| = 2; \\ 40 & \textit{if } S = \{1\} \textit{ or } S = \{2\}; \\ 30 & \textit{if } S = \{3\} \textit{ or } S = \{4\}. \end{cases}$$

One can check that u is a 3-preconvex game ($k = 3$). We now argue that u is 3-semicohesive. Since $k = 1$, we have $n - k + 1 = 2$; and it hence suffices to find a two-player coalition satisfying (5). Taking $T = \{3, 4\}$, note that $u(3) + u(4) = 60 \leq 60 = u(34)$; and hence v is 3-semicohesive. On the other hand, v is ***not*** *3-cohesive: taking $T = \{1, 2\}$ gives $u(1) + u(2) = 80 > 60 = u(12)$.*

Example 5. *Considering any 4-player game v that is 2-preconvex, we have $N = \{1, 2, 3, 4\}$, $n = 4$, and $k = 2$, and hence $n - k + 1 = 3$. Recalling (5), v is 3-cohesive if and only if $v(ij) + v(ik) + v(jk) \leq 2v(ijk)$ for any distinct $i, j, k \in N$. As an illustration, the TU game introduced in Example 1 (which is indeed a 4-player, 2-preconvex game) is 2-cohesive: for any distinct i, j, k, remark that $v(ij) + v(ik) + v(jk) = 60 + 60 + 60 = 180 \leq 2\ v(ijk) = 2 \times 100 = 200$.*

Next, considering a 4-player game w that is 1-preconvex, we have $N = \{1, 2, 3, 4\}$, $n = 4$, $k = 1$; and hence $n - k + 1 = 4$. From (5) and $\mathbf{C}_4 = \{N\}$, note that w is 1-cohesive if and only if $w(123) + w(124) + w(134) + w(234) \leq 3w(1234)$. Ob-

serve for instance that the TU game w introduced in Example 2 (which is indeed a 4-player, 1-preconvex game) is 1*-cohesive:* $w(123)+w(124)+w(134)+w(234) = 151+149+149+149 = 598 \leq 3\ w(1234) = 3 \times 200 = 600$.

Interestingly, we prove below that k-semicohesiveness is a sufficient condition for balancedness of our k-preconvex games; and the stronger condition k-cohesiveness is necessary and sufficient for every one of our adjusted marginal values to fall in the core of the considered k-preconvex game.

Theorem 1. *Let $v \in \mathcal{S}_k$, where $k \in \{1,\dots,n-1\}$. Then the following properties hold true:*

$$\textbf{\textit{(a)}}\quad [v \textit{ is } k\textit{-semicohesive}] \Rightarrow [Core(v) \neq \emptyset];$$
$$\textbf{\textit{(b)}}\quad [v \textit{ is } k\textit{-cohesive}] \Leftrightarrow [\,{}^k\tilde{x}^{\pi}(v) \in Core(v), \forall \pi \in \Pi(N)].$$

The following theorem claims that, for any preconvex game that is k-cohesive, our solution selects a core allocation.

Theorem 2. *Let $v \in \mathcal{S}_k$, where $k \in \{1,\dots,n-1\}$. Then the following implication holds:*

$$[v \textit{ is } k\textit{-cohesive}] \Rightarrow [{}^k\tilde{x}(v) \in Core(v) \neq \emptyset];$$

The following example illustrates Theorem 2.

Example 6. *Recall that TU game w introduced in Example 2 is k-cohesive. From Theorem* 1*, the 1-adjusted π-marginal value falls in the core (for every permutation $\pi \in \Pi(N)$). All these values are listed in the column ${}^k\tilde{x}^{\pi}(w)$ in Table 1. One can indeed check that all of them are in the core of game w. Moreover, as stated in Theorem* 2*, the average of these values ${}^k\tilde{x} = (50\frac{1}{2}, 50\frac{1}{2}, 50\frac{1}{2}, 48\frac{1}{2})$ (i.e., the 1-adjusted average marginal value) also falls in the core.*

In the case of a k-preconvex game that is not k-cohesive, note that the k-adjusted average marginal value may or may not fall in the core (even if the core is nonempty).

5 Axioms and characterization result

Let $\mathcal{S}^\star = \bigcup_{k=1}^{n} \mathcal{S}_k$ denote the set of preconvex games (for any degree $k = 1,\dots,n$); and recall that S_n coincides with the set of convex games. The solutions considered in this section are values of the type $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$. We introduce below three axioms that are natural properties in our framework.

5.1 Anonymity

First, we define the well-known Anonymity property in our context. Given a permutation $\sigma \in \Pi(N)$ and a game $v \in \mathcal{S}^\star$, define the TU game σv as follows: $\forall S \in 2^N$, $\sigma v(S) = v(\sigma(S))$. Fixing a permutation σ and a degree $k = 1, \ldots, n$, it is not difficult to see from Definition 1 that, if v is k-preconvex ($v \in \mathcal{S}_k$), then the permuted game σv is also k-preconvex ($\sigma v \in \mathcal{S}_k$). We may thus formally define this axiom as follows.

Definition 4. *A value $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$ satisfies **Anonymity**, for all $\sigma \in \Pi(N)$, $v \in \mathcal{S}^\star$ and $i \in N$, it holds that $\hat{x}_i(\sigma v) = \hat{x}_{\sigma(i)}(v)$.*

Anonymity essentially says that a value should not use the identity (or label) of any given player in assigning the payoffs. In particular, if two players have their respective roles in the game swapped, then this should result in a swap of their payoffs as well.

5.2 Cone Additivity

Next, we use a restricted additivity property. Note that requiring a value to satisfy full additivity (in the sense of Shapley (1953)) would not make sense in our context, since the set $\mathcal{S}^\star = \bigcup_{k=1}^{n} \mathcal{S}_k$ is not closed under addition. However, remark that each $\mathcal{S}_k$ (for any $k = 1, \ldots, n$) is a convex cone (within the vector space $\mathcal{R}^{2^N}$). In other words, we have $(\alpha v + \beta w) \in \mathcal{S}_k$ whenever $v, w \in \mathcal{S}_k$ and $\alpha, \beta \geq 0$. It is therefore natural to require the following property.

Definition 5. *A value $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$ satisfies **Cone Additivity** if we have*

$$\hat{x}(v + w) = \hat{x}(v) + \hat{x}(w) \text{ if } v, w \in \mathcal{S}_k \text{ (for some } k = 1, \ldots, n).$$

In words, a value is cone-additive if, for any two games v and w that belong to the same cone of preconvex games, the payoffs vector for the sum $v + w$ is exactly the sum of the respective payoff vectors associated with v and w. The following remark states two results that obtain by repeatedly applying Cone Additivity.

Remark 4. *Suppose that $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$ is cone-additive. Then for all $k \in \{1, \ldots, n\}$, $v \in \mathcal{S}_k$, and $p, q \in \{1, 2, 3, \ldots\}$, we have: **(a)** $\hat{x}(pv) = p\hat{x}(v)$; and **(b)** $\hat{x}(\frac{p}{q}v) = \frac{p}{q}\hat{x}(v)$.*

5.3 Sequentially Marginal Equivalence

We adapt to our context the axiom of marginality, which was introduced by Young (1985) and subsequently used by Chun (1989) and De Clippel and Serrano (2008), just to name a few. Our adaptation of the vector of marginal contributions restricts attention to increments associated with sets whose cardinality is above the inflection point of the preconvex game at hand.

Given any $i \in N$, $T \in \mathbf{C}_{n-k}^{-i}$ and $v \in \mathcal{S}_k$ (for some $k = 1, \ldots, n$), define the vector of increments $\partial_i^T v \in \mathbb{R}^{P_T^i}$, where $P_T^i = \{S \in 2^{N \setminus i} : S \subseteq T \text{ or } T \subseteq S\}$.

For all $S \in P_T^i$,

$$\partial_i^T v(S) = \begin{cases} mc_i^v(S) & \text{if } T \subsetneq S \text{ or } k = n; \\ mc_i^v(T) & \text{if } \emptyset \neq S \subseteq T \text{ and } k \neq n; \\ v(T \cup i) - \sum_{j \in T} mc_j^v((T \cup i) \setminus j) & \text{otherwise.} \end{cases} \tag{7}$$

Note that $\partial_i^T v$ represents a list of adjusted increments of v (with respect to player i) for all nonempty subsets that are comparable to T (in the sense of set inclusion). Observe from (7) that the purpose of the adjustment is to correct for any violations of the property of increasing marginal contributions (exhibited by convex games). We will also use the notation $\partial_i v := (\partial_i^T v)_{T \in \mathbf{C}_{n-k}^{-i}}$. We say that $\partial_i v$ is the *adjusted vector of marginal contributions* of i in v.

Definition 6. *A value $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$ satisfies* ***Sequentially Marginal Equivalence*** *(or SM-Equivalence, for short) if, for all $v \in \mathcal{S}^\star$ and $i \in N$, we have the implication*

$$[\exists w \in \mathcal{S}_n^\star \text{ s.t. } \partial_i v = \partial_i w] \Rightarrow [\hat{x}_i(v) = \hat{x}_i(w)].$$

The definition of SM-Equivalence can be summarized as follows: if player i's adjusted vector of marginal contributions in v coincides with her vector of marginal contributions in some convex game w, then player i's payoff should be the same in v and w.

5.4 Characterization

We are now ready to axiomatize our solution $x^* : \mathcal{S}^\star \to \mathbb{R}^N$, which is defined by $x^*(v) = {}^k\tilde{x}(v)$ [for all $v \in \mathcal{S}^\star$ and $k = 1, \ldots, n$ s.t. $v \in \mathcal{S}_k$], as the only value on $\mathcal{S}^\star = \bigcup_{k=1}^{n} \mathcal{S}_k$ that meets all three axioms. Recall that ${}^k\tilde{x}(v)$ was introduced in Definition 3-(ii).

Theorem 3. *The adjusted average marginal value x^* is the unique value on $\mathcal{S}^\star$ that satisfies Anonymity, Cone Additivity and SM-Equivalence.*

6 The case of anticonvex games

Having stated general results in the previous sections, we now provide more specific results for the special case of anticonvex games, which have the lowest possible degree of preconvexity ($k = 1$). In particular, we derive the necessary and sufficient condition for the core to be non-empty; and we provide a full description of the core of an anticonvex game.[4]

As noted in Example 5, the 1-cohesiveness property is equivalent to 1-semicohesiveness (as $\mathbf{C}_n = \{N\}$). The property is also a balancedness condition (see Bondareva (1963); Shapley (1971)) and thus necessary for the core to be non-empty. We can then state the following result.

Theorem 4. *Let $v \in \mathcal{S}_1$, Then the following equivalence holds:*

$$[v \textit{ is } 1\textit{-cohesive}] \Leftrightarrow [{}^1\tilde{x}(v) \in Core(v) \neq \emptyset].$$

To study the core of an anticonvex game, it is useful to define an associated TU game. For any game v, define the game V as follows: for all $S \subseteq N$,

$$V(S) \equiv \begin{cases} \sum\limits_{i \in N \setminus S} v(N \setminus i) - (n - s - 1)v(N) & \text{if } 1 \leq s \equiv |S| \leq n-1, \\ v(S) & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 1. *If v is 1-preconvex then we have ${}^1\tilde{x}^\pi(v) = x^\pi(V)$, for all $\pi \in \Pi(N)$.*

An immediate corollary of Lemma 1 is that ${}^1\tilde{x}(v) = x^{Sh}(V)$, that is to say, our adjusted allocation ${}^1\tilde{x}(v)$ coincides with the Shapley allocation of the associated game V. Interestingly, we also prove that the core of any 1-cohesive anticonvex games v is exactly the core of the associated game V: this result allows a full description of the core of v.

Theorem 5. *For any $v \in \mathcal{S}_1$, we have $Core(v) = Core(V)$. Moreover, if v is balanced, $Core(v)$ is precisely the convex hull of the family of adjusted marginal vectors $\{{}^1\tilde{x}^\pi(v)\}_{\pi \in \Pi(N)}$.*

Our final result states that, for anticonvex games, our 1-adjusted average marginal value is related to the well-known nucleolus.

[4] Atay and Trudeau (2024) study queueing games with a variable number of machines. When agents must queue on a single machine and pay a fixed cost for that machine, we obtain anticonvex games: the gain from sharing the fixed cost is mitigated by congestion on the machines, which worsens as coalitions grow.

For any subset $S \in 2^N \setminus \{\emptyset, N\}$ and any allocation $x \in \mathbb{R}^N$, call the quantity $e^v(S,x) = v(S) - x_S$ the *excess* of S at x. Let $\tilde{e}^v(x) \in \mathbb{R}^{2^n-2}$ be the vector listing, in a non-increasing manner, the excesses of the respective coalitions at x. Given a game v, the *nucleolus* $\eta(v)$ is defined as the (unique) allocation that lexicographically minimizes this ordered vector of excesses $\tilde{e}^v(x)$ —see Schmeidler (1969). We can now state the following result for anticonvex games.

Theorem 6. *If $v \in \mathcal{S}_1$ then ${}^1\tilde{x}(v) = \eta(V)$. Moreover, if v is 1-cohesive, then ${}^1\tilde{x}(v) = \eta(v)$.*

For completeness, we provide below an example of game v that is not 1-cohesive and such that ${}^1\tilde{x}(v) \neq \eta(v)$.

Example 7. *Let $N = \{1,2,3,4\}$ and consider the TU game v defined as follows.*

S	$v(S)$	S	$v(S)$	S	$v(S)$
$\{1\}$	0	$\{1,2\}$	*35*	$\{1,2,3\}$	*62*
$\{2\}$	0	$\{1,3\}$	*54*	$\{1,2,4\}$	*106*
$\{3\}$	0	$\{1,4\}$	*76*	$\{1,3,4\}$	*125*
$\{4\}$	0	$\{2,3\}$	*54*	$\{2,3,4\}$	*125*
		$\{2,4\}$	*76*	$\{1,2,3,4\}$	*128*
		$\{3,4\}$	*76*		

One can easily check that v is 1-preconvex. Moreover, since

$$(n-1)v(N) - \sum_{k \in N} v(N \backslash k) = -34 < 0,$$

the game v is not 1-cohesive. Our computations give ${}^1\tilde{x}(v) = (11.5, 11.5, 30.5, 74.5)$, and one can verify that the vector of excesses related to $x' = (12, 12, 31, 73)$ lexicographically dominates the one related to ${}^1\tilde{x}(v)$. In particular, the largest excesses for both allocations are for coalitions $\{i,j\}$, with $i,j \in \{1,2,3\}$. For ${}^1\tilde{x}(v)$ they are of 12, but for x' they are 11. Thus, ${}^1\tilde{x}(v) \neq \eta(v)$.

7 Concluding comments

We have introduced and studied the family of preconvex games, thus extending the well-known class of convex games. Even though some preconvex games may have an empty core, we have defined a verifiable condition that is sufficient to guarantee the balancedness of a preconvex game (and this condition has in fact been shown to be both necessary and sufficient for balancedness in the special case of anticonvex games).

Our proposed solution, the adjusted average marginal value, has been characterized by means of three natural properties in the context of preconvex games: Anonymity, Cone Additivity, and SM-Equivalence. We argue that the three axioms used in Theorem 3 are logically independent. First, note that the Shapley value meets Anonymity and Cone additivity, but it violates SM-Equivalence. Second, letting π be the natural order of the player set (*i.e.*, $\pi(i) = i$ for all $i \in N$), define the value $\tilde{x}^{\pi}$ as follows: $\tilde{x}^{\pi}(v) = {}^{k}\tilde{x}^{\pi}(v)$ for all $v \in \mathcal{S}^{\star}$ and $k = 1, \ldots, n$ s.t. $v \in \mathcal{S}^{\star}_{k}$ —where ${}^{k}\tilde{x}^{\pi}(v)$ comes from Definition 3.(i). It is easy to see that $\tilde{x}^{\pi}$ satisfies Cone Additivity and SM-Equivalence; but it obviously fails Anonymity. Finally, considering the case of three players ($n = 3$, for simplicity) and letting $\hat{x}$ be the truncated value yielding the Shapley allocation $x^{Sh}(v)$ if the considered game is anticonvex and **not** 1-cohesive [and yielding the adjusted average marginal allocation $x^{*}(v)$ otherwise], one can check that $\hat{x}$ satisfies Anonymity and SM-Equivalence, but it violates Cone Additivity.

Our family of preconvex games is one of various extensions (of the class of convex games) that have been proposed in the literature. For instance, in permutationally convex games (Granot and Huberman, 1982) the monotonicity of marginal contributions is required with respect to a single ordering of the agents; and in average convex games. Inarra and Usategui (1993) compare average marginal contributions rather than marginal contributions [more precisely, the sum of marginal contributions of agents in S must be weakly larger when joining $S \cup T$ than when joining S]. The concept was recently generalized to weighted games (Skoda and Venel, 2023). There is no significant overlap between either of these sets of games and the set of preconvex games.

A second stream of literature generalizing convexity is based on the idea of superadditivity (see Young (1985) and Solymosi (1999)), the notion that disjoint groups generate at least as much value together than separately. The property is necessary for convexity, but not sufficient to generate a non-empty core. Recently, Bahel (2021) refined the concept and defined hyperadditive games, which generalize convex games while preserving a non-empty core. The refinement is based on the notion of reduced games:[5] agents are removed sequentially, with the remaining coalitions offered the choice between (a) keeping their original worth and (b) using their worth with the last agent served, net of his marginal contribution. Hyperadditive games are such that all such reduced games are superadditive. There is no inclusion relation between the family of preconvex games and that of hyperadditive games. Indeed, some preconvex games have an empty core —and hence those

[5]See Davis and Maschler (1965) or Núñez and Rafels (1998) for a formal definition of reduced games.

games cannot be hyperadditive.[6] On the other hand, some hyperadditive games —such as non-convex veto games (Bahel, 2016)– are not preconvex.

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A Proofs

A.1 Theorem 1

Proof. Fix $k \in \{1, \ldots, n-1\}$ and consider $v \in \mathcal{S}_k$. Looking at (4), remark that

$$\begin{aligned}\sum_{l=1}^{k-1} {}^k\tilde{x}^{\pi}_{\pi^{-1}(l)}(v) =& [v(N) - v(F^{\pi}_2)] + \ldots + [v(F^{\pi}_{n-k+1} \cup \pi^{-1}(k-1)) - v(F^{\pi}_{n-k+1})] \\ =& v(N) - v(F^{\pi}_{n-k+1}),\ \forall \pi \in \Pi(N). \end{aligned} \tag{9}$$

Substituting (9) into the efficiency property $\sum\limits_{l=1}^{n} {}^k\tilde{x}^{\pi}_{\pi^{-1}(l)}(v) = v(N))$ thus gives

$$ {}^k\tilde{x}^{\pi}_{F^{\pi}_{n-k+1}}(v) = \sum_{l=k}^{n} {}^k\tilde{x}^{\pi}_{\pi^{-1}(l)}(v) = v(F^{\pi}_{n-k+1}),\ \forall \pi \in \Pi(N). \tag{10}$$

We will use (10) to prove the two statements (a) and (b) below.

(a) Suppose that v is k-semicohesive. This means that there exists $T \in \mathbf{C}_{n-k+1}$ such that v is k-semicohesive at T. We will prove that, for all $\pi \in \Pi(N)$,

$$[F^{\pi}_{n-k+1} = T] \Rightarrow [\ {}^k\tilde{x}^{\pi}(v) \in Core(v)]. \tag{11}$$

Fixing $\pi \in \Pi(N)$ such that $F^{\pi}_{n-k+1} = T$, we must show that $v(S) \leq\ {}^k\tilde{x}^{\pi}_S(v)$ for all $S \in 2^N \setminus \{\emptyset\}$. Consider then $S \in 2^N \setminus \{\emptyset\}$. We will discuss two cases; and the proof of Case 2 exploits the result shown in Case 1.

Case 1: $S \subseteq T$.
We distinguish three subcases that are mutually exclusive and cover Case 1.

[6] Even some balanced preconvex games fail hyperadditivity. For instance, recall that the TU game u of Example 4 is 3-semicohesive and hence balanced (by Theorem 1); but it is neither superadditive nor hyperadditive.

Subcase 1.1: $S = T$. In this first subcase, observe that ${}^k\tilde{x}^\pi_S(v) = {}^k\tilde{x}^\pi_T(v) = v(F^\pi_{n-k+1}) = v(S)$, where the second equality comes from Equation (10). This means that coalitional rationality is satisfied with equality for S.

Subcase 1.2: $S \subsetneq T$ and $\pi^{-1}(n) \in S$. Note in this subcase that there necessarily exists $i \in T \backslash S$ (since $S \subsetneq T$). Given that $\pi^{-1}(n) \in S$, we have $T \backslash S \subseteq T \backslash \pi^{-1}(n)$; and it thus follows from (4) that

$$ {}^k\tilde{x}^\pi_{T\backslash S}(v) = \sum_{i \in T\backslash S} m_i^{\pi,k}(v). \tag{12} $$

Without loss of generality, let us write $T \backslash S = \{i_1, \ldots\ldots, i_L\}$, where $L = |T \backslash S| \geq 1$. Then we have

$$ v(S \cup i_1) = v(S) + [v(S \cup i_1) - v(S)] \geq v(S) + [v(T) - v(T \setminus i_1)] = v(S) +^k \tilde{x}^\pi_{i_1}(v), \tag{13} $$

where the inequality stems from the fact that v is k-preconvex —recall property (b) in Definition 1 and the fact that $S \subseteq T \setminus i_1 = F^\pi_{n-k+1} \setminus i_1$. Remark also that the last equality in (13) stems from (4) because $i_1 \neq \pi^{-1}(n)$ and $T = F^\pi_{n-k+1}$.

Next, if $L \geq 2$, one can use the same reasoning to show that $v(S \cup i_1 i_2) \geq v(S \cup i_1) + {}^k\tilde{x}^\pi_{i_2}(v)$; and combining this inequality with $v(S \cup i_1) \geq v(S) + {}^k\tilde{x}^\pi_{i_1}(v)$ thus yields

$$ v(S \cup i_1 i_2) \geq v(S) + {}^k\tilde{x}^\pi_{i_1}(v) + {}^k\tilde{x}^\pi_{i_2}(v). $$

Repeating this argument for all players $i \in T \setminus S = \{i_1, \ldots\ldots, i_L\}$, we can write

$$ v(T) = v(S \cup i_1 \ldots i_L) \geq v(S) + {}^k\tilde{x}^\pi_{i_1}(v) + \ldots + {}^k\tilde{x}^\pi_{i_L}(v). \tag{14} $$

Combining (12) and (14) thus gives

$$ v(T) - \underbrace{[{}^k\tilde{x}^\pi_{i_1}(v) + \ldots + {}^k\tilde{x}^\pi_{i_L}(v)]}_{{}^k\tilde{x}^\pi_{T\backslash S}(v)} = v(T) -^k \tilde{x}^\pi_{T\backslash S}(v) \geq v(S). \tag{15} $$

Finally, since $T = F^\pi_{n-k+1}$, recall from (10) that $v(T) = \sum_{i \in T} {}^k\tilde{x}^\pi_i(v)$; and it hence follows that $v(T) - {}^k\tilde{x}^\pi_{T\backslash S}(v) = {}^k\tilde{x}^\pi_S(v)$. Substituting this in (15) gives $v(S) \leq {}^k\tilde{x}^\pi_S(v)$.

Subcase 1.3: $S \subsetneq T$ and $\pi^{-1}(n) \notin S$. In this subcase, given that $\pi^{-1}(n) \notin S$, it comes from (4) that

$$ {}^k\tilde{x}^\pi_S(v) = \sum_{i \in S} m_i^{\pi,k}(v). \tag{16} $$

Since v is cohesive at T [i.e., (A.4) holds], we can write $v(T) \leq (n - k + 1)\ v(T) -$

$\sum_{i\in T} v(T\setminus i) = \sum_{i\in T}(v(T) - v(T\setminus i))$. Rearranging the terms, we get

$$v(T) - \sum_{i\in T\setminus\pi^{-1}(n)} (v(T) - v(T\setminus i)) \leq v(T) - v(T\setminus\pi^{-1}(n)). \tag{17}$$

Recalling from (10) that $v(T) = \sum_{i\in T} {}^k\tilde{x}_i^\pi(v)$, and observing from (4) that ${}^k\tilde{x}_i^\pi(v) = v(T) - v(T\setminus i)$ for all $i \in T\setminus\pi^{-1}(n)$, we get

$${}^k\tilde{x}^\pi_{\pi^{-1}(n)}(v) = v(T) - \sum_{i\in T\setminus\pi^{-1}(n)} (v(T) - v(T\setminus i)). \tag{18}$$

Combining (17) and (18) gives

$${}^k\tilde{x}^\pi_{\pi^{-1}(n)}(v) \leq v(T) - v(T\setminus\pi^{-1}(n)). \tag{19}$$

Inequality (19) means that the adjusted payment to the last player $[\pi^{-1}(n)]$ under the permutation π is less than this player's marginal contribution to T. Recalling that $S \subsetneq T$ in this subcase, and using Definition 1-(b), we have

$$v(T) - v(T\setminus\pi^{-1}(n)) \leq v(S\cup\pi^{-1}(n)) - v(S). \tag{20}$$

Combining (19) and (20) yields

$${}^k\tilde{x}^\pi_{\pi^{-1}(n)}(v) \leq v(S\cup\pi^{-1}(n)) - v(S). \tag{21}$$

In subcase 1.2 we have proved that $v(S') \leq {}^k\tilde{x}^\pi_{S'}(v)$, for any $S' \subsetneq T$ such that $\pi^{-1}(n) \in S'$. Taking then $S' = S\cup\pi^{-1}(n)$, we can write

$$v(S\cup\pi^{-1}(n)) \leq {}^k\tilde{x}^\pi_{S\cup\pi^{-1}(n)}(v) = {}^k\tilde{x}^\pi_S(v) + {}^k\tilde{x}^\pi_{\pi^{-1}(n)}(v). \tag{22}$$

Summing up (21) and (22), and rearranging, one finally obtains

$$v(S) \leq {}^k\tilde{x}^\pi_S(v). \tag{23}$$

Case 2: $S\setminus T \neq \emptyset$.
We discuss two subcases that are mutually exclusive and cover Case 2.

Subcase 2.1: $S\setminus T \neq \emptyset$ and $S\cap T = \emptyset$. Note in this case that we have $S \subseteq \{\pi^{-1}(1), \ldots, \pi^{-1}(k-1)\} = N\setminus T$ (since $S\cap T = \emptyset$). Letting $L = |S| \geq 1$, we write (without loss of generality) $S = \{i_1, \ldots, i_L\}$, where $\pi(i_1) < \ldots < \pi(i_L) \leq k-1$.

Note that $v(S)$ can be decomposed as the following sum of increments:

$$v(S) = v(i_L) + [v(i_{L-1}i_L) - v(i_L)] \ldots + [v(S) - v(S \setminus i_1)]. \tag{24}$$

Combining (4), the k-preconvexity of v [see Definition 1-(a)] and the fact that $S \subseteq \bar{F}^{\pi}_{i_1}$, observe that

$$v(S) - v(S \setminus i_1) \leq v(\bar{F}^{\pi}(i_1)) - v(\bar{F}^{\pi}_{i_1} \setminus i_1) = {}^k\tilde{x}^{\pi}_{i_1}(v).$$

Using an argument along the same lines, one similarly obtains

$$v(i_1 \ldots i_l) - v(i_1 \ldots i_l \setminus i_l) \leq v(\bar{F}^{\pi}(i_l)) - v(\bar{F}^{\pi}(i_l) \setminus i_l) = {}^k\tilde{x}^{\pi}_{i_l}(v), \forall l \in \{1, \ldots L\}. \tag{25}$$

Substituting (25) in (24), it thus comes that $v(S) \leq {}^k\tilde{x}^{\pi}_S(v)$.

Subcase 2.2: $S \setminus T \neq \emptyset$ and $S \cap T \neq \emptyset$. Note in this subcase that we can write $S = \bar{S} \cap \hat{S}$, where $\bar{S} = S \setminus T$ and $\hat{S} = S \cap T$. Letting $L = |\bar{S}|$, write $\bar{S} = \{i_1, \ldots, i_L\}$, where $\pi(i_1) < \ldots < \pi(i_L) \leq k - 1$.

Remark that $v(S)$ can be written as the following sum

$$v(S) = v(\hat{S}) + [\underbrace{v(\hat{S} \cup i_L) - v(\hat{S})}_{\leq {}^k\tilde{x}^{\pi}_{i_L}(v)}] + \ldots + [\underbrace{v(\hat{S} \cup \bar{S}) - v((\hat{S} \cup \bar{S}) \setminus i_1)}_{\leq {}^k\tilde{x}^{\pi}_{i_1}(v)}], \tag{26}$$

where the inequalities follow from the combination of (4), the k-preconvexity of v [Definition 1-(a)] and the fact that $\hat{S} \cup i_L \ldots i_l \subseteq \bar{F}^{\pi}_{i_l}$ (for $l = 1, \ldots, L$).

It thus follows from (26) that

$$v(S) \leq v(\hat{S}) + {}^k\tilde{x}^{\pi}_{\bar{S}}(v). \tag{27}$$

But since $\hat{S} \subseteq T$, we know from the discussion of Case 1 that $v(\hat{S}) \leq {}^k\tilde{x}^{\pi}_{\hat{S}}(v)$ must hold; and substituting this inequality in (27) gives $v(S) \leq {}^k\tilde{x}^{\pi}_{\hat{S}}(v) + {}^k\tilde{x}^{\pi}_{\bar{S}}(v) = {}^k\tilde{x}^{\pi}_S(v)$. We have thus shown that ${}^k\tilde{x}^{\pi}(v) \in Core(v)$; and this proves that $Core(v) \neq \emptyset$.

(b) Suppose now that v is k-cohesive. We will prove each of the two implications stated in Theorem 1-(b).

($\Rightarrow$) For all $\pi \in \Pi(N)$, define $T_\pi := F^{\pi}_{n-k+1}$. Given that v is k-cohesive and $|T_\pi| = n - k + 1$, one can assert that v is cohesive at T_π. Hence, the property (11) [whose proof is described in the previous paragraphs] applies to the triple (v, π, T_π); and this means that ${}^k\tilde{x}^{\pi}(v) \in Core(v)$, for all $\pi \in \Pi(N)$.

($\Leftarrow$) Suppose now that ${}^k\tilde{x}^\pi(v) \in Core(v)$, for all $\pi \in \Pi(N)$. We prove below that v is k-cohesive.

Consider $T \in \mathbf{C}_{n-k+1}$. Notice that T contains at least 2 players (since $k \leq n-1$). Pick then a permutation $\pi \in \Pi(N)$ such that every member of T is served after any member of $N \setminus T$ (if $N \setminus T \neq \emptyset$), that is to say,[7]

$$[(i,j) \in T \times (N \setminus T)] \Rightarrow [\pi(i) > \pi(j)]. \tag{28}$$

Remark from (28) that $T = F_k^\pi$; and it then follows from the combination of (4) and (10) that

$${}^k\tilde{x}_T^\pi(v) = \sum_{i \in T \setminus \pi^{-1}(n)} m_i^{\pi,k}(v) + {}^k\tilde{x}_{\pi^{-1}(n)}^\pi(v) = v(T). \tag{29}$$

Since ${}^k\tilde{x}^\pi(v) \in Core(v)$, notice that rationality of the coalition $T \setminus \pi^{-1}(n)$ implies ${}^k\tilde{x}_{\pi^{-1}(n)}^\pi(v) \leq m_{\pi^{-1}(n)}^{\pi,k}(v)$. Substituting the last inequality in Equation (29) gives $v(T) \leq \sum\limits_{i \in T \setminus \pi^{-1}(n)} m_i^{\pi,k}(v) + m_{\pi^{-1}(n)}^{\pi,k}(v) = \sum\limits_{i \in T} m_i^{\pi,k}(v)$, which is equivalent to saying that $v(T) \leq \sum\limits_{i \in T} [v(T) - v(T \setminus i)]$. Rewriting this inequality then gives the desired result. We have thus shown $\sum_{i \in T} v(T \setminus i) \leq (n-k)\ v(T), \forall T \in \mathbf{C}_{n-k+1}$. □

A.2 Theorem 2

Proof. Let $k \in \{1, \ldots, n-1\}$; and suppose that $v \in \mathcal{S}_k$ is k-cohesive. Since the core is a convex set and the k-adjusted average marginal value ${}^k\tilde{x}$ picks the average (i.e., a convex combination) of all allocations ${}^k\tilde{x}^\pi(v)$ (for all $\pi \in \Pi(N)$), it follows from the equivalence stated in Theorem 1-(b) that ${}^k\tilde{x}(v) \in Core(v)$. □

A.3 Theorem 3

Proof. It is not difficult to see that x^* satisfies Anonymity, Cone-Additivity and SM-Equivalence. Conversely, considering now a value $\hat{x} : \mathcal{S}^\star \to \mathbb{R}^N$ that satisfies Anonymity, Cone-Additivity and SM-Equivalence, we must show that $\hat{x} = x^*$. Let us proceed in three steps to prove this equality.

Let $Q \in 2^N \setminus \{\emptyset\}$. We call v_Q the *carrier game* associated with Q if, for all $S \in 2^N$, we have

$$v_Q(S) = \begin{cases} 1 & \text{if } Q \subseteq S, \\ 0 & \text{otherwise .} \end{cases} \tag{30}$$

[7] Observe that the product $|T|!(n-|T|)!$ is s valid natural number; and it gives the number of permutations $\pi \in \Pi(N)$ satisfying (28).

Note from (30) that carrier games are convex games. We call a game v *superconvex* if it is a linear combination of carrier games with positive coefficients, that is to say, if $v = \lambda_1 v_{Q_1} + \ldots + \lambda_T v_{Q_T}$ for some $T \geq 1$, $Q_1, \ldots, Q_T \in 2^N \setminus \{\emptyset\}$, and $\lambda_1, \ldots, \lambda_T > 0$.

Step 1: $\hat{x}(v) = x^*(v)$, for any superconvex game $v = \sum_{t=1}^{T} \lambda_t v_{Q_t}$.
Suppose that $v = \sum_{t=1}^{T} \lambda_t v_{Q_t}$, where $\lambda_1, \ldots, \lambda_T > 0$. First, observe that Anonymity of $\hat{x}$ requires $\hat{x}_i(\lambda_t v_{Q_t}) = \hat{x}_j(\lambda_t v_{Q_t})$, for all $t \in \{1, \ldots, T\}$ and $i, j \in Q_t$. Second, since $x_1(\mathbf{0}) = \ldots = x_n(\mathbf{0}) = 0$ (by Anonymity and efficiency of $\hat{x}$), note that SM-Equivalence and the fact that the null game $\mathbf{0}$ is convex imply $\hat{x}_i(\lambda_t v_{Q_t}) = \hat{x}_i(\mathbf{0}) = 0$ for any $i \in N \setminus Q_t$. Combining these results with the efficiency of $\hat{x}$ then yields

$$\hat{x}_i(v_Q) \begin{cases} \frac{\lambda_Q}{|Q|} & \text{if } i \in Q, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Since x^* also satisfies Anonymity and SM-Equivalence (in addition to efficiency), the same reasoning gives

$$x_i^*(v_Q) = \begin{cases} \frac{\lambda_Q}{|Q|} & \text{if } i \in Q, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Therefore, combining (31)-(32) with Cone Additivity (of $\hat{x}$ and x^*), it follows that

$$\hat{x}(v) = \sum_{t=1}^{T} \hat{x}(\lambda_t v_{Q_t}) = \sum_{t=1}^{T} x^*(\lambda_t v_{Q_t}) = x^*(v). \tag{33}$$

Step 2: $\hat{x}(v) = x^*(v)$, for any convex game $v \in \mathcal{S}_n$.
Fix a convex game $v \in \mathcal{S}_n$ that is not superconvex. It is well known that the set of carrier games is a basis of the vector space of TU games; and since v is not superconvex, its decomposition in this basis must exhibit at least one negative coordinate. That is to say,

$$v = \sum_{t=1}^{T} \lambda_t v_{Q_t} - \sum_{t=1}^{T'} \beta_t v_{P_t}, \tag{34}$$

for some $T, T' \geq 1$, $Q_1, \ldots, Q_T, P_1, \ldots, P_{T'} \in 2^N \setminus \{\emptyset\}$, $\lambda_1, \ldots, \lambda_T, \beta_1, \ldots, \beta_{T'} > 0$.

Letting then $u = \sum\limits_{t=1}^{T} \lambda_t v_{Q_t}$ and $w = \sum\limits_{t=1}^{T'} \beta_t v_{P_t}$, it comes from (34) that $u = v + w$.

Recalling Step 1, we can write

$$\hat{x}(u) = x^*(u); \tag{35}$$
$$\hat{x}(w) = x^*(w). \tag{36}$$

Combining (35)-(36), $u = v + w$, and Cone Additivity (of $\hat{x}$ and x^*) finally gives $\hat{x}(v) = \hat{x}(u) - \hat{x}(w) = x^*(u) - x^*(w) = x^*(v)$.

Step 3: $\hat{x}(v) = x^*(v)$, for any preconvex game $v \in \mathcal{S}_k$ (where $k = 1, \ldots, n-1$).

Fix now a degree of preconvexity k such that $1 \leq k \leq n-1$. We discuss two substeps under this third step.

Substep 3.1: We first consider only k-cohesive games.

Suppose then that $v \in \mathcal{S}_k$ is k-cohesive. We show without loss of generality that $\hat{x}_1(v) = x_1^*(v)$ (the argument is identical, *mutatis mutandis*, for any $i \neq 1$). We discuss two cases.

Case 1: Suppose that the $(n-1)$ players $2, 3, \ldots, n$ are all symmetric in v. In other words, we have

$$v(S \cup i) = v(S \cup j), \tag{37}$$

$\forall i, j \in N \backslash 1$, $\forall S \in 2^{N \backslash \{i,j\}}$. Note from (37) that there must exist two nondecreasing functions $f, g : \{0, \ldots, n-1\} \to \mathbb{R}_+$ such that, for all $S \in 2^N$,

$$v(S) = \begin{cases} f(s) & \text{if } 1 \in S, \\ g(s) & \text{if } 1 \notin S, \end{cases} \tag{38}$$

where $s = |S \setminus 1|$. Let us use the shorthand $h'(s) = h(s) - h(s-1)$, for all $h \in \{f, g\}$, $s \in \{1, \ldots, n-1\}$. Next, consider the game $\dot{v}$ defined as follows: for all $S \in 2^N \setminus \{\emptyset\}$,

$$\dot{v}(S) = \begin{cases} v(S) & \text{if } S \in \bar{C}_{n-k}; \\ f(n-k) - (n-k-s) f'(n-k) & \text{if } s < n-k \text{ and } 1 \in S; \\ g(n-k) - (n-k-s) \min\{f'(n-k), g'(n-k-1)\} & \text{if } s < n-k \text{ and } 1 \notin S. \end{cases} \tag{39}$$

Note from (39) [and the fact that $v \in \mathcal{S}_k$ is k-cohesive] that the game $\dot{v}$ is convex.

Moreover, combining (37) and (39), observe that

$$\partial_1^T \dot{v}(S) = \partial_1^T v(S) = \begin{cases} mc_1^v(S) & \text{if } s \geq n-k, \\ f(n-k) - g(n-k) & \text{if } 1 \leq s < n-k-1, \\ f(n-k) - (n-k) f'(n-k) & \text{if } s = 0, \end{cases} \tag{40}$$

$\forall T \in \mathbf{C}_{n-k}^{-1}$ and $S \in P_T^1$.

It hence follows that

$$\hat{x}_1(v) = \hat{x}_1(\dot{v}) = x_1^*(\dot{v}) = x_1^*(v). \tag{41}$$

The first equality in (41) comes from SM-Equivalence of $\hat{x}$; the second equality holds by Step 2 (since $\dot{v}$ is convex); and the last equality obtains by combining (40) and SM-Equivalence of x^*. This concludes Case 1.

Case 2: Suppose now that not all players $2, \ldots, n$ are symmetric.

For all $v \in \mathcal{S}_k$, define $M_2^v = \{j \in \{2, \ldots, n\} : j \text{ and } 2 \text{ are symmetric in } v\}$. Note that $2 \in M_2^v$ always holds. Let then $m_2^v := |M_2^v| \geq 1$ be the cardinality of M_2^v. Remark that Case 1 applies to any k-cohesive game $v \in \mathcal{S}_k$ such that $m_2^v = n-1$. Hence, we may use backward induction over m_2^v to prove the result.

Fix then an integer $\bar{m}$ such that $1 \leq \bar{m} \leq n-2$; and assume by backward induction that $\hat{x}_1(w) = x_1^*(w)$ for all k-cohesive $w \in \mathcal{S}_k$ such that $m_2^w \geq \bar{m}+1$.

Consider now a k-cohesive $v \in \mathcal{S}_k$ such that $m_2^v = \bar{m}$. Since $m_2^v = \bar{m} \leq n-2$ and $2 \in M_2^v$, there exists $j \in \{3, \ldots, n\} \setminus M_2^v \neq \emptyset$. We will now construct a k-cohesive game w such that

$$M_2^w \supseteq j \cup M_2^v \text{ and } \hat{x}_1(w) = \hat{x}_1(v). \tag{42}$$

Let v^{23} be the game where the roles of player 2 and player 3 are permuted, that is to say, for all $S \in 2^N$, we have

$$v^{23}(S) = \begin{cases} v((S \setminus 2) \cup 3) & \text{if } 2 \in S \text{ and } 3 \notin S; \\ v((S \setminus 3) \cup 2) & \text{if } 2 \notin S \text{ and } 3 \in S; \\ v(S) & \text{otherwise .} \end{cases} \tag{43}$$

Note from (43) that $v^{23} \in \mathcal{S}_k$ and v^{23} is k-cohesive (since $v \in \mathcal{S}_k$ and v is k-cohesive). Moreover, since x^* and $\hat{x}$ are both anonymous (and $v^{23} = \sigma v$, where the permutation σ preserves player 1's label, while swapping only those of player 2 and player 3), we must have

$$\begin{cases} x_1^*(v) = x_1^*(v^{23}), \\ \hat{x}_1(v) = \hat{x}_1(v^{23}). \end{cases}$$

The same reasoning gives the following equalities:

$$\begin{cases} x_1^*(v) = x_1^*(v^{lt}), \\ \hat{x}_1(v) = \hat{x}_1(v^{lt}), \end{cases} \tag{44}$$

for all $l, t \in \{2, \ldots, n\}$ such that $l \neq t$ and $v^{lt} \in \mathcal{S}_k$.

Define then the game

$$w = \frac{1}{m_2^v + 1}(v + \sum_{i \in M_2^v} v^{ij}). \tag{45}$$

Recall that j is a player that is not symmetric to player 2 in game v. The game w, which is defined in Equation (45), obtains by swapping player j with each player that is symmetric to player 2 in game v (i.e. in M_2^v) and then taking the average of all these games. In the new game w, it is not difficult to see that j is symmetric to every player $j \in M_2^v$; and it follows that $M_2^w \supseteq j \cup M_2^v$. That is to say, $m_2^w = |M_2^w| \geq m_2^v + 1 = \bar{m} + 1$. Furthermore, noting that the set of k-cohesive games is a convex cone (in the vector space $\mathbb{R}^{2^N}$), remark from (45) that w is k-cohesive.

Combining (44) and (45) with the Cone Additivity of $\hat{x}$ and x^*, one can write

$$\hat{x}_1(w) = \hat{x}_1(v), \tag{46a}$$

$$x_1^*(w) = x_1^*(v). \tag{46b}$$

Note that, together, the results $M_2^w \supseteq j \cup M_2^v$ and (46a) validate the postulate made in (42). It hence follows that

$$\hat{x}_1(v) = \hat{x}_1(w) = x_1^*(w) = x_1^*(v). \tag{47}$$

The first equality comes from (46a); the second equality stems from the induction hypothesis (since we have established $m_2^w \geq \bar{m} + 1$ and the k-cohesiveness of w); and the last equality holds by our result (46b).

Substep 3.2: We now consider games $v \in \mathcal{S}_k$ that are **not** k-cohesive.

Fix then a game $v \in \mathcal{S}_k$ that is not k-cohesive. That is to say,

$$\mathbf{T} = \left\{ T \in \mathbf{C}_{n-k+1} : (n-k)v(T) - \sum_{i \in T} v(T \setminus i) < 0 \right\} \neq \emptyset. \tag{48}$$

Next, pick a game $w \in \mathcal{S}_k$ satisfying the following property:[8]

$$(n-k)w(T) - \sum_{i \in T} w(T \setminus i) > 0, \ \forall T \in \mathbf{C}_{n-k+1}. \tag{49}$$

Note from (A.4) and (49) that w is k-cohesive.

Consider now the game $u_\alpha = \alpha w + (1-\alpha)v$ (for any $\alpha \in [0,1]$); and for every $T \in \mathbf{C}_{n-k+1}$, define the function $g_T : [0,1] \to \mathbb{R}$ as follows:

$$\begin{aligned} g_T(\alpha) =& (n-k)u_\alpha(T) - \sum_{i \in T} u_\alpha(T \setminus i) \\ =& \left[(n-k)(w(T)-v(T)) - \sum_{i \in T}(w(T \setminus i) - v(T \setminus i))\right]\alpha \\ &+ (n-k)v(T) - \sum_{i \in T} v(T \setminus i). \end{aligned} \tag{50}$$

Observing from (50) that g_T is an affine function of α, we conclude that it is continuous. Hence, for any $T \in \mathbf{T}$, the intermediate value theorem [since $g_T(0) < 0$ by (48), and $g_T(1) > 0$ by (49)] and the fact that the function g_T is affine guarantee the existence of a unique $\alpha_T \in (0,1)$ such that

$$g_T(\alpha_T) = 0; \text{ and } g_T(\alpha) > 0, \forall \alpha \in (\alpha_T, 1). \tag{51}$$

Since n (and hence $\mathbf{T}$) is finite, we can define $\bar{\alpha} := \max\{\alpha_T, T \in \mathbf{T}\}$. Note from what precedes that we must have $\bar{\alpha} \in (0,1)$; and it then follows from (51) that

$$g_T(\alpha) > 0, \forall \alpha \in (\bar{\alpha}, 1), \forall T \in \mathbf{T}. \tag{52}$$

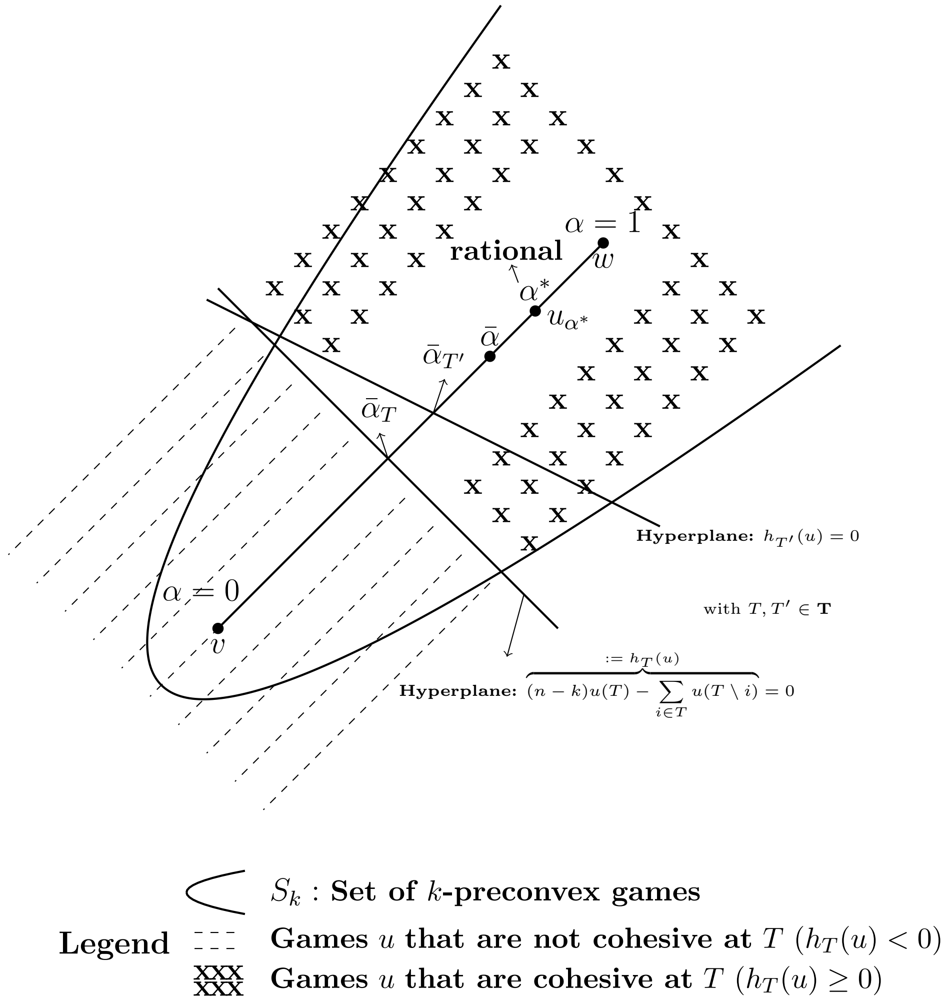
Recalling that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, we can use (52) to claim the existence of a rational number $\alpha^* \in (\bar{\alpha}, 1) \cap \mathbb{Q}$ such that

$$g_T(\alpha^*) > 0, \forall T \in \mathbf{T}. \tag{53}$$

Moreover, if $\mathbf{T} \subsetneq \mathbf{C}_{n-k+1}$, then for any $T \in \mathbf{C}_{n-k+1} \setminus \mathbf{T}$, combining $g_T(0) > 0$

[8]It is not difficult to see that there exist infinitely many games w meeting (49). For instance, it suffices to choose a fully symmetric game $w \in \mathcal{S}_k$ such that $w(S) = \begin{cases} 2 & \text{if } S \in \mathbf{C}_{n_k+1}, \\ 1 & \text{if } S \in \mathbf{C}_{n-k} \end{cases}$ to guarantee $(n-k)v(T) = 2(n-k) > n-k+1 = \sum_{i \in T} v(T \setminus i)$ for all $T \in \mathbf{C}_{n-k+1}$ —and thus satisfying (49).

Figure 1: Construction of the game u_{α^*}



[by (48)] and $g_T(1) > 0$ [by (49)] with the fact that g_T is affine, we may write

$$g_T(\alpha) > 0, \forall \alpha \in (0,1), \forall T \in \mathbf{C}_{n-k+1} \setminus \mathbf{T}. \tag{54}$$

Remark from (53)-(54) and $\alpha^* \in (0,1)$ that $g_T(\alpha^*) > 0, \forall T \in \mathbf{C}_{n-k+1}$; and recalling (50) thus allows to see that the game u_{α^*} is k-cohesive.[9] Figure 1 illustrates the construction of u_{α^*}.

Since $u_{\alpha^*} = \alpha^* w + (1 - \alpha^*)$, applying Cone Additivity [and its consequence,[10]

[9]Given that $\mathcal{S}_k$ is a convex cone, note that $u_{\alpha^*} = \alpha^* w + (1-\alpha^*)v \in \mathcal{S}_k$ (as the convex combination of two preconvex games).

[10]Note that the rational number $\alpha^* \in (0,1) \cap \mathbb{Q}$ can be written in the form $\alpha^* = \frac{p}{q}$, with $p, q \in \{1, 2, 3, \ldots\}$. Hence, Remark 4-(b) indeed applies respectively to $\alpha^* w$ and $(1-\alpha^*)v$.

stated in Remark 4-(b)] to both values $\hat{x}$ and x^* yields:

$$\hat{x}(u_{\alpha^*}) = \alpha^* \hat{x}(w) + (1-\alpha^*)\hat{x}(v) \text{ and } x^*(u_{\alpha^*}) = \alpha^* x^*(w) + (1-\alpha^*)x^*(v).$$

That is to say,

$$\begin{aligned} \hat{x}(v) =& \frac{1}{1-\alpha^*}[\hat{x}(u_{\alpha^*}) - \alpha^* \hat{x}(w)] \\ x^*(v) =& \frac{1}{1-\alpha^*}[x^*(u_{\alpha^*}) - \alpha^* x^*(w)]. \end{aligned} \tag{55}$$

Since we have already established that both w and u_{α^*} are k-cohesive, it comes from Substep 3.1 that $\hat{x}(w) = x^*(w)$ and $\hat{x}(u_{\alpha^*}) = x^*(u_{\alpha^*})$. Substituting these two equalities in (55), we finally get the desired result: $\hat{x}(v) = x^*(v)$. □

A.4 Theorem 4

Proof. Fix $v \in \mathcal{S}_1$ satisfying 1-cohesiveness. By Theorem 2, it comes that ${}^1\tilde{x}(v) \in Core(v) \neq \emptyset$. Conversely, consider now a TU game $v \in \mathcal{S}_1$ such that ${}^1\tilde{x}(v) \in Core(v) \neq \emptyset$. By the Bondareva-Shapley theorem, the following balancedness inequality must be satisfied

$$\sum_{i\in N} v(N\setminus i) \leq (n-1)\ v(N),$$

which is precisely 1-cohesiveness. □

A.5 Lemma 1

Proof. Fix a permutation $\pi \in \Pi(N)$. For all $i \in N$, Equation (4) gives

$${}^1\tilde{x}_i^\pi(v) = v(N) - v(N\backslash i) \text{ if } 1 \leq \pi(i) \leq n-1; \tag{56}$$

and it then comes from efficiency that

$${}^1\tilde{x}_i^\pi(v) = \sum_{j\in N\backslash i} v(N\backslash j) - (n-2)\, v(N), \text{ if } \pi(i) = n. \tag{57}$$

On the other hand, for all $S \subsetneq N$, (8) yields

$$V(S\cup i) - V(S) = \begin{cases} v(N) - v(N\setminus i) & \text{if } 1 \leq s \equiv |S| \leq n-1, \\ \sum_{j\in N\backslash i} v(N\backslash j) - (n-2)v(N) & \text{if } S = \emptyset. \end{cases} \tag{58}$$

Combining (56)-(58) allows to see that ${}^1\tilde{x}_i^\pi(v) = mc_i^V(F^\pi(i)) = x_i^\pi(v),\ \forall i \in N.$ $\square$

A.6 Theorem 5

Proof. Suppose that v is 1-preconvex. We first show that $Core(v) = Core(V)$. Pick $i, j \in N$. In particular, the core conditions imply $x(N\backslash i) \geq v(N\backslash i)$ and $x(N\backslash j) \geq v(N\backslash j)$. Summing these up, and observing that $x(N\backslash i) + x(N\backslash j) = x(N) + x(N\backslash \{i,j\})$, we obtain $x(N) + x(N\backslash \{i,j\}) \geq v(N\backslash i) + v(N\backslash j)$, or

$$x(N\backslash \{i,j\}) \geq v(N\backslash i) + v(N\backslash j) - v(N).$$

Since v is 1-preconvex, we have that $v(N\backslash i) - v(N\backslash \{i,j\}) \geq v(N) - v(N\backslash j)$, or $v(N\backslash i) + v(N\backslash j) - v(N) \geq v(N\backslash \{i,j\})$. That is, the constraint we have found above supplants the usual core constraint $x(N\backslash \{i,j\}) \geq v(N\backslash \{i,j\})$.

Pick $i, j, k \in N$. The core conditions imply $x(N\backslash \{i,j\}) \geq v(N\backslash i) + v(N\backslash j) - v(N)$ (by the result above) and $x(N\backslash k) \geq v(N\backslash k)$. Summing these up, we obtain $x(N) + x(N\backslash \{i,j,k\}) \geq v(N\backslash i) + v(N\backslash j) + v(N\backslash k) - v(N)$, or

$$x(N\backslash \{i,j,k\}) \geq v(N\backslash i) + v(N\backslash j) + v(N\backslash k) - 2v(N).$$

Then, from the result above, we know that $v(N\backslash i)+v(N\backslash j)-v(N) \geq v(N\backslash \{i,j\})$. By 1-preconvexity, we have that $v(N\backslash k) - v(N) \geq v(N\backslash \{i,j,k\}) - v(N\backslash \{i,j\})$. Summing these 2 inequalities, we obtain $v(N\backslash i) + v(N\backslash j) + v(N\backslash k) - 2v(N) \geq v(N\backslash \{i,j,k\})$, and thus the constraint we have found above supplants the usual core constraint.

We continue in this manner until we have found new constraints for $v(S)$, with $s = 1, ..., n-2$. We have that $x(N\backslash S) \geq \sum_{i\in S} v(N\backslash i) - (s-1)v(N) = V(N\backslash S)$, for all $S \subsetneq N$, $s \geq 2$.

Thus, $Core(v) = Core(V)$ as desired.

Let us now prove the second part in the statement of Theorem 5. Considering then a balanced TU games $v \in \mathcal{S}_1$, note from Theorem that v must be 1-cohesive (*i.e.*, cohesive at N), and it thus follows from (5) that $\sum\limits_{i\in N} v(N \setminus i) \leq (n-1)\ v(N)$. Rewriting this inequality gives, $\sum\limits_{j\in N} v(N\backslash j)-(n-2)v(N) \leq v(N)-v(N\backslash i), \forall i \in N$; and therefore, recalling (58), one can write: for all $i \in N$ and $S \in \overline{\mathbf{C}}_2^{-i}$,

$$V(i) = \sum_{j\in N} v(N \setminus j) - (n-2)v(N) \leq v(N) - v(N \setminus i) = V(S) - V(S \setminus i).$$

Note from the above equation that the game V is convex. Hence, it comes from

(Ichiishi, 1981) that $Core(V)$ is the convex hull of V's marginal worth vectors $\{x^{\pi}(V)\}_{\pi\in\Pi(N)}$. Finally, recalling Lemma 1 gives the desired result: $Core(v) = Core(V) = \text{Conv}(\{{}^1\tilde{x}^{\pi}(v)\}_{\pi\in\Pi(N)})$. $\square$

A.7 Theorem 6

Proof. Suppose that $v \in \mathcal{S}_1$.

First step: We show that ${}^1\tilde{x}(v) = \eta(V)$.

Computing the excess of a coalition S at the allocation ${}^1\tilde{x}(v)$ gives:

$$\begin{aligned} e^V(S,\ {}^1\tilde{x}(v)) &= V(S) - \frac{s}{n}\left(v(N) + \sum_{j\in N} v(N\backslash j)\right) + \sum_{j\in S} v(N\backslash j) \qquad (59) \\ &= \left(1 - \frac{s}{n}\right)\sum_{j\in N} v(N\backslash j) - \left(n - s - 1 + \frac{s}{n}\right)v(N), \end{aligned}$$

for all S such that $0 < |S| \equiv s < n-1$. We can see from (59) that $e^V(S,\ {}^1\tilde{x}(v))$ depends only on the cardinality of S. For any $i \in N$, taking $S = N \setminus i$ in (59) yields

$$e^V(N\backslash i,\ {}^1\tilde{x}(v)) = \left(\sum_{j\in N} v(N\backslash j) - (n-1)v(N)\right)/n. \qquad (60)$$

Suppose first that v is 1-cohesive, i.e. $(n-1)v(N) - \sum_{j\in N} v(N\backslash j) \geq 0$. Then $e^V(N\backslash i,\ {}^1\tilde{x}(v)) \leq 0$ [from (60)] and $e^V(S,\ {}^1\tilde{x}(v))$ is an increasing (or constant) function of the cardinality of S [from (59)]. Thus, all coalitions of size $n-1$ achieve the largest excess at ${}^1\tilde{x}(v)$. Since $\sum\limits_{i\in N} x_{N\backslash i} = (n-1)v(N)$ by efficiency; the sum $\sum\limits_{i\in N} x_{N\backslash i}$ is invariable across all allocations x, and it thus follows that any other allocation $x \neq {}^1\tilde{x}(v)$ will have at least one coalition $N \setminus i$ such that $e^V(N\backslash i, x) > e^V(N\backslash i,\ {}^1\tilde{x}(v)) \geq e^V(S,\ {}^1\tilde{x}(v))$ for $\emptyset \neq S \subsetneq N$. Hence, we conclude that ${}^1\tilde{x}(v)$ lexicographically minimizes (from largest coordinate to smallest) the vector of excesses for the game V.

Suppose next that v is not 1-cohesive, i.e. $(n-1)v(N) - \sum_{j\in N} v(N\backslash j) < 0$. Then $e^V(N\backslash i,\ {}^1\tilde{x}(v)) > 0$ and $e^V(S,\ {}^1\tilde{x}(v))$ is decreasing in the cardinality of S. Thus, the singletons achieve the largest excess at ${}^1\tilde{x}(v)$. Since $\sum\limits_{i\in N} x_i = v(N)$ for all allocations x, notice that any other allocation $x \neq {}^1\tilde{x}(v)$ will have at least one $i \in N$ such that such that $e^V(i, x) > e^V(i,\ {}^1\tilde{x}(v)) \geq e^V(S,\ {}^1\tilde{x}(v))$ for $\emptyset \neq S \subsetneq N$. Therefore, ${}^1\tilde{x}(v)$ is the nucleolus of V, that is, ${}^1\tilde{x}(v) = \eta(V)$.

Second step: if v is 1-cohesive then ${}^1\tilde{x}(v) = \eta(v) \in Core(v)$.

Suppose that v is 1-cohesive, that is $(n-1)v(N) - \sum_{j\in N} v(N\backslash j) \geq 0$. From

(8) and the fact that v is 1-preconvex, remark that (a) $v(S) \leq V(S)$ for all $S \subsetneq N$, and (b) $v(N\backslash i) = V(N\backslash i)$ for all $i \in N$. Thus, we can write (c) $e^v(S,\, ^1\tilde{x}(v)) \leq e^V(S,\, ^1\tilde{x}(v))$ for all $S \subsetneq N$, and (d) $e^v(N\backslash i,\, ^1\tilde{x}(v)) = e^V(N\backslash i,\, ^1\tilde{x}(v))$ for all $i \in N$. Since we have shown that the largest excesses at $^1\tilde{x}(v)$ for V are obtained with coalitions of size $n-1$, the properties (c)-(d) mean that

$$e^v(N\backslash i\, ^1\tilde{x}(v)) = e^v(N\backslash j,\, ^1\tilde{x}(v)) \geq e^v(S,\, ^1\tilde{x}(v)), \forall i, j \in N, \forall S \in 2^N \setminus \{\emptyset, N\}. \quad (61)$$

Exploiting (61) and replicating the argument used above for V, one can conclude in the same way that $^1\tilde{x}(v) = \eta(v)$. □

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