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Minimum cost spanning tree problems as value sharing problems

Christian Trudeau (University of Windsor)

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Christian Trudeau
Economics Department, University of Windsor

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Abstract

Minimum cost spanning tree (mcst) problems study situations in which agents must connect to a source to obtain a good, with the cost of building an edge being independent of the number of users. We reinterpret mcst problems as value sharing problems, and show that the folk and cycle-complete solutions, two of the most studied cost-sharing solutions for mcst problems, do not share values in a consistent way. More precisely, two mcst problems yielding the same value sharing problem might lead to value being shared in different ways. However, they satisfy a weaker version of the property that applies only to elementary problems, in which the cost on an edge can only be 0 or 1. The folk solution satisfies the version related to the public approach, while the cycle-complete solution satisfies the one related to the private approach, which differ depending if we allow a group to use the nodes of other agents or only their own nodes. We then build axiomatizations built on these properties. While the two solutions are usually seen as competitors in the private approach, the results point towards a different interpretation: the two solutions are based on different interpretations of the mcst problem, but are otherwise conceptually very close.

JEL Classification: C71, D63

Keywords: Minimum cost spanning tree; value sharing; cycle-complete solution; folk solution.

1 Introduction

The minimum cost spanning tree (mcst) problem is a classic OR problem that has received considerable attention in the economics literature. Mcst problems model a situation where agents are located at different points and need to be connected to a source in order to obtain a good or information. Agents do not care if they are connected directly to the source or indirectly through other agents who are. The cost to build a link between two agents or an agent and the source is a fixed number, meaning that the cost is the same whether one or ten agents use that particular link. Mcst problems can be used to model various real-life problems, from telephone and cable TV to water supply networks. We are interested in the cost sharing problem related to mcst problems. Once agents decide to build the network, the common cost of construction must be split among the participants. There are (at least) two natural interpretations transforming the mcst problem into a cooperative game. The so-called private approach supposes that a coalition can only use the nodes of its members to connect to the source, while the public approach allows the use of nodes belonging to agents outside of the coalition.

The application of the Shapley value to the private mcst problem, first studied in Bird (1976) and known as the Kar solution (Kar (2002)), has interesting properties but might be outside of the core, meaning that some coalitions might be better off leaving the group and undertaking the project themselves. The folk (first suggested by Feltkamp et al. (1994) and rediscovered independently by Bergantinos and Vidal-Puga (2007)) and cycle-complete (Trudeau (2012)) solutions offer similar remedies. Both modify the cost matrix before applying the Shapley value on the modified problem. To obtain the modifications needed to compute the folk solution, for each pair of agents (or for each agent and the source), we find the path between them for which the most expensive edge is as cheap as possible. To obtain the irreducible cost matrix, we then assign that cost to this pair of agents. For the cycle-complete solution, we proceed in the same manner, but look at cycles instead of paths to obtain the cycle-complete cost matrix. Various characterizations of the methods have been proposed, see Trudeau and Vidal-Puga (2019) for a review.

The approach taken here is novel in that we consider the mcst problem as a value sharing problem, using the habitual way of constructing a value sharing game from a cost sharing game. To calculate the value generated by a coalition, we sum the stand-alone costs of its individual members and subtract the stand-alone cost of the coalition itself: the amount obtained is naturally interpreted as the savings generated by the coalition. We then consider a very natural property, called Value Equivalence: if two mcst problem yield the same value function, then the value should be shared in the same manner. This natural property is not satisfied by the folk and cycle-complete solutions, as shown in the following example. Players are identified in the circles, and the cost of each edge is next to it. The source is identified as 0.

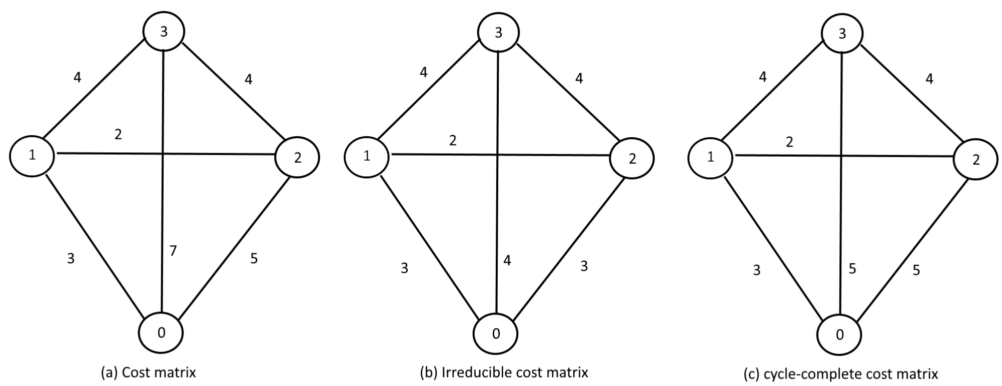

Figure 1: Example showing that neither the folk or the cycle-complete solutions satisfy the Value Equivalence properties

In (a), we have a cost matrix, with its corresponding irreducible and cycle-complete counterparts in (b) and (c) (see section 2 for formal definitions). It is immediate that in this example the public and private approaches yield the same cost for all coalitions. We have that $C(\{1\}) = 3$, $C(\{2\}) = C(\{1,2\}) = 5$, $C(\{3\}) = C(\{1,3\}) = 7$, $C(\{2,3\}) = C(\{1,2,3\}) = 9$. In terms of value, we obtain values of 0 for any singleton, 3 for each pair and 6 for the grand coalition. The folk solution is the Shapley value of the cost function induced by the irreducible cost matrix represented in (b). It yields cost shares of $\left(2\frac{1}{2}, 2\frac{1}{2}, 4\right)$, corresponding to values shares of $\left(\frac{1}{2}, 2\frac{1}{2}, 3\right)$. The cycle-complete solution is the Shapley value of the cost function induced by the cycle-complete cost matrix represented in (c). It yields cost shares of $\left(1\frac{1}{3}, 3\frac{1}{3}, 4\frac{1}{3}\right)$, corresponding to values shares of $\left(1\frac{2}{3}, 1\frac{2}{3}, 2\frac{2}{3}\right)$. Consider a different cost matrix c' such that $c'_{0i} = 10$, and $c'_{jk} = 7$ for all $i, j, k \in N$. Once again, there is no distinction between the public and private approaches, and any singleton connects to the source at a cost of 10, any pair at a cost of 17 and the grand coalition at a cost of 24, yielding the same value function as for the cost matrix in (a). But since both the folk and cycle-complete solutions satisfy Symmetry, they propose the cost allocation $(8, 8, 8)$, corresponding to value shares of $(2, 2, 2)$. Thus, even though the two problems are identical in terms of value created, the two methods treat them differently.

The property is however satisfied by classic solutions for cooperative games like the Shapley value, the nucleolus (Schmeidler (1969)) and the permutation-weighted average of extreme core allocations (Trudeau and Vidal-Puga (2017)). The classic solutions however come with drawbacks: the Shapley value does not always propose allocations in the core, while the nucleolus and the permutation-weighted average of extreme core allocations are much more difficult to compute. It is disappointing but perhaps not surprising that the folk and cycle-complete solutions, which adapted the Shapley value specifically to the context of mcst problems, fail Value Equivalence. There is, however, a saving grace: these two solutions satisfy a weak version of Value Equivalence over elementary cost matrices, in which the cost on each edge can only take one of two values, 0 or 1. It has been shown that any cost matrix can be represented as a sum of elementary cost matrices (Bergantinos and Vidal-Puga (2009)), and the folk and cycle-complete solutions both satisfy the Piecewise Linearity property, which states that the cost shares can also be obtained as the sum of the cost shares on the corresponding problems with elementary cost matrices.

Consider the simple 2-player examples in Figure 1. Represented edges have a cost of zero, others a cost of one.

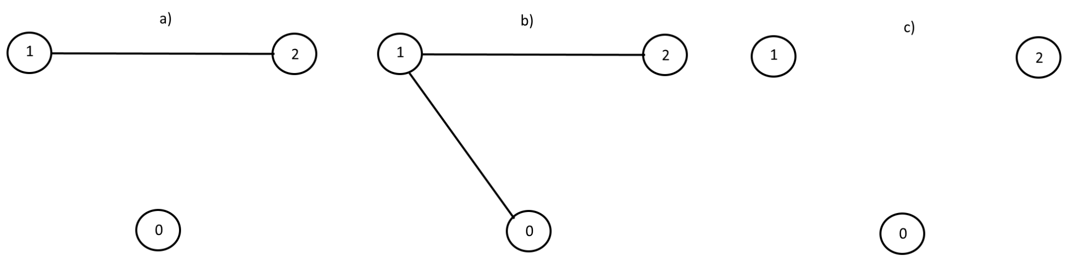

Figure 2: 2-player examples

Consider first the private mcst problems generated by a) and b). In both cases, there is a value of 1 generated by the cooperation between agents 1 and 2. In a), the sum of their stand-alone costs is 2, and they can jointly connect at a cost of 1. In b), the sum of their stand-alone costs is 1, and they can jointly connect at a cost of 0. The folk solution splits the value created equally in the first case, but gives it all to agent 2 in the second case. The cycle-complete solution shares the value equally in both cases. Consider next the public mcst problems generated by b) and c). In both cases, there is no value generated by the cooperation between agents 1 and 2. In b), their stand-alone costs and the joint cost are all zero. In c), the sum of their stand-alone costs is 2, as is their joint cost. The folk solution assigns a value share of zero in both cases. The cycle-complete solution does the same in c), but assigns value shares of $\frac{1}{2}$ to agent 1 and $-\frac{1}{2}$ to agent 2 in b). In section 3, we show formally that the folk solution satisfies the weak form of Value Equivalence when applied to public mcst problems, while the cycle-complete solution satisfies the weak form of Value Equivalence when applied to private mcst problems.

In section 4, we propose axiomatizations of the folk and cycle-complete solutions using weak Value Equivalence. The folk solution is obtained by combining the public version of weak Value Equivalence with Piecewise Linearity, Symmetry and a simple separation property (Group independence). The characterization for the cycle-complete solution is more evolved, as private mcst problems offer more possibilities than public ones. We use a stronger version of Value Equivalence, called Core Equivalence in Values, that goes further by supposing that the problems for which the value functions have the same core should share the values in the same way. If we are concerned about stability, than the core is the most important information, and thus problems with the same core are seen as equivalent. Given that reducing the cost of the edge between and agent and the source by a results in the potential value created by any coalition containing that agent to be reduced by a, we strengthen the property so that it applies to affine transformation of the core. In addition to Symmetry and Piecewise Linearity, we use a new property called Clique Decomposition, which applies to a sub-family of mcst problems in which the value created can be directly attributed to cliques. A clique is a coalition of players such that the addition of a member of the coalition always creates the same value, as long as it joins at least one member of the coalition. In such cases, since the value created is decomposable in the cliques, the cost shares should also be similarly decomposable. Clique games were introduced in Trudeau and Vidal-Puga (2020).

Section 2 formally presents mcst problems. Section 3 introduces Value Equivalence, its weaker variants, and shows compliance/non-compliance by the folk and cycle-complete solutions. In section 4, the axiomatizations of the folk and cycle-complete solutions are presented. Section 5 provides concluding remarks.

2 Minimum cost spanning tree problems

Let $\mathcal{N} = \{1, 2, ...\}$ be the set of potential participants and $N \subseteq \mathcal{N}$ be the set of agents that actually need to be connected to the source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set $Z \subseteq \mathcal{N} \cup \{0\}$, define Z^p as the set of all non-ordered pairs (i, j) of elements of Z. In our context, any element (i, j) of Z^p represents the edge between i and j. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with c_e representing the cost of edge e. Let $\Gamma(N)$ be the set of all cost vectors when the set of agents is N, with $N \subseteq \mathcal{N}$. Let Γ be the set of all cost vectors, for all possible N. Since c assigns cost to all edges e, we often abuse language and call c a cost matrix. A minimum cost spanning tree problem is (N, c), with $N \subseteq \mathcal{N}$ and $c \in \Gamma(N)$.

Denote by Γ^e the set of elementary cost matrices where all connection costs are either 0 or 1 : $\Gamma^e(N) = \{c \in \Gamma(N) : c_e \in \{0, 1\} \text{ for all } e \in N_0^p\}$. Let $F(c)$ be the set of free edges, and let $N^F(c)$ be the set of agents such that $c_{0i} = 0$.

A cycle p_{ll} is a set of $K \geq 3$ edges (i_k, i_{k+1}), with $k \in [0, K-1]$ and such that $i_0 = i_K = l$ and $i_1, ..., i_{K-1}$ distinct and different than l. A path p_{lm} between l and m is a set of K edges (i_k, i_{k+1}), with $k \in [0, K-1]$, containing no cycle and such that $i_0 = l$, $i_K = m$ and $i_1, ..., i_{K-1}$ distinct and different from l and m. Let $P_{lm}(N_0)$ be the set of all such paths between l and m. For a set of edges $Y \subseteq N_0^p$, we say that Y is in $S \subseteq N_0$ if for all $(i, j) \in Y$, $i, j \in S$. We say that Y contains a cycle in S if, for all $i \in S$, there exists a cycle p_{ii} in S such that all elements of p_{ii} are also in Y. We say that a path p_{lm} is a free path if $c_e = 0$ for all $e \in p_{lm}$.

A spanning tree is a non-orientated graph without cycles that connects all elements of N_0. A spanning tree t is identified by the set of its edges. Its associated cost is $\sum_{e \in t} c_e$. The spanning trees with the minimum cost are called minimum cost spanning trees. It is well known that we can find a mcst in polynomial time (Prim (1957), Kruskal (1956)). Let $C(N, c)$ be the cost of these mcst.

Let c^S be the restriction of the cost matrix c to the coalition $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the mcst of the problem (S, c^S). Given these definitions, we say that C is the stand-alone cost function associated with the private mcst problem (N, c). For all $S \subset N$, let $C^{PUB}(S, c) = \min_{T \supseteq S} C(T, c)$. We say that C^{PUB} is the stand-alone cost function associated with the public mcst problem (N, c).

For a problem (N, c), a cost allocation $y \in \mathbb{R}^N$ assigns a cost share to each agent, and the budget balance condition is $\sum_{i \in N} y_i = C(N, c) = C^{PUB}(N, c)$. We define the core as the set of cost allocations such that no coalition pays more than its stand-alone cost:

$$Core(C) = \left\{ y \in \mathbb{R}^N \middle| \sum_{i \in S} y_i \leq C(S) \text{ for all } S \subset N, \ \sum_{i \in N} y_i = C(N) \right\}.$$

$Core(C^{PUB})$ is defined in the same manner.

A cost sharing solution assigns a cost allocation $y(N, c)$ to any admissible mcst problem (N, c). A well-known solution is the Shapley value. For each $i \in N$, we have

$$Sh_i(N, C) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! \, (|N| - |S| - 1)!}{|N|!} \left(C(S \cup \{i\}) - C(S) \right).$$

From any cost matrix c, we define the irreducible cost matrix $\bar{c}$ as follows:

$$\bar{c}_{ij} = \min_{p_{ij} \in P_{ij}(N_0)} \max_{e \in p_{ij}} c_e \text{ for all } i, j \in N_0.$$

From any cost matrix c, we define the cycle-complete cost matrix c^* as follows:

$$\begin{aligned} c^*_{ij} &= \max_{k\in N\setminus\{i,j\}} \overline{c^{N\setminus\{k\}}_{ij}} \text{ for all } i,j\in N \\ c^*_{0i} &= \max_{k\in N\setminus\{i\}} \overline{c^{N\setminus\{k\}}_{ij}} \text{ for all } i\in N. \end{aligned}$$

where $\overline{c^{N\setminus\{k\}}_{ij}}$ indicate the cost of edge (i,j) on the matrix that we first restricted to agents in $N\setminus\{k\}$ before transforming into an irreducible matrix.

The cycle complete matrix can also be defined using cycles (Trudeau (2012)): for edge (i,j), we look at cycles that go through i and j. If there is one such cycle such that its most expensive edge is cheaper than a direct connection on edge (i,j), we assign this cost to edge (i,j).

The folk solution is the Shapley value of $C(\cdot,\bar{c})$ while the cycle-complete solution is the Shapley value of $C(\cdot,c^*)$.

For any mcst problem (N,c), let $V(\cdot,c)$ be the value function built from $C(\cdot,c)$. For each $S\subseteq N$, $V(S,c)=\sum_{i\in S}C(\{i\},c)-C(S,c)$. $V^{PUB}(\cdot,c)$ is defined in the same manner from $C^{PUB}(\cdot,c)$. For the value sharing problem, an allocation $y\in\mathbb{R}^N$ assigns a value share to each agent, and the budget balance condition is $\sum_{i\in N}y_i=V(N,c)$. We define the core as the set of value allocations such that no coalition receives less than its stand-alone value:

$$Core(V)=\left\{y\in\mathbb{R}^N \,\middle|\, \sum_{i\in S}y_i\geq V(S) \text{ for all } S\subset N,\ \sum_{i\in N}y_i=V(N)\right\}.$$

$Core(V^{PUB})$ is defined in the same manner.

3 Value equivalence

We formalize the property discussed in the introduction, defining a version for the private approach and one for the public approach. It requires that whenever two mcst problems yield the same value function, the value shares should be the same. Given our definition of the value game, the share in value of agent i is $C(\{i\},c)-y_i(N,c)$, the savings compared to the stand-alone cost.

$C-$Value Equivalence: For any mcst problems (N,c) and (N,c') with $c,c'\in\Gamma$, if $V(\cdot,c)=V(\cdot,c')$, then $C(\{i\},c)-y_i(N,c)=C(\{i\},c')-y_i(N,c')$ for all $i\in N$.

$C^{PUB}-$Value Equivalence: For any mcst problems (N,c) and (N,c') with $c,c'\in\Gamma$, if $V^{PUB}(\cdot,c)=V^{PUB}(\cdot,c')$, then $C^{PUB}(\{i\},c)-y_i(N,c)=C^{PUB}(\{i\},c')-y_i(N,c')$ for all $i\in N$.

As discussed in the introduction, we also consider weaker versions, that apply only on elementary cost matrices.

Weak $C-$Value Equivalence: For any mcst problems (N,c) and (N,c') with $c,c'\in\Gamma^e$, if $V(\cdot,c)=V(\cdot,c')$, then $C(\{i\},c)-y_i(N,c)=C(\{i\},c')-y_i(N,c')$ for all $i\in N$.

Weak $C^{PUB}-$Value Equivalence: For any mcst problems (N,c) and (N,c') with $c,c'\in\Gamma^e$, if $V^{PUB}(\cdot,c)=V^{PUB}(\cdot,c')$, then $C^{PUB}(\{i\},c)-y_i(N,c)=C^{PUB}(\{i\},c')-y_i(N,c')$ for all $i\in N$.

Theorem 1 *i) The folk and cycle-complete solutions fail the $C^{PUB}-$Value Equivalence and $C-$Value Equivalence properties.*

ii) The folk solution satisfies the Weak $C^{PUB}-$Value Equivalence property but fails the Weak $C-$Value Equivalence property.

iii) The cycle-complete solution satisfies the Weak $C-$Value Equivalence property but fails the Weak $C^{PUB}-$Value Equivalence property.

Proof. i) See Example 1.

ii) Example 2 shows that the folk solution fails the Weak $C-$Value Equivalence property. We show that it satisfies the Weak $C^{PUB}-$Value Equivalence property. Suppose that $y \in Core(C^{PUB})$. Then, for all $S \subseteq N$, $y(S) \leq C^{PUB}(S, c)$, or equivalently,

$$\begin{aligned} \sum_{i \in S} \left(C^{PUB}(\{i\}, c) - y_i(N, c) \right) &\geq \sum_{i \in S} C^{PUB}(\{i\}, c) - C^{PUB}(S, c) \\ &= V^{PUB}(S, c). \end{aligned}$$

Stated otherwise, $y \in Core(C^{PUB})$ if and only if $x \in Core(V^{PUB})$, with $x_i = C^{PUB}(\{i\}, c) - y_i(N, c)$ for all $i \in N$. Suppose that $V^{PUB}(\cdot, c) = V^{PUB}(\cdot, c')$. Trudeau and Vidal-Puga (2017) show that the folk solution is the permutation-weighted average of the extreme points of $Core(C^{PUB})$, which, given the correspondence with $Core(V^{PUB})$, must also be permutation-weighted average of extreme points of $Core(V^{PUB})$. Thus, we must have $C^{PUB}(\{i\}, c) - y_i(N, c) = C^{PUB}(\{i\}, c') - y_i(N, c')$ for all $i \in N$.

iii) Identical to ii). ■

4 Axiomatizations

We propose axiomatizations of the folk and cycle-complete solutions based on the Weak Value Equivalence properties. While we can think of the folk and cycle-complete solutions as being conceptually very close to each other, with the folk solution based on the public game, and the cycle-complete solution on the private game, we obtain a much simpler characterization for the folk solution. The reason is straightforward: elementary cost matrices generate a much smaller set of public cost functions than private ones. For instance, as soon as there is a way to connect the grand coalition at a cost of zero, the corresponding cost and value functions are zero for all coalitions. This might explain why the literature has offered much more axiomatizations for the folk solution than the cycle-complete solution. Here, we need stronger versions of the same properties to axiomatize the cycle-complete solution.

The three additional properties needed for the axiomatization of the folk solution are already known. Piecewise Linearity says that if we can decompose a cost matrix into submatrices where the cost of all edges are ordered in the same manner as the original matrix, then the cost allocation on the original cost matrix should equal the sum of the cost allocations on the submatrices. This property (or similar versions), a weaker version than the classical Additivity property in the general setting (first proposed by Shapley (1953)), has been used in Bergantinos and Vidal-Puga (2009), Bogomolnaia and Moulin (2010), Branzei et al. (2004) and Tijs et al. (2006). Piecewise Linearity generates a rich class of solutions having a simple structure. Cost shares can be defined on simple elementary matrices where costs of all edges are either 0 or 1, making it particularly appealing.

Piecewise Linearity: For any mcst problems (N, c) and (N, c'), if there exists an order of the edges $\sigma : N_0^p \to \left\{1, ..., \frac{n(n+1)}{2}\right\}$ such that for any $e, e' \in N_0^p$, if $\sigma(e) \leq \sigma(e')$, we have $c_e \leq c_{e'}$ and $c'_e \leq c'_{e'}$, then, $y(N, c + c') = y(N, c) + y(N, c')$.

Symmetry is a simple property that imposes that two agents with symmetrical locations pay the same share.

Symmetry: For any mcst problem (N, c), and any $i, j \in N$, if $c_{ik} = c_{jk}$ for all $k \in N_0 \backslash \{i, j\}$, then $y_i(N, c) = y_j(N, c)$.

Group Independence states that if two groups of agents never have any incentives to cooperate - even their subcoalitions have no such interest - then the shares can be computed separately on each group.

Group Independence: For any mcst problem (N, c), if there is $S \subset N$ such that $c_{ij} \geq \max\{c_{0i}, c_{0j}\}$ for all $i \in S$, $j \in N \backslash S$, then $y_i(N, c) = y_i(S, c^S)$ for all $i \in S$ and $y_j(N, c) = y_j(N \backslash S, c^{N \backslash S})$ for all $j \in N \backslash S$.

We are now ready for the characterization of the folk solution.

Theorem 2 *The folk solution is the only solution that satisfies Weak $C^{PUB}-$Value Equivalence, Piecewise Linearity, Symmetry and Group Independence.*

Proof. It is well known that the folk solution satisfies Piecewise Linearity, Symmetry and Group Independence. Theorem 1 shows that it satisfies Weak $C^{PUB}-$Value Equivalence. We show that the four properties imply a unique solution. Suppose that y satisfies Piecewise Linearity, Symmetry, Weak $C^{PUB}-$Value Equivalence and Group Independence.

Take $c \in \Gamma^e$. Suppose that we can partition N into $N_1, ..., N_K$ such that there is a free path between $i, j \in N$ if and only if there is N_k such that $\{i, j\} \subseteq N_k$. If $K > 1$, by Group Independence, if $i \in N_k$ then $y_i(N, c) = y_i(N_k, c^{N_k})$.

Thus, suppose that there is a free path between any $i, j \in N$. We have two possibilities. 1) there is a free path between all $i \in N$ and 0. 2) There is no free path between any $i \in N$ and 0. We consider each case in turn.

1) there is a free path between all $i \in N$ and 0. Then, $C^{PUB}(S, c) = V^{PUB}(S, c) = 0$ for all $S \subseteq N$. Consider c^0 such that $c^0_e = 0$ for all e. It is immediate that $C^{PUB}(S, c^0) = V^{PUB}(S, c^0) = 0$ for all $S \subseteq N$. Thus, by Weak $C^{PUB}-$Value Equivalence, $C^{PUB}(\{i\}, c) - y_i(N, c) = C^{PUB}(\{i\}, c^0) - y_i(N, c^0)$ for all $i \in N$, and since $C^{PUB}(\{i\}, c) = C^{PUB}(\{i\}, c^0) = 0$, we have $y_i(N, c) = y_i(N, c^0)$. By Symmetry, $y_i(N, c^0) = 0$ for all $i \in N$.

2) There is no free path between any $i \in N$ and 0. Then, $C^{PUB}(S, c) = 1$ and $V^{PUB}(S, c) = |S| - 1$ for all $S \subseteq N$. Consider c^1 such that $c^1_{ij} = 0$ and $c^1_{0k} = 1$ for all $i, j, k \in N$. It is immediate that $C^{PUB}(S, c^1) = 1$ and $V^{PUB}(S, c^1) = |S| - 1$ for all $S \subseteq N$. Thus, by Weak $C^{PUB}-$Value Equivalence, $C^{PUB}(\{i\}, c) - y_i(N, c) = C^{PUB}(\{i\}, c^1) - y_i(N, c^1)$ for all $i \in N$, and since $C^{PUB}(\{i\}, c) = C^{PUB}(\{i\}, c^1) = 1$, we have $y_i(N, c) = y_i(N, c^1)$. By Symmetry, $y_i(N, c^1) = \frac{1}{|N|}$ for all $i \in N$.

We thus have a unique allocation for any $c \in \Gamma^e$. By Piecewise Linearity, we have a unique allocation for any $c \in \Gamma$. ■

We use Piecewise Linearity and Symmetry in the axiomatization of the cycle-complete solution, together with strengthened versions of Weak $C-$Value Equivalence and Group Independence.

Value Equivalence imposes that for mcst problems generating the same value function, the value created should be shared in the same manner. We push the property further, asking that if the cores of the corresponding value functions are the same, then the allocations of the value created be the same. The justification is based on the idea that if stability is our premier objective, then the core contains all the information needed to decide how to split the value. Finally, since $V^c(S) = \sum_{i \in S} C(\{i\}, c) - C(S, c)$, the value created is highly sensible to the cost of connecting an agent directly to the source. We thus allow for the property to apply when there is a translation of the core. We illustrate with the examples in Figure 3, again with the represented edges having a cost of zero, all others having a cost of one.

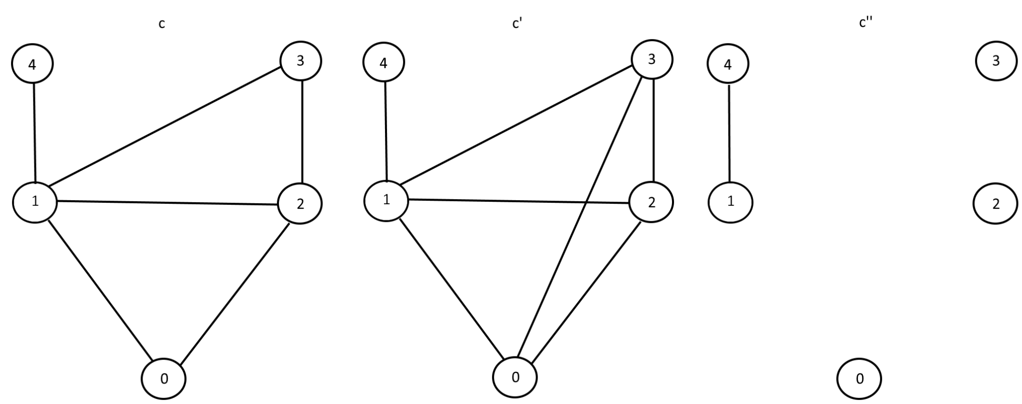

Figure 3: Examples illustrating Core equivalence in values

In (N, c), there is a value of 1 created by the pairs $\{1, 3\}, \{2, 3\}$ and $\{1, 4\}$, with the value created by larger coalitions simply being the sum of the value created by the pairs it contains. It is easy to show that $Core(V(\cdot, c)) = (\alpha, 0, 1, 1-\alpha)$, with $\alpha \in [0,1]$. In (N, c'), only the pair $\{1,4\}$ (and its supersets) creates a value of one. Thus, $Core(V(\cdot, c')) = (\alpha, 0, 0, 1-\alpha)$. The value function and the core are the same for (N, c'') as for (N, c'), as obviously only the pair $\{1,4\}$ creates value. We want the property to imply that the value be shared in the same manner in problems (N, c') and (N, c''), but we also want the property to link the shares of (N, c) and (N, c'). Since the core of the value functions differ only by $a = (0,0,1,0)$, we simply translate the shares in the same way as the core. Notice also that in (N, c') the reduction of agent 3's core allocation by 1 is caused by a reduction of her cost of connection to the source by 1.

Core equivalence in values: For any mcst problems (N, c) and (N, c') with $c, c' \in \Gamma^e$, if $Core(V(\cdot, c)) = Core(V(\cdot, c')) + a$, with $a \in \mathbb{R}^N$, then for all $i \in N$, $C(\{i\}, c) - y_i(N, c) = C(\{i\}, c') - y_i(N, c') + a_i$.

It is immediate that Core equivalence in values implies Weak $C-$Value Equivalence.

Our second property is based on the concept of clique mcst problems, as introduced in Trudeau and Vidal-Puga (2020). In these problems, the agents are divided into cliques, with each agent belonging to at least one clique. Within a clique, all agents have the same cost to connect to each other, and almost all of them have the same cost to connect to the source: a single agent can have a different, lower cost. In keeping with the theme of this paper, the associated value function (for the private mcst problem) is quite simple: the value is created by the cooperation of members of a clique, with the value linearly increasing with the number of members of the clique, starting with the second one. In clique mcst problems, it is thus easy to see how the value is created. Since that value is additively separable over cliques, we propose that cost shares also be.

We first formally define clique mcst problems. We say that $\mathcal{Q} = \{Q^1, \ldots, Q^K\}$ is a cover of N if $Q^k \subseteq N$ for $k = 1, ..., K$ and $\cup_{k=1}^K Q^k = N$. A *path* between Q^k and Q^l is a sequence $P^{kl} = \{Q^{k_1}, \ldots, Q^{k_M}\}$ of different coalitions such that $Q^{k_1} = Q^k$, $Q^{k_M} = Q^l$ and $\left|Q^{k_m} \cap Q^{k_{m+1}}\right| = 1$ for all $m = 1, \ldots, M-1$. A cover $\mathcal{Q}$ is a clique-cover if for all $k, l \in \{1, ..., K\}$, there is at most one path between Q^k and Q^l. For a clique-cover $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{R}$, (N, c) is a clique mcst problem if

a) $c_{ij} = c_Q$ for all $Q \in \mathcal{Q}$ and all $i, j \in Q$;

b) $c_{ij} \geq \max\{c_{0i}, c_{0j}\}$ for all $i, j \in N$ such that there exists no $Q \in \mathcal{Q}$ with $i, j \in Q$;

c) for all $Q \in \mathcal{Q}$, if $c_{0,Q}^{min}, c_Q < c_{0,Q}^{max}$ then $|\arg\max_{j \in Q} c_{0j}| = |Q| - 1$, where $c_{0,Q}^{max} = \max_{j \in Q} c_{0j}$ and $c_{0,Q}^{min} = \min_{j \in Q} c_{0j}$.

We then say that (N, c) is a clique-mcst problem associated with $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}}$. For all $i \in N$, let $\mathcal{Q}(i) = \{Q \in \mathcal{Q} \,|\, i \in Q\}$, and for all $S \subseteq N$, let $\mathcal{Q}(S) = \cup_{i \in S} \mathcal{Q}(i)$. It is straightforward to verify that the cost for any coalition S is:

$$C(S, c) = \sum_{i \in S} c_{0i} + \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)(c_Q - c_{0,Q}^{max}).$$

We then have that $V(S, c)$ is such that

$$V(S, c) = \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)(c_{0,Q}^{max} - c_Q).$$

Clique decomposition: For $c \in \Gamma$, if there exists $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}}$ such that (N, c) is a clique-mcst problem associated with $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}}$, then for all $i \in N$, $c_{0i} - y_i(N, c) = \sum_{Q \in \mathcal{Q}^i} \left(c_{0i} - y_i(Q, c^Q)\right)$.

The following figure illustrates a clique-mcst problem associated with $\mathcal{Q} = \{\{1,2,3\}, \{1,4\}\}$ and $c_{\{1,2,3\}} = 6$ and $c_{\{1,4\}} = 1$. Cost are represented next to the edges. Unrepresented edges have a cost of 8.

We can see that the cooperation of members of $\{1,2,3\}$ creates a value of 6 for each added member, starting with the second. Agents 1 and 2 can jointly connect at a cost of 7, compared to 13 separately. The three of them can connect at a cost of 9, compared to 21 individually. Agents 1 and 4 create a

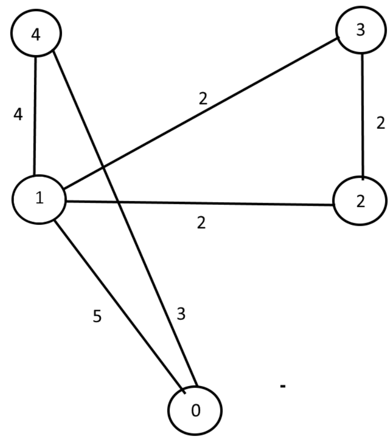

Figure 4: Example illustrating Clique Decomposition.

value of 1 (joint cost of 7, compared to stand alone costs of 8). Given how we can separate the value created in cliques, Clique Decomposition imposes that the value shares for this problem be the sum of the value shares for the problems restricted to the cliques. Here, we would look at the problem of agents 1,2,3 and the problem of 1 and 4, and sum the value shares.

We are now ready for the axiomatization of the cycle-complete solution.

Theorem 3 *The cycle-complete solution is the only solution that satisfies Core Equivalence in values, Piecewise Linearity, Symmetry and Clique Decomposition.*

Proof. It is well known that the cycle-complete solution satisfies Piecewise Linearity and Symmetry (Trudeau and Vidal-Puga (2019)). Core Equivalence in values follows from the fact that for elementary matrices, the cycle-complete solution is the permutation-weighted average of the extreme points of the core (Trudeau and Vidal-Puga (2017)). Suppose that there exists $\mathcal{Q}$ and $\{c_Q\}_{Q\in\mathcal{Q}}$ such that (N,c) is a clique-mcst problem associated with $\mathcal{Q}$ and $\{c_Q\}_{Q\in\mathcal{Q}}$. It is straightforward to verify that if (N,c) is a clique-mcst problem, then c is cycle-complete. For all $i\in N$, we have $y_i^{cc}(N,c)=Sh_i(N,C(\cdot,c))=c_{0i}-Sh_i(N,V(\cdot,c))$. Since $V(S,c)=\sum_{Q\in\mathcal{Q}(S)}(|Q\cap S|-1)\left(c_{0,Q}^{max}-c_Q\right)$ for all $S\subseteq N$ and by the properties of the Shapley value, we have $Sh(N,V(\cdot,c))=\sum_{Q\in\mathcal{Q}}Sh(N,V(\cdot,c^Q))$. For all $Q\in\mathcal{Q}$ and all $i\notin Q$, we have $Sh_i(N,V(\cdot,c^Q))=0$. From that, it is easy to see that $Sh_i(N,V(\cdot,c^Q))=Sh_i(Q,V(\cdot,c^Q))$ for all $i\in Q$. Combining, we obtain that $y_i^{cc}(N,c)=c_{0i}-\sum_{Q\in\mathcal{Q}(i)}Sh_i(Q,V(\cdot,c^Q)=c_{0i}-\sum_{Q\in\mathcal{Q}(i)}\left(c_{0i}-y_i^{cc}(Q,c^Q)\right)$. Rearranging, we obtain $c_{0i}-y_i^{cc}(N,c)=\sum_{Q\in\mathcal{Q}(i)}\left(c_{0i}-y_i^{cc}(Q,c^Q)\right)$ as desired, and y^{cc} satisfies Clique Decomposition.

We show that there is a unique solution satisfying the four properties. Suppose that y satisfies Core Equivalence in values, Piecewise Linearity, Symmetry and Clique Decomposition.

For $r\in\{0,1,2,...,|N|\}$, let Γ_r^e be the set of elementary cost matrices in which r agents in N have a cost of zero to directly connect to the source.

Consider $c\in\Gamma_0^e$. Construct $R(c)$ in the following way: $R(c)$ contains all singletons of N, contains $\{i,j\}$ if $\{i,j\}\in F(c)$ and contains S if $F(c)$ contains a cycle in S, for all S such that $|S|>2$.

Let $\mathcal{Q}(c)=\{S\in R(c)\,|\nexists T\supset S \text{ such that } T\in R(c)\}$. Let $\tilde{c}$ be such that $\tilde{c}_{ij}=0$ if there exists $S\in\mathcal{Q}$ such that $i,j\in S$, and $\tilde{c}_{ij}=1$ otherwise. Let $\tilde{c}_{0i}=1$ for all $i\in N$. It is obvious that $\tilde{c}$ is the cycle-complete version of c. Since for elementary matrices the cycle-complete solution is the permutation-weighted average of extreme core allocations, it is immediate that $Core(C(\cdot,c))=Core(C(\cdot,\tilde{c}))$, and thus that $Core(V(\cdot,c))=Core(V(\cdot,\tilde{c})$. By Core Equivalence in values, we thus have that $c_{0i}-y_i(N,c)=\tilde{c}_{0i}-y_i(N,\tilde{c})$ for all $i\in N$. Since $c_{0i}=\tilde{c}_{0i}=1$ for all $i\in N$, we have $y(N,c)=y(N,\tilde{c})$.

Since $(N, \tilde{c})$ is a clique-mcst problem associated with $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}}$,[1] with $c_Q = 0$ for all $Q \in \mathcal{Q}$, by Clique Decomposition we have $\tilde{c}_{0i} - y_i(N, \tilde{c}) = \sum_{Q \in \mathcal{Q}^i} \left(1 - y_i(Q, \tilde{c}^Q)\right)$. For any $Q \in \mathcal{Q}$, by Symmetry we have $y_i(Q, \tilde{c}^Q) = \frac{1}{|Q|}$ for all $i \in Q$. Thus, we have a unique value for $\tilde{c}$, and thus for c. Since c was chosen arbitrarily in Γ_0^e, we have a unique value for all $c \in \Gamma_0^e$.

Consider $c \in \Gamma_1^e$. There is a single agent $i \in N$ such that $c_{0i} = 0$. Let c' be such that $c'_{0i} = 1$ and $c'_e = c_e$ else. It is immediate that $V(\cdot, c) = V(\cdot, c')$, and thus, by Core equivalence in values, that for all $j \in N$, $c_{0j} - y_j(N, c) = c'_{0j} - y_j(N, c')$. For all $j \neq i$, this implies $y_j(N, c) = y_j(N, c')$. For i, it implies that $y_i(N, c) = y_i(N, c') - 1$. Thus, we have a unique solution for c. Since c was chosen arbitrarily in Γ_1^e, we have a unique value for all $c \in \Gamma_1^e$.

Consider $c \in \Gamma_k^e$, for $k > 1$. Then $\left|N^F(c)\right| = k$ and $c_{0i} = 0$ for all $i \in N^F(c)$. Consider c' to be such that $c'_{ij} = 0$ if $c_{0i} = c_{0j} = 0$ and $c'_{ij} = c_{ij}$ otherwise. It is immediate that $C(\cdot, c) = C(\cdot, c')$, and thus that $V(\cdot, c) = V(\cdot, c')$. By Core Equivalence in values, we thus have that $c_{0i} - y_i(N, c) = c'_{0i} - y_i(N, c')$. Since $c_{0i} = c'_{0i}$ for all $i \in N$, we have $y(N, c) = y(N, c')$.

Let $\hat{c}$ be such that $\hat{c}_{0i} = 1$ for all $i \in N$ and $\hat{c}_e = c'_e$ otherwise. Consider $\mathcal{Q}(\hat{c})$. By construction, there must be a unique $S^{N^{F(c)}} \in \mathcal{Q}(\hat{c})$ such that $N^{F(c)} \subseteq S^{N^{F(c)}}$. Define $\mathring{c}$ such that $\mathring{c}_{0i} = 0$ for all $i \in S^{N^{F(c)}}$ and $\mathring{c}_e = c'_e$ otherwise. Let $a \in \mathbb{R}^N$ be such that $a_i = 1$ if $i \in S^{N^{F(c)}} \backslash N^{F(c)}$ and $a_i = 0$ otherwise. It is immediate that $\mathring{c}$ is the cycle-complete version of c', and because of that, $Core(V(\cdot, c') = Core(V(\cdot, \mathring{c})) + a$. By Core Equivalence in values, we thus have that $c'_{0i} - y_i(N, c') = \mathring{c}_{0i} - y_i(N, \mathring{c}) + a_i$. Construct $\mathring{\mathcal{Q}}$ as follows: start with $\mathcal{Q}(\hat{c})$, remove $S^{N^{F(c)}}$ and for each $i \in S^{N^{F(c)}}$, if $\mathcal{Q}(i) = \left\{S^{N^{F(c)}}\right\}$, add $\{i\}$ to $\mathring{\mathcal{Q}}$. Let $\dot{c}$ be such that $\dot{c}_{ij} = 0$ if $i, j \in Q$ for some $Q \in \mathring{\mathcal{Q}}$, and $\dot{c}_{ij} = 1$ otherwise. Let $\dot{c}_{0i} = 1$ for all $i \in N$. By construction, $V(\cdot, \mathring{c}) = V(\cdot, \dot{c})$ and by Core equivalence in values we have $\mathring{c}_{0i} - y_i(N, \mathring{c}) = \dot{c}_{0i} - y_i(N, \dot{c})$. Since $\dot{c} \in \Gamma_0^e$, $y(N, \dot{c})$ is uniquely defined. Thus, so are $y(N, \mathring{c})$, $y(N, c')$ and $y(N, c)$. Since c was chosen arbitrarily in Γ_k^e, we have a unique value for all $c \in \Gamma_k^e$.

We thus have a unique allocation for any $c \in \Gamma^e$. By Piecewise Linearity, we have a unique allocation for any $c \in \Gamma$. ■

5 Concluding remarks

Theorem 2 could be rewritten with the equivalent of Core Equivalence in values and Clique Decomposition for the public mcst problem, but it would be overkill. As mentioned, among public mcst problems obtained from elementary cost matrices for which all agents are connected together to the source, there are only two possibilities: either all coalitional costs are zero, or they are all one. Then, the values functions are the same, and thus their cores also are, and Core Equivalence in values simplifies to Weak Value Equivalence.[2] In the same way, the set of public mcst problems obtained from elementary cost matrices for which the public value function can be decomposed into cliques can also be decomposed using Group Independence. Clique Decomposition (its public equivalent) and Group independence do not imply the same thing for non-elementary problems, but combined with Piecewise Linearity, only their implications for elementary problems matter.

The transformation of an elementary cost matrix into a cycle-complete cost matrix is a necessary and sufficient condition for the resulting private mcst problem to generate a concave cost game (Trudeau (2012)), which in turn guarantees that the Shapley value is in the core. However, for a non-elementary cost matrix, the transformation is only a sufficient condition. There is a similar result for the irreducible matrix: the transformation is necessary and sufficient for the private mcst problem to generate a concave and monotone cost function. The monotonicity condition guarantees that the private and public approaches generate the same game. The condition is sufficient but not necessary in general.

[1] Trudeau and Vidal-Puga (2020) show that all private mcst problem resulting from an elementary cycle-complete matrix is a clique-mcst problem. By construction, it is a clique mcst-problem associated with $\mathcal{Q}$ and $\{c_Q\}_{Q \in \mathcal{Q}}$.

[2] In fact, we could use a property stating that two mcst problems yielding the same cost function should share cost in the same way. In keeping with the theme of the paper, we stick with the value interpretation.

While these modifications guarantee that the solutions are in the core, they break the link between cost and value representations for non-elementary matrices. In addition, the folk solution is equivalent to the nucleolus and the permutation-weighted average of the extreme core allocations for the public mcst problem resulting from an elementary cost matrix, with the same being true for the cycle-complete solution for the private mcst game resulting from an elementary cost matrix (Trudeau and Vidal-Puga (2017), Trudeau and Vidal-Puga (2020)). Thus, we can see the folk and cycle-complete solutions as approximations of the concepts of nucleolus and permutation-weighted average of extreme core allocations. Combined with Theorems 2 and 3, it shows clearly that the folk solution is grounded in the logic of the public mcst problem, while the cycle-complete solution is its equivalent for the private mcst problem. It is worth noting that it does not preclude using the folk solution for private mcst problems (in particular the core of a public mcst problem is a subset of the core of the corresponding private mcst problem), but it should be acknowledged that the folk solution is conceptually designed for the public mcst problem.

Independence of properties for Theorems 2 and 3 is shown in Appendix.

A Appendix: Independence of Properties

A.1 Theorem 2

The cycle-complete solution satisfies Piecewise Linearity, Symmetry and Group Independence, but fails Weak $C^{PUB}-$Value Equivalence.

$Sh(C^{PUB})$ satisfies Weak $C^{PUB}-$Value Equivalence, Symmetry and Group Independence., but fails Piecewise Linearity (see example 1).

Take an order π of N. For each $c \in \Gamma^e$ let $y^{\pi}(N, c)$ be the extreme core allocation of C^{PUB} according to π, i.e. we lexicographically maximize among core allocations according to π. Let $\bar{y}^{\pi}$ be the Piecewise-linear extension of y^{π}. $\bar{y}^{\pi}$ satisfies Weak $C^{PUB}-$Value Equivalence, Piecewise Linearity and Group Independence but fails Symmetry.

For each $c \in \Gamma^e$ let $y^{eq}(N, c)$ be such that $y_i^{eq}(N, c) = \frac{C^{PUB}(N,c)}{|N|}$ for all $i \in N$. Let $\bar{y}^{eq}$ be the piecewise linear extension of y^{eq}. $\bar{y}^{eq}$ satisfies Weak $C^{PUB}-$Value Equivalence, Piecewise Linearity and Symmetry but fails Group Independence.

A.2 Theorem 3

$Sh(c)$ satisfies Piecewise Linearity, Symmetry and Clique Decomposition but not Core Equivalence in values.

The permutation-weighted average of extreme core allocations of C satisfies Core Equivalence in values, Symmetry and Clique Decomposition but not Piecewise Linearity.

Take an order π of N. For each $c \in \Gamma^e$ let $x^{\pi}(N, c)$ be the extreme core allocation of C according to π, i.e. we lexicographically maximize among core allocations according to π. Let $\bar{x}^{\pi}$ be the Piecewise-linear extension of y^{π}. $\bar{x}^{\pi}$ satisfies Core Equivalence in values, Piecewise Linearity and Clique Decomposition but not Symmetry.

For each $c \in \Gamma^e$ let $x^{eq}(N, c)$ be such that $x_i^{eq}(N, c) = c_{0i} - \frac{V(N,c)}{|N|}$ for all $i \in N$. Let $\bar{x}^{eq}$ be the piecewise linear extension of x^{eq}. $\bar{x}^{eq}$ satisfies Core Equivalence in values, Piecewise Linearity and Symmetry but not Clique Decomposition.

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