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ON GROVES MECHANISMS FOR COSTLY INCLUSION

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Abstract

We investigate Groves mechanisms for economies where (i) a social outcome specifies a group of winning agents, and (ii) a cost function associates each group with a monetary cost. In particular, we characterize both (i) the class of cost functions for which there are Groves mechanisms such that the agents cover the costs through voluntary payments, and (ii) the class of cost functions for which there are envy-free Groves mechanisms. It follows directly from our results that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments.

Keywords: costly inclusion, Groves mechanism, pivot mechanism, Vickrey auction, free-rider problem, labor markets

JEL Codes: D82, D47, D24, D44, H41, D61, D63

1 Introduction

We consider the problem of selecting a group of agents to receive a service and then allocating the associated costs, a remarkably prevalent problem faced by a variety of public enterprises and private firms. Across the rich collection of applications considered in the literature,¹ some common themes have emerged: the group of beneficiaries should maximize surplus (*production efficiency*), the costs should be covered without outside subsidy (*no-deficit*), nobody should contribute more than he is willing (*voluntary participation*), and the allocation should be fair (which can be formalized many ways; we consider *no-envy*). In this article, we thoroughly investigate the implementation of these objectives in dominant strategy equilibrium (*strategy-proofness*) when costs are known but valuations are private information.

For the first-best scenario—where both costs and valuations are known—it is a classical insight that in order to maximize some notion of social welfare, (i) marginal cost pricing is optimal when costs are convex and symmetric, due to standard arguments involving supply and demand; and (ii) deviations from marginal cost pricing may be optimal otherwise, which has been particularly investigated in the context of pricing public utilities (Ramsey, 1927; Manne, 1952; Boiteux, 1956; Baumol and Bradford, 1970). Moreover, if it is known that each agent’s valuation is sufficiently high, then the cost of serving all agents can be directly allocated on the basis of normative principles; this is the approach taken in the large literature on axiomatic cost allocation (see Young, 1985a and Algaba, Fragnelli, and Sánchez-Soriano, 2019 for overviews). That said, these approaches are problematic when valuations are private information, as we cannot directly set prices using demand information and it may be inefficient to serve all agents unconditionally.

For the scenario we consider—where costs are known but valuations are not—there is an active literature on mechanism design for costly inclusion environments that has grown out of its seminal contributions (Young, 1985b; Young, 1998; Moulin, 1999; Moulin and Shenker, 2001). It is well-known that the mechanisms satisfying *production efficiency* and *strategy-proofness* are precisely the Groves mechanisms in a variety of models, including this one (Groves, 1973; Green and

¹Examples include the overhead costs faced by branches of a firm (Shubik, 1962), the runway costs faced by airlines (Littlechild and Owen, 1973), the costs of common water resources such as multi-purpose reservoirs and Tennessee Valley Authority projects (Straffin and Heaney, 1981; Young, Okada, and Hashimoto, 1982), and the costs of networks such as those for cable television (Sharkey, 1995) and multicast messages (Herzog, Shenker, and Estrin, 1997; Feigenbaum, Papadimitriou, and Shenker, 2001).

Laffont, 1979; Holmström, 1979), and early in this literature it was established that when the cost function is both monotone and concave (but not additive), no Groves mechanism satisfies *no-deficit*, satisfies *voluntary participation*, and never assigns positive transfers (Moulin and Shenker, 2001). Since then, the literature has primarily focused on monotone and concave cost functions (possibly with additional structure) while abandoning *production efficiency* in favor of precisely equating total contributions to total costs (Deb and Razzolini, 1999a; Deb and Razzolini, 1999b; Ohseto, 2000; Mutuswami, 2004; Ohseto, 2005; Yu, 2007; Ohseto, 2009; Massó, Nicolò, Sen, Sharma, and Ülkü, 2015; Hashimoto and Saitoh, 2016).²

A few simple observations suggest that Groves mechanisms merit further investigation for costly inclusion environments. First, if costs are additive, then the classical prescription of marginal cost pricing (adapted to allow for personalized prices) trivially yields a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and that moreover satisfies *no-envy* when costs are symmetric: simply ask each agent if he is willing to pay his constant inclusion cost. Second, if we expand the model to allow for infinite costs modeling infeasibility, then we include auctions with identical objects where agents have unit demand, in which case the Vickrey mechanism (Vickrey, 1961) is a Groves mechanism that achieves all of our objectives.³ These positive observations are mitigated by the fact that the expanded model also allows for binary public goods, for which a well-known version of the free-rider problem states that no Groves mechanism satisfies *no-deficit* and *voluntary participation* (see, for example, Green and Laffont, 1979). Taken together, these observations establish that Groves mechanisms have a limited but non-trivial scope for success. What, then, is this scope?

To answer this question, we introduce *inclusion cost coverage* as a generalization of non-decreasing average costs from quantity-based cost functions to set-based cost functions; a similar generalization was done in the context of cooperative games for convexity (Shapley, 1971). We also introduce *exclusion pivot mechanisms*, variants of standard pivot mechanisms⁴ that differ only in that when

²As notable exceptions, *group strategy-proofness* has been investigated without any cost function restrictions (Juarez, 2013), and approximate versions of axioms have been thoroughly investigated by computer scientists (see Feigenbaum and Shenker, 2004 for an overview).

³In fact, there is a large literature on fair Groves mechanisms for environments with indivisible objects and money. Our paper is most closely related to those that have investigated *envy-free* Groves mechanisms (Pápai, 2003; Ohseto, 2006; Yengin, 2012; Yengin, 2017); the primary difference is that we allow for production.

⁴The family of exclusion pivot mechanisms is essentially single-valued: each exclusion pivot mechanism is distinguished by a method of breaking ties when multiple groups maximize surplus, which has no welfare consequences. Similarly, the family of standard pivot mechanisms is essentially single-valued. We remark that in other models, it is common to refer to the standard

an agent's peers maximize their surplus without him, they are restricted to selecting a group that excludes him. First, we show that the cost function satisfies inclusion cost coverage if and only if there are Groves mechanisms that satisfy *no-deficit* and *voluntary participation* if and only if each exclusion pivot mechanism is such a mechanism (Theorem 1). Second, we show that the cost function is convex and symmetric if and only if there are *envy-free* Groves mechanisms if and only if each exclusion pivot mechanism is such a mechanism (Theorem 2). Because convexity is a stronger requirement than inclusion cost coverage, it follows directly that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments (see Figure 1). Remarkably, this is precisely the case of monotonic supply curves first considered by the classical literature, and our proof involves establishing that for these environments, the exclusion pivot mechanisms have considerable structure involving the supply curve and the *reported* demand curve.

Altogether, our results both (i) reinforce the justification for abandoning *production efficiency* in favor of balancing the budget when costs are concave, and (ii) suggest Groves mechanisms merit further study when costs are convex. Our findings apply to auctions, public goods, natural monopolies, two-sided markets, minimum cost spanning trees, and labor markets that involve revenue functions instead of cost functions; see Section 4.

2 Model

2.1 Costly inclusion environments

We consider the following class of production economies with incomplete information and no consumption externalities:

DEFINITION: A (*costly inclusion*) *environment* is a pair (N, C) , where

- N is a finite and nonempty set of *agents*. For convenience, we write $N = \{1, 2, \dots, n\}$.
- For each $i \in N$, a *bundle* $(t_i, x_i) \in \mathbb{R} \times \{0, 1\}$ specifies (i) a transfer of *money*, and (ii) whether or not i is a *winner*, with 1 representing that he is. An *allocation* $(t, W) \in \mathbb{R}^N \times 2^N$ specifies both a list of monetary transfers t

pivot mechanism as the VCG mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973).

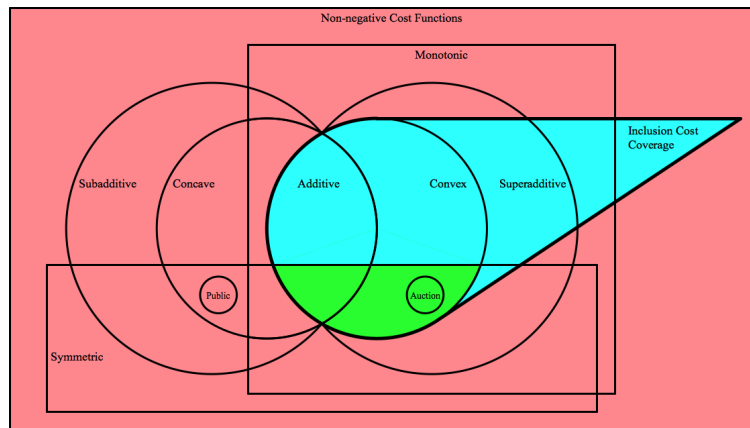
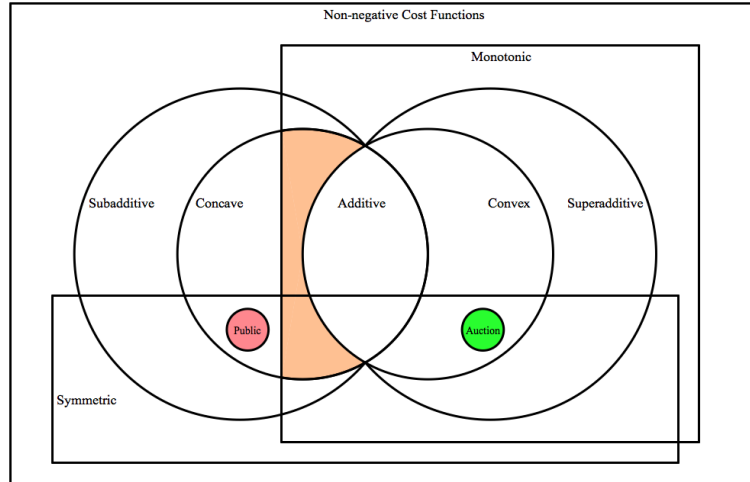


Figure 1: *Summary of results.* (a) For simple auction environments, the Vickrey mechanism is a Groves mechanism that satisfies *no-deficit*, *voluntary participation*, and *no-envy* (Vickrey, 1961). When providing a binary public good, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*; this is a version of the classic free-rider problem (see, for example, Green and Laffont, 1979). When costs are monotone and concave (but not additive), no Groves mechanism satisfies *no-deficit*, satisfies *voluntary participation*, and never assigns positive transfers (Moulin and Shenker, 2001). (b) For any cost function in the red region, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. For any cost function in the blue region, the exclusion pivot mechanisms satisfy *no-deficit* and *voluntary participation*, but no Groves mechanism satisfies *no-envy*. For any cost function in the green region, the exclusion pivot mechanisms satisfy *no-deficit*, *voluntary participation*, and *no-envy*. We establish logical relations justifying how the blue region is drawn in Appendix 3.

and a group of winners W ; this can also be written $(t, x) \in \mathbb{R}^N \times \{0, 1\}^N$ to emphasize the bundle each agent receives.

- Money can be used to produce a group of *winners*; we call an agent who does not win a *loser*. This is summarized by the *cost function*, $C : 2^N \rightarrow \mathbb{R} \cup \{\infty\}$ with $C(\emptyset) = 0$, which specifies the monetary cost of producing each winning group. We let $\mathcal{W}$ denote the collection of groups with finite cost.⁵
- Each $i \in N$ has a *valuation* $v_i \in \mathbb{R}$, which represents the preference relation R_i over bundles such that for each pair $(t_i, x_i), (t'_i, x'_i) \in \mathbb{R} \times \{0, 1\}$,

$$(t_i, x_i) R_i (t'_i, x'_i) \text{ if and only if } x_i v_i + t_i \geq x'_i v_i + t'_i.$$

Each agent's valuation is his private information. We use V_i to denote $\mathbb{R}$ when emphasizing that it is the set of possible valuations for i , and we use $V \equiv \times V_i$ to denote the set of possible valuation profiles.⁶

For convenience, whenever we refer to a generic environment we will implicitly assume all notation introduced thus far.

Costly inclusion environments use infinite costs to model infeasible groups and negative costs to model revenues, and therefore cover many familiar environments from the literature. For example, if there is $k \in \{1, 2, \dots, n\}$ such that for each $W \subseteq N$,

$$C(W) = \begin{cases} 0, & |W| \leq k, \\ \infty, & |W| > k, \end{cases}$$

then we have an auction environment with k identical objects where agents have unit demand. As a second example, if there is $\kappa \in [0, \infty)$ such that

$$C(W) = \begin{cases} 0, & W = \emptyset, \\ \kappa, & W = N, \\ \infty, & \text{else,} \end{cases}$$

then we have a public good environment. Finally, if all costs are negative, and if the associated revenue function satisfies gross substitutes, then we have a Kelso-

⁵We use ∞ to denote positive infinity and $-\infty$ to denote negative infinity; thus for each $x \in \mathbb{R}$, we have $\infty > x > -\infty$. We are careful to only use addition and subtraction for real numbers.

⁶We consider environments where each agent's set of possible valuations is $\mathbb{R}_+$ in [Section 4.3](#).

Crawford labor market environment with a single firm (Kelso and Crawford, 1982). We discuss the implications of our results for these examples and others in Section 4.

2.2 Mechanisms and axioms

Because we focus on dominant strategy implementation, it is without loss of generality to focus on (*direct*) *mechanisms*, where the agents simultaneously report their preferences to an administrator who processes the reports to determine an allocation (Gibbard, 1973; Myerson, 1981):

DEFINITION: Fix an environment. A *transfer policy* is a mapping $\tau : V \rightarrow \mathbb{R}^N$. A *winner policy* is a mapping $\varphi : V \rightarrow 2^N$. A *mechanism* (τ, φ) consists of a transfer policy τ and a winner policy φ . Given a winner policy φ and an agent i , we sometimes speak of the associated policy for determining whether or not i is a winner, using $\varphi_i : V \rightarrow \{0, 1\}$ for the mapping

$$\varphi_i(v) \equiv \begin{cases} 1, & i \in \varphi(v), \\ 0, & i \notin \varphi(v). \end{cases}$$

We are interested in mechanisms that satisfy desirable properties, or *axioms*. In particular, we consider the following five standard axioms:

DEFINITION: Fix an environment. For each $v \in V$ and each $W \subseteq N$, the *surplus of W at v* , $\sigma_v(W)$, is given by:

$$\sigma_v(W) \equiv \begin{cases} \sum_{i \in W} v_i - C(W), & W \in \mathcal{W}, \\ -\infty, & \text{else.} \end{cases}$$

A mechanism (τ, φ) satisfies

- *production efficiency* if and only if for each $v \in V$ and each $W \subseteq N$, we have $\sigma_v(\varphi(v)) \geq \sigma_v(W)$;
- *strategy-proofness* if and only if for each $i \in N$, each $v \in V$, and each $v'_i \in V_i$, we have $\varphi_i(v)v_i + \tau_i(v) \geq \varphi_i(v'_i, v_{-i})v_i + \tau_i(v'_i, v_{-i})$;
- *no-deficit* if and only if for each $v \in V$, we have $0 \geq \sum \tau_i(v) + C(\varphi(v))$;
- *voluntary participation* if and only if for each $i \in N$ and each $v \in V$, we have $\varphi_i(v)v_i + \tau_i(v) \geq 0$; and

- *no-envy* if and only if for each pair $i, j \in N$ and each $v \in V$, we have $\varphi_i(v)v_i + \tau_i(v) \geq \varphi_j(v)v_i + \tau_j(v)$.

We say that φ is *surplus-maximizing* if and only if (τ, φ) satisfies *production efficiency*.

It follows from [Holmström \(1979\)](#) that the class of mechanisms satisfying *production efficiency* and *strategy-proofness* is precisely the class of Groves mechanisms; see [Appendix 1](#) for the familiar definition. We are interested in (i) identifying the costly inclusion environments that have Groves mechanisms satisfying *no-deficit* and *voluntary participation*, and (ii) identifying the costly inclusion environments that have *envy-free* Groves mechanisms.

3 Results

3.1 Preliminaries

The collections of axioms we investigate are logically compatible for some environments but not for others, and this compatibility is determined by certain properties of the cost function. We introduce the properties that prove focal to our analysis here.

DEFINITION: Fix an environment. For each nonempty $W \subseteq N$ and each $i \in W$, define the *inclusion cost for i in W* , $\mathcal{I}_i(W)$, by

$$\mathcal{I}_i(W) \equiv \begin{cases} C(W) - C(W \setminus \{i\}), & W, W \setminus \{i\} \in \mathcal{W}, \\ \infty, & W \notin \mathcal{W}, \\ -\infty, & \text{else.} \end{cases}$$

The cost function satisfies

- *convexity* if and only if for each $i \in N$ and each pair $W, W' \subseteq N$ such that $\{i\} \subseteq W \subseteq W'$, we have $\mathcal{I}_i(W') \geq \mathcal{I}_i(W)$;
- *symmetry* if and only if for each pair $W, W' \subseteq N$ such that $|W| = |W'|$, we have $C(W) = C(W')$; and
- *inclusion cost coverage* if and only if for each nonempty $W \in \mathcal{W}$, we have
 - (i) for each $i \in W$, $W \setminus \{i\} \in \mathcal{W}$, and
 - (ii) $\sum_{i \in W} \mathcal{I}_i(W) \geq C(W)$.

Both *convexity* and *symmetry* are standard properties, while *inclusion cost coverage* is novel. It is easy to verify that *inclusion cost coverage* is (i) satisfied by any auction environment with identical objects where agents have unit demand, and (ii) violated by any public good environment. Moreover, as we discuss in [Section 4.2](#), it follows from results in the literature that *inclusion cost coverage* is satisfied by any Kelso-Crawford labor market environment with a single firm.

In [Appendix 3](#), we investigate the relationship between these cost function properties and three additional standard properties: *monotonicity*, *superadditivity*, and (for *symmetric* cost functions) *non-decreasing average costs*. In particular, we establish seven propositions providing logical relationships among these cost function properties, most of which are illustrated in [Figure 1](#).

Both of our main results also involve variants of the classic *pivot mechanism* ([Clarke, 1971](#)), where each agent pays the difference between (i) the maximum surplus his peers could receive were he absent, and (ii) the surplus his peers do in fact receive. To highlight the distinction, we define both the standard version and our variants:

DEFINITION: Fix an environment. A mechanism (τ, φ) is a *standard pivot mechanism* if and only if (i) φ is surplus-maximizing; and (ii) for each $i \in N$ and each $v \in V$,

$$\tau_i(v) = ([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) - [\max_{W \subseteq N} \sigma_{(0, v_{-i})}(W)].$$

A mechanism (τ, φ) is an *exclusion pivot mechanism* if and only if (i) φ is surplus-maximizing; and (ii) for each $i \in N$ and each $v \in V$,

$$\tau_i(v) = ([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)].$$

The two kinds of pivot mechanisms differ in how they calculate the maximum surplus that the peers of i could receive were he absent. In particular, though in both cases the calculation treats the valuation of i as zero by using the profile $(0, v_{-i})$, the mechanisms differ in what they allow the peers of i to select: a standard pivot mechanism allows them to select any social alternative, while an exclusion pivot mechanism requires them to select a group that excludes i . We remark that due to this difference, standard pivot mechanisms are defined for more general environments where social alternatives are not groups of agents, while

exclusion pivot mechanisms are not. It is not hard to see that standard pivot mechanisms coincide with exclusion pivot mechanisms when costs are monotonic, as in this case the peers of i cannot improve their own surplus by including i , but the two versions differ in general—for example, they differ for public good provision.

Finally, each of our main results concerns one sub-collection of our five axioms, and we frequently refer to these sub-collections in our proof sketches and proofs. For convenience, we introduce the following shorthand terms for these sections:

DEFINITION: A mechanism is

- *autonomous* if and only if it satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*; and
- *equitable* if and only if it satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

Equivalently, an *autonomous* mechanism is a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and an *equitable* mechanism is an *envy-free* Groves mechanism.

3.2 Characterizations

Our first result characterizes the costly inclusion environments for which there are Groves mechanisms satisfying *no-deficit* and *voluntary participation*:

THEOREM 1: Fix an environment. The following are equivalent:

- there is a mechanism that satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*;
- each exclusion pivot mechanism satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*; and
- the cost function satisfies *inclusion cost coverage*.

Proof sketch. See [Appendix 1](#) for the full proof. After verifying that each of our exclusion pivot mechanisms satisfies *production efficiency*, *strategy-proofness*, and *voluntary participation* ([Lemma 1.1](#)), the proof consists of three lemmas. Note that we also provide a more detailed proof sketch for each of these lemmas in

[Appendix 1](#). Recall that for our proofs, we use *autonomous* mechanism as a shorthand for Groves mechanism satisfying *no-deficit* and *voluntary participation*.

We begin by simplifying our problem to the study of exclusion pivot mechanisms. In particular, we first prove that if there is an *autonomous* mechanism (τ, φ) , then each exclusion pivot mechanism is *autonomous* ([Lemma 1.2](#)). By [Lemma 1.1](#), we simply need to show that each exclusion pivot mechanism satisfies *no-deficit*. We first argue that the exclusion pivot mechanism with winner policy φ must satisfy *no-deficit*, as otherwise we can construct a profile where a loser pays under (τ, φ) , contradicting *voluntary participation*; we then take advantage of the structure of exclusion pivot transfer policies to establish the result.

To complete the proof, we therefore simply need to characterize the class of cost functions for which the exclusion pivot mechanisms satisfy *no-deficit*. As a general observation, notice that for each cost function and each valuation profile, any exclusion pivot mechanism requires (i) each loser pays nothing, and (ii) each winner pays at least his inclusion cost. Indeed, (i) follows directly from the definition, while to see (ii), let v be a profile where W^* is selected and let $i \in W^*$ be a winner. If $W^* \setminus \{i\}$ has infinite cost, then i has an inclusion cost of negative infinity and we are done. Otherwise, the optimal surplus that the peers of i can achieve without i is at least what they can achieve at $W^* \setminus \{i\}$, or

$$\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W) \geq \sigma_{(0, v_{-i})}(W^* \setminus \{i\}).$$

It is easy to verify that if the two terms above are equal, then i pays his inclusion cost, while if the left hand side is greater, then i pays more than his inclusion cost. This simple observation establishes [Lemma 1.3](#): if the cost function satisfies *inclusion cost coverage*, then each exclusion pivot mechanism is *autonomous*.

It is less obvious that if the cost function violates *inclusion cost coverage*, then each exclusion pivot mechanism violates *no-deficit*—after all, even when we cannot cover costs by charging each agent his inclusion cost, perhaps we sometimes cover costs anyway because some agents pay more. If all costs are finite, then we can directly confront this issue by considering a profile where all winners have extremely high valuations and all losers have extremely low valuations: in this case, each winner’s peers optimize their surplus without him by simply dropping him from the group of winners, and therefore each winner pays exactly his inclusion cost. This strategy does not work for a public good cost function, however, such

as the following example for $N = \{1, 2, 3\}$:

$$C(W) \equiv \begin{cases} 0, & W = \emptyset, \\ \infty, & \emptyset \subsetneq W \subsetneq N, \\ 5, & W = N. \end{cases}$$

For this example, we cannot construct profiles where we select N and charge winners their inclusion costs because these inclusion costs are each negative infinity. Notice, however, that at the profile $(2, 2, 2)$, each agent pays 1—which is already not enough to cover costs—while at the profile $(100, 100, 100)$, matters are strictly worse as each agent *receives* 195. Indeed, as all valuations increase, each winner is more desperately needed by the others, and is thus compensated by the exclusion pivot mechanisms accordingly. In the proof of [Lemma 1.4](#), we generalize this insight to arbitrary cost functions: if there is even a single agent $i \in W^*$ for which $W^* \setminus \{i\}$ has infinite cost, then across the profiles where we must pick W^* , the sum of the transfers *to* the winners is unbounded. By applying this argument if there is such an agent i and applying the finite-cost argument if there is no such agent i , we establish that when the cost function violates *inclusion cost coverage*, the exclusion pivot mechanisms violate *no-deficit*.

Our second result characterizes the costly inclusion environments for which there are *envy-free* Groves mechanisms:

THEOREM 2: Fix an environment. The following are equivalent:

- there is a mechanism that satisfies *production efficiency*, *strategy-proofness*, and *no-envy*;
- each exclusion pivot mechanism satisfies *production efficiency*, *strategy-proofness*, and *no-envy*; and
- the cost function satisfies *convexity* and *symmetry*.

Proof sketch. See [Appendix 2](#) for the full proof, which consists of three lemmas. Note that we also provide a more detailed proof sketch for each of these lemmas in [Appendix 2](#). Recall that for our proofs, we use *equitable* mechanism as a shorthand for *envy-free* Groves mechanism.

Though with [Theorem 1](#) we began by simplifying our problem to characterizing when the exclusion pivot mechanisms satisfy *no-deficit*, our proof of [Theorem 2](#) has

a completely different structure. In principle, it seems that there could be some cost function for which the exclusion pivot mechanisms violate *no-envy* while some other Groves mechanism satisfies *no-envy*, and in fact we only conclude that this is not the case when we complete the proof of the theorem.

Instead, we first argue that if the cost function violates *convexity*, then there are no *equitable* mechanisms (Lemma 2.1). Consider the following example for $N = \{1, 2, 3\}$, which satisfies *symmetry* and *inclusion cost coverage* but violates *convexity*:

$$C(W) \equiv \begin{cases} 0, & |W| = 0, \\ 1, & |W| = 1, \\ 8, & |W| = 2, \\ 12, & |W| = 3. \end{cases}$$

In this case, one of the violations of *convexity* is that agent 1 has an inclusion cost of 7 for $\{1, 2\}$, while he has a lower inclusion cost of 4 for the larger group $\{1, 2, 3\}$. This means that whenever 1 has a valuation between 4 and 7, it may be optimal to exclude him from the smaller group by picking $\{2\}$ or it may be optimal to include him in the larger group by picking $\{1, 2, 3\}$. In particular, if 2 has a sufficiently high valuation, then the common member of both these groups must be included, and thus with his report agent 3 determines whether *both* he and 1 are included or excluded, regardless of what 1 reports between 4 and 7. It is this feature—that an agent has no voice even when changing his report *while* another controls their shared fate—that is incompatible with *equitable* mechanisms.

To see this, let 1 have low and high valuations of 5 and 6, let 2 always have the valuation 20, and let 3 have low and high valuations of 0 and 15. We use these valuations to construct four profiles, where the first superscript suggests the valuation of agent 1 and the second suggests the valuation of agent 3:

- $v^{++} = (6, 20, 15)$,
- $v^{+-} = (6, 20, 0)$,
- $v^{-+} = (5, 20, 15)$, and
- $v^{--} = (5, 20, 0)$.

By *production efficiency*, we must select $\{1, 2, 3\}$ at both v^{++} and v^{-+} , so by *no-envy* we have $\tau_1(v^{++}) = \tau_3(v^{++})$ and $\tau_1(v^{-+}) = \tau_3(v^{-+})$, and by *strategy-proofness* we have $\tau_1(v^{++}) = \tau_1(v^{-+})$, so altogether we have $\tau_3(v^{++}) = \tau_3(v^{-+})$. Similarly,

by *production efficiency*, we must select $\{2\}$ at both v^{+-} and v^{--} , and it follows that $\tau_3(v^{+-}) = \tau_3(v^{--})$. This implies that the *price* of winning for agent 3 does not depend on the valuation of agent 1: $\tau_3(v^{+-}) - \tau_3(v^{++}) = \tau_3(v^{--}) - \tau_3(v^{-+})$. But in a Groves mechanism, this price must be precisely the externality imposed on the other agents by changing the social outcome, and this externality changes when agent 1 changes his valuation; thus we have a contradiction. In the proof of [Lemma 2.1](#), we show that this insight extends to any cost function with any violation of *convexity*.

We next argue that if the cost function is *convex* but not *symmetric*, then there are no *equitable* mechanisms ([Lemma 2.2](#)). It is intuitive that asymmetry in opportunities as expressed by the cost function might lead to asymmetry in outcomes, as illustrated by the following example for $N = \{1, 2\}$:

$$C(W) \equiv \begin{cases} 0, & W = \emptyset, \\ 1, & W = \{1\}, \\ 4, & W = \{2\}, \\ 10, & W = \{1, 2\}. \end{cases}$$

In this case, the group $\emptyset$ has the feature that the cost of adding agent 1 is 1 while the cost of adding agent 2 is 4. Moreover—and this is the role played by *convexity*—it is costly to sidestep this situation by simply including both agents. It therefore may be optimal to include 1 and exclude 2 even though 2 has the higher valuation, and indeed this is the case at the profile $v = (2, 3)$. At this profile, *no-envy* requires both $2 + \tau_1(v) \geq \tau_2(v)$ (so that 1 does not envy 2) and $\tau_2(v) \geq 3 + \tau_1(v)$ (so that 2 does not envy 1), but this is impossible. In the proof of [Lemma 2.2](#), we argue that if there is any violation of *symmetry*, then we can use *convexity* to identify such a situation.

Finally, we argue that if the cost function is *convex* and *symmetric*, then each exclusion pivot mechanism satisfies *no-envy* ([Lemma 2.3](#)). It is easy to show that each loser pays nothing, and therefore no loser is envied; it remains to show that no loser envies a winner and that no winner envies another winner. Notice that by *convexity* and *symmetry*, there is a (fixed) non-decreasing supply curve, and by properly arranging the valuations there is a (reported) non-increasing demand curve. By the standard insight, *production efficiency* requires that we serve an agent with the highest valuation if and only if this valuation exceeds the lowest marginal cost, then serve an agent with the second-highest valuation if and only if this valuation exceeds the second-lowest marginal cost, and so on. It therefore

follows that for each agent i and each profile v_{-i} , there is a group W^* that will surely be included among the winners, and thus by *strategy-proofness* i faces one of two prices: (i) the marginal cost of joining W^* , if nobody else is willing to pay, or (ii) the highest valuation of somebody willing to pay to join W^* , whom i must outbid. We argue that these individual prices are in fact together compatible with a common price faced by all winners and losers, which can be interpreted as a generalized Vickrey price for non-decreasing supply curves.

4 Discussion

4.1 Costs

Because convexity implies inclusion cost coverage, [Theorem 1](#) and [Theorem 2](#) together imply that whenever production efficient and envy-free allocations can be implemented in dominant strategies, this can moreover be done while funding production through voluntary payments. Remarkably, in this model, implementing production efficiency while providing equal opportunities never requires outside funding (such as subsidies from neighboring communities or borrowing from future generations) or redistribution through taxation. That said, this goal can only be achieved when production is both convex and symmetric.

We conclude this section by discussing some implications of our results in various settings.

APPLICATION 1: AUCTIONS. As discussed in [Section 2](#), the problem of allocating k identical objects among agents with unit demand is modeled by a cost function that (i) assigns zero to any group with at most k members, and (ii) assigns infinity to any group with more than k members. Such a cost function is both *convex* and *symmetric*, so by [Theorem 1](#) and [Theorem 2](#), the exclusion pivot mechanism is a Groves mechanism that satisfies *no-deficit*, *voluntary participation*, and *no-envy*. Indeed, this is simply the Vickrey mechanism, and it is well-known that it satisfies these properties ([Vickrey, 1961](#)).

APPLICATION 2: PUBLIC GOODS. As discussed in [Section 2](#), the problem of providing a pure public good is modeled by a cost function that (i) assigns some $\kappa \in [0, \infty)$ to serving everybody, (ii) assigns zero to serving nobody, and (iii) assigns infinity to serving any strict subset of the agents that is nonempty. Such a cost function violates *inclusion cost coverage*, so by [Theorem 1](#) and [Theorem 2](#), no

Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. Indeed, this is a well-known formalization of the classic free-rider problem (Green and Laffont, 1979). As a particular example, the Lindahl mechanism does not satisfy either collection of axioms because it is not *strategy-proof*.

APPLICATION 3: NATURAL MONOPOLIES. A cost function is *subadditive* if and only if for each disjoint pair $W, W' \subseteq N$, we have $C(W \cup W') \leq C(W) + C(W')$. The cost function is moreover *strictly subadditive* if and only if the inequality is strict for some pair of groups. It has been argued that these are the technologies that lead naturally to monopolies: if one firm produces W and another produces W' , then the two firms can increase profits by merging (Baumol, 1977). Because these cost functions violate *inclusion cost coverage*, thus by Theorem 1 and Theorem 2, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*. As discussed in Section 1, this strengthens a previous impossibility for the subclass of subadditive cost functions that are monotone and concave but not convex (Moulin and Shenker, 2001).

APPLICATION 4: ONE-SIDED COLLUSION IN TWO-SIDED MARKETS. Consider an economy with money and indivisible objects, where each individual desires at most one object and has preferences summarized by his valuation. Suppose that there are n buyers, each endowed with no object, and m sellers, each endowed with one object. In such an economy, there are Walrasian price vectors (Koopmans and Beckmann, 1957; Gale, 1960), and moreover these vectors form a bounded lattice (Shapley and Shubik, 1972). It is well-known that each *minimum-price Walrasian mechanism*, which uses reported buyer valuations to (i) calculate the minimum Walrasian price vector, and (ii) assign the buyers what they would receive in an associated Walrasian outcome, is *strategy-proof* (Demange, 1982; Leonard, 1983).

Our analysis applies to the special case where the objects are identical; the classic example is a market for horses (Böhm-Bawerk, 1888). Indeed, suppose that the n buyer valuations are private information while the m seller valuations are common knowledge, and suppose that the buyers wish to coordinate on an outcome by means of a mechanism. If the seller valuations are ordered $S_1 \leq S_2 \leq \dots \leq S_m$, then feasibility is modeled by the cost function that (i) assigns zero to the empty set, (ii) assigns $\sum_{q'=1}^q S_{q'}$ to any group of size $q \leq m$, and (iii) assigns infinity to any group with more than m members. This cost function is both *convex* and *symmetric*, so by Theorem 1 and Theorem 2, the exclusion pivot mechanisms are

Groves mechanisms that satisfy *no-deficit*, *voluntary participation*, and *no-envy*. Indeed, in this case each exclusion pivot mechanism is a minimum-price Walrasian mechanism.

APPLICATION 5: MINIMUM COST SPANNING TREES. Suppose there is a *source* denoted 0, an *edge* is a distinct pair $i, j \in N \cup \{0\}$, and for each edge ij there is an associated cost $c_{ij} \in \mathbb{R}_+$. Moreover, for each $W \subseteq N$,

- a *spanning tree* (for W) is a collection of edges such that for each $i \in W$, there is a unique path from 0 to i ;
- the cost of a spanning tree is the sum of the costs of its edges; and
- $C(W)$ is the minimum cost of a spanning tree.

Such *minimum cost spanning tree* cost functions (Claus and Kleitman, 1973) have been used to model the provision of utilities (such as electricity, water, sewage, and gas) and multicast transmissions.

Since Bird (1976), the literature has largely focused on the problem of sharing the cost $C(N)$ using cooperative game theory, implicitly requiring all agents to be connected. By contrast, our analysis applies when agents have finite valuations that are private information, in which case it may be inefficient to connect everybody. It is not difficult to see that, unless connecting everybody directly to the source always minimizes costs, the cost function does not satisfy *inclusion cost coverage*, so by Theorem 1 and Theorem 2, no Groves mechanism satisfies *no-deficit* and *voluntary participation*, and no Groves mechanism satisfies *no-envy*.

4.2 Revenues

When all costs are non-negative, it is natural to imagine that $C(W)$ units of divisible money are used as an input, producing an indivisible good for members of W to consume. When all costs are non-positive, however, it is more natural to imagine that members of W provide indivisible labor as an input, producing $-C(W)$ units of divisible money. In this case we can interpret N as a set of workers and interpret the set of winners W as the group of workers that are employed.⁷

Indeed, for each environment (N, C) , each valuation profile v , and each agent i , we can define the *revenue function* $R \equiv -C$ and the *reservation wage* $r_i \equiv -v_i$; observe that if i is willing to pay v_i to work, then he is willing to receive r_i to

⁷We thank an anonymous referee for suggesting this application.

work. This alternative notation allows us to re-write our axioms—for example, *production efficiency* requires that the group of winners maximizes the difference between revenue and the sum of the reservation wages, and *no-deficit* requires that the sum of the transfers to the agents is no greater than the produced revenue. As this change of notation has no impact on the logic of our arguments, our results can be applied to labor market environments that are specified by revenue (or “gross product”) functions. To avoid duplicating all of our definitions with this alternative notation, we simply illustrate this point using the *gross substitutes* condition of Kelso and Crawford (1982):

DEFINITION: Fix a set of agents $N = \{1, 2, \dots, n\}$. A *revenue function* is a mapping $R : 2^N \rightarrow \mathbb{R} \cup \{-\infty\}$ with $R(\emptyset) = 0$. In the context of a revenue function, we let $\mathcal{W}$ denote the collection of groups with finite revenue. For each salary vector $s \in \mathbb{R}^N$, we let $D(s)$ denote the firm’s demand set, $\arg \max_{W \in \mathcal{W}} R(W) - \sum_{i \in W} s_i$. We say that R satisfies

- *gross substitutes* if and only if for each pair $s, s' \in \mathbb{R}^N$ such that for each $i \in N$ we have $s'_i \geq s_i$, and for each $W \in D(s)$, there is a superset of $\{i \in W | s_i = s'_i\}$ in $D(s')$; and
- *concavity* if and only if for each $i \in N$ and each pair $W, W' \subseteq N$ such that $\{i\} \subseteq W \subseteq W'$ and $W' \in \mathcal{W}$, we have
 - (i) $W \in \mathcal{W}$, and
 - (ii) $R(W') - R(W' \setminus \{i\}) \leq R(W) - R(W \setminus \{i\})$.

If a revenue function satisfies *gross substitutes*, then it satisfies *concavity*.⁸ That said, the two conditions are not equivalent: Kelso and Crawford (1982) provide a revenue function that satisfies *concavity* but violates *gross substitutes*.

To simplify the interpretation, suppose that there is a single firm, and suppose that $R(W)$ specifies the revenue generated by this firm when it hires the members of W . If R satisfies *gross substitutes*, if the firm makes offers to workers

⁸This implication was first established under *monotonicity* and the requirement that all groups have finite revenue (Gul and Stacchetti, 1999; Lemma 5), and has since been shown to hold even without *monotonicity* (Paes Leme, 2017; Theorem 2.2). In fact, this implication holds in general for our model; we briefly sketch the straightforward argument. First, argue that *gross substitutes* implies that if W belongs to $\mathcal{W}$, then all of its subsets belong to $\mathcal{W}$. Second, use the argument of Paes Leme (2017) to show that *gross substitutes* implies that for each distinct pair $i, j \in N$ and each $W \subseteq N \setminus \{i, j\}$ such that $W \cup \{i, j\} \in \mathcal{W}$, we have $R(W \cup \{i, j\}) - R(W \cup \{i\}) \leq R(W \cup \{j\}) - R(W)$. Finally, show that this version of submodularity implies *concavity*.

who either tentatively accept or reject these offers, and if this is done iteratively in accordance with the *salary-adjustment process*, then this process converges to a discrete generalization of a competitive equilibrium in a finite number of steps (Kelso and Crawford, 1982). We are interested in comparing this dynamic procedure to a static alternative, where workers report their reservation wages to the firm, which then uses these reports to determine whom to hire and how much to pay each worker.

In fact, it follows immediately from our results that if the revenue function satisfies *gross substitutes*, then (i) each exclusion pivot mechanism is a Groves mechanism that satisfies *no-deficit* and *voluntary participation*, and (ii) there is an *envy-free* Groves mechanism if and only if workers are alike in production in that R is a symmetric function. Indeed, since R satisfies *gross substitutes*, thus R is *concave*, so the associated cost function $C = -R$ is *convex*, so C satisfies *inclusion cost coverage*; the conclusions then follow from Theorem 1 and Theorem 2. On the other hand, if there are *envy-free* Groves mechanisms, then the revenue function satisfies *gross substitutes*: by Theorem 2 we have that C satisfies *symmetry* and *convexity*, so R satisfies *symmetry* and *concavity*, so R satisfies *gross substitutes* (Kelso and Crawford, 1982; Theorem 6).⁹ Finally, if the revenue function satisfies *concavity* but not *gross substitutes*, then the dynamic procedure need not work, but the exclusion pivot mechanism satisfies all of our axioms except *no-envy*.

4.3 Preferences

For our main results, we assume that each agent has quasi-linear preferences, and that moreover the set of admissible valuations is $\mathbb{R}$. How sensitive are our results to these assumptions? We consider both smaller preference domains and larger preference domains.¹⁰

First, for many economic applications, it is common knowledge that inclusion is a good that is costly to provide; thus in Appendix 4, we provide analogues to our main results that apply when (i) each agent's set of admissible valuations is $\mathbb{R}_+$, and (ii) each group is necessarily associated with a non-negative cost. To state our results, we first introduce the modified cost function that associates each

⁹Kelso and Crawford (1982) consider revenue functions for which all groups have finite revenue, but their proof extends directly to our setting.

¹⁰We thank two anonymous referee for suggesting that we consider these two directions.

group with the cost its members would face if they were free to add any outsiders:

$$C^\supseteq(W) = \min_{\{W' \subseteq N \mid W' \supseteq W\}} C(W').$$

In [Appendix 4](#), we prove that for non-negative valuations and non-negative costs, (i) there are autonomous mechanisms if and only if each *standard* pivot mechanism is autonomous if and only if $C^\supseteq$ satisfies inclusion cost coverage ([Theorem 3](#)), and (ii) there are equitable mechanisms if and only if each *standard* pivot mechanism is equitable if and only if $C^\supseteq$ is convex and symmetric ([Theorem 4](#)). It follows that when costs are moreover monotonic, our main results apply without modification.

Second, suppose we allow for income effects by relaxing quasi-linearity. In particular, suppose we assume only that for each agent i , (i) there is a *willingness to pay* w_i such that he is indifferent between $(0, 0)$ and $(-w_i, 1)$, and (ii) more money is better. It is clear that whenever there is no desirable mechanism with quasi-linearity, there is no desirable mechanism with this larger preference domain: otherwise the restriction to quasi-linear profiles would be desirable. On the other hand, whenever there are desirable quasi-linear mechanisms, one of them is an exclusion pivot mechanism; in this case we can simply ask each agent to report his willingness to pay and use the exclusion pivot mechanism treating the reports as quasi-linear valuations.¹¹ With the exception of *production efficiency*, it is not hard to see that this extended mechanism inherits all axioms satisfied by its quasi-linear restriction: *no-envy* follows from the fact that losers always receive zero transfer, while the other axioms are trivial. The catch is that without quasi-linearity, maximizing the sum of the willingness to pay reports minus the cost does not imply there is no Pareto improvement that burns the same amount of money; this offers an intriguing direction for future work.

Appendix 1

In this appendix, we prove [Theorem 1](#). We begin by stating (a corollary of a modification of) a fundamental theorem:

DEFINITION: Fix an environment. A mechanism (τ, φ) is a *Groves mechanism* if and only if (i) φ is surplus-maximizing, and (ii) for each $i \in N$, there is $\theta_i : V_{-i} \rightarrow \mathbb{R}$

¹¹This method of extending mechanisms beyond quasi-linear domains using the fixed reference bundle $(0, 0)$ has been previously considered for auctions ([Saitoh and Serizawa, 2008](#); [Sakai, 2008](#)), while an interesting variant in which reference bundles differ across agents been previously considered for public goods ([Hashimoto and Saitoh, 2016](#)).

such that for each $v \in V$,

$$\tau_i(v) = \left(\left[\max_{W \subseteq N} \sigma_v(W) \right] - \varphi_i(v)v_i \right) + \theta_i(v_{-i}).$$

- **THEOREM H** ([HOLMSTRÖM, 1979](#)):¹² Fix an environment. A mechanism is *production efficient* and *strategy-proof* if and only if it is a Groves mechanism.

Our first lemma states that each exclusion pivot mechanism satisfies all requirements to be *autonomous* except possibly *no-deficit*. This is essentially a corollary of [Theorem H](#); we include the formal statement, together with its short proof, for completeness.

- **LEMMA 1.1:** Fix an environment. Each exclusion pivot mechanism satisfies *production efficiency*, *strategy-proofness*, and *voluntary participation*.

PROOF: Let (τ, φ) be an exclusion pivot mechanism. For $i \in N$ and each $v_{-i} \in V_{-i}$, we can define $\theta_i(v_{-i}) \equiv -[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)]$; as $i \in N$ was arbitrary, thus by [Theorem H](#), (τ, φ) is *production efficient* and *strategy-proof*.

To see that (τ, φ) satisfies *voluntary participation*, let $i \in N$ and let $v \in V$. Then

$$\begin{aligned} \max_{W \subseteq N} \sigma_v(W) &\geq \max_{W \subseteq N \setminus \{i\}} \sigma_v(W) \\ &= \max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W), \end{aligned}$$

so $\tau_i(v) \geq -\varphi_i(v)v_i$. It follows immediately that (τ, φ) satisfies *voluntary participation*. ■

Our second lemma states that the existence of *autonomous* mechanisms can always be determined using the exclusion pivot mechanisms. We begin by assuming there is an *autonomous* mechanism (τ, φ) ; the proof then consists of two steps.

First, we show that the exclusion pivot mechanism with winner policy φ , $(\tau^{\text{pl}\varphi}, \varphi)$, is *autonomous*. This is shown by contradiction: if not, then by the

¹²Though the original theorem of [Holmström \(1979\)](#) does not involve cost functions, the straightforward adaptation of its proof goes through line-by-line for the modified version with cost functions. Our statement is in fact a corollary of the modified theorem, which follows because the domain $\times_N \mathbb{R}$ is a topologically connected space.

previous lemma $(\tau^{\text{pl}\varphi}, \varphi)$ satisfies all requirements but *no-deficit*, so there is some profile v and some agent i such that i receives a higher transfer at v under $(\tau^{\text{pl}\varphi}, \varphi)$ than under (τ, φ) . But then we can construct a profile (x_i, v_{-i}) where i loses and pays under (τ, φ) , contradicting that this mechanism satisfies *voluntary participation*.

Second, we show that each exclusion pivot mechanism $(\tau^{\text{p}}, \varphi^{\text{p}})$ is *autonomous*. By taking advantage of the structure of exclusion pivot transfer policies, we write the difference in net transfers at an arbitrary profile between $(\tau^{\text{pl}\varphi}, \varphi)$ and $(\tau^{\text{p}}, \varphi^{\text{p}})$ as the difference in the costs of the selected groups, from which we can conclude that neither mechanism runs a deficit at the given profile. By the previous lemma, we are done.

• **LEMMA 1.2:** Fix an environment. If there is an *autonomous* mechanism, then each exclusion pivot mechanism is *autonomous*.

PROOF: Assume (τ, φ) is *autonomous*. By [Theorem H](#), since (τ, φ) satisfies *production efficiency* and *strategy-proofness*, thus for each $i \in N$, there is $\theta_i : V_{-i} \rightarrow \mathbb{R}$ such that for each $v \in V$,

$$\tau_i(v) = ([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) + \theta_i(v_{-i}).$$

Let $(\tau^{\text{pl}\varphi}, \varphi)$ be the exclusion pivot mechanism whose winner policy is φ .

◦ **STEP 1:** The mechanism $(\tau^{\text{pl}\varphi}, \varphi)$ is *autonomous*.

Assume, by way of contradiction, that $(\tau^{\text{pl}\varphi}, \varphi)$ is not *autonomous*. Then by [Lemma 1.1](#), $(\tau^{\text{pl}\varphi}, \varphi)$ violates *no-deficit*.

Since (τ, φ) satisfies *no-deficit* and $(\tau^{\text{pl}\varphi}, \varphi)$ does not, thus there is $v \in V$ such that

$$\begin{aligned} \sum \tau_i^{\text{pl}\varphi}(v) &> -C(\varphi(v)) \\ &\geq \sum \tau_i(v). \end{aligned}$$

Then there is $i \in N$ such that

$$\begin{aligned}
([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W)] &= \tau_i^{\text{pl}\varphi}(v) \\
&> \tau_i(v) \\
&= ([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) + \theta_i(v_{-i}).
\end{aligned}$$

It cannot be that $\varphi(v) \subseteq N \setminus \{i\}$; else as φ is surplus-maximizing, thus for each $W \subseteq N \setminus \{i\}$,

$$\begin{aligned}
\sigma_{(0,v_{-i})}(\varphi(v)) &= \sigma_v(\varphi(v)) \\
&\geq \sigma_v(W) \\
&= \sigma_{(0,v_{-i})}(W),
\end{aligned}$$

so $[\max_{W \subseteq N} \sigma_v(W)] = \sigma_v(\varphi(v)) = \sigma_{(0,v_{-i})}(\varphi(v)) = [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W)]$ and thus $\tau_i(v) < \tau_i^{\text{pl}\varphi}(v) = 0$, contradicting that (τ, φ) satisfies *voluntary participation*.

Thus $i \in \varphi(v)$. Define

$$\sigma^* \equiv \max_{\{W \subseteq N \mid i \in W\}} \sigma_{(0,v_{-i})}(W).$$

Since $i \in \varphi(v)$ and φ is surplus-maximizing, thus

$$\begin{aligned}
\sigma^* &\geq \sigma_{(0,v_{-i})}(\varphi(v)) \\
&= \sigma_v(\varphi(v)) - v_i \\
&\geq \sigma_v(\emptyset) - v_i \\
&= -v_i,
\end{aligned}$$

so $\sigma^* > -\infty$. Define $x_i \in V_i$ by $x_i \equiv -(\sigma^* + 1)$.

By construction, at profile (x_i, v_{-i}) , each group that includes i receives negative surplus and thus cannot be chosen over $\emptyset$, so surplus is maximized at a subset of $N \setminus \{i\}$. But then since $\theta_i(v_{-i}) < -[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W)]$, thus

$$\begin{aligned}
\tau_i(x_i, v_{-i}) &= ([\max_{W \subseteq N} \sigma_{(x_i, v_{-i})}(W)] - 0) + \theta_i(v_{-i}) \\
&< ([\max_{W \subseteq N} \sigma_{(x_i, v_{-i})}(W)] - 0) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0,v_{-i})}(W)] \\
&= ([\max_{W \subseteq N \setminus \{i\}} \sigma_{(x_i, v_{-i})}(W)] - 0) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(x_i, v_{-i})}(W)] \\
&= 0,
\end{aligned}$$

contradicting that (τ, φ) satisfies *voluntary participation*. $\square$

◦ STEP 2: Each exclusion pivot mechanism is *autonomous*.

Let (τ^P, φ^P) be an exclusion pivot mechanism and let $v \in V$. Then the net transfer at v according to $(\tau^{P|\varphi}, \varphi)$ can be written

$$\begin{aligned} \sum \tau_i^{P|\varphi}(v) &= \left[\sum \left(\left[\max_{W \subseteq N} \sigma_v(W) \right] - \varphi_i(v)v_i \right) - \left[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W) \right] \right] \\ &= \left[\sum \left(\left[\max_{W \subseteq N} \sigma_v(W) \right] - \left[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W) \right] \right) \right] - \left[\sum \varphi_i(v)v_i \right]. \end{aligned}$$

Similarly, the net transfer at v according to (τ^P, φ^P) can be written

$$\sum \tau_i^P(v) = \left[\sum \left(\left[\max_{W \subseteq N} \sigma_v(W) \right] - \left[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W) \right] \right) \right] - \left[\sum \varphi_i^P(v)v_i \right].$$

The difference between these net transfers is therefore

$$\sum \tau_i^{P|\varphi}(v) - \sum \tau_i^P(v) = \sum \varphi_i^P(v)v_i - \sum \varphi_i(v)v_i.$$

Since φ and φ^P are both surplus-maximizing, thus

$$\sum \varphi_i(v)v_i - C(\varphi(v)) = \sum \varphi_i^P(v)v_i - C(\varphi^P(v)),$$

so altogether the difference between net transfers is

$$\sum \tau_i^{P|\varphi}(v) - \sum \tau_i^P(v) = C(\varphi^P(v)) - C(\varphi(v)).$$

As $(\tau^{P|\varphi}, \varphi)$ satisfies *no-deficit*, thus

$$\begin{aligned} \sum \tau_i^P(\varphi(v)) + C(\varphi^P(v)) &= \sum \tau_i^{P|\varphi}(v) + C(\varphi(v)) \\ &\leq 0, \end{aligned}$$

from which it follows that (τ^P, φ^P) runs no deficit at v . Since $v \in V$ was arbitrary, thus (τ^P, φ^P) satisfies *no-deficit*, so by [Lemma 1.1](#), (τ^P, φ^P) is *autonomous*. $\blacksquare$

Our third lemma states that *inclusion cost coverage* guarantees each exclusion pivot mechanism is *autonomous*. In the proof, we argue that if the cost function satisfies this condition, then at each profile, any exclusion pivot mechanism requires both that (i) each loser pays nothing, and (ii) each winner pays at least his

inclusion cost; it then follows that there is no deficit, and by our first lemma we are done.

• **LEMMA 1.3:** Fix an environment. If the cost function satisfies *inclusion cost coverage*, then each exclusion pivot mechanism is *autonomous*.

PROOF: Fix an environment whose cost function satisfies *inclusion cost coverage*, and let (τ, φ) be an exclusion pivot mechanism. By [Lemma 1.1](#), we need only prove that (τ, φ) satisfies *no-deficit*. Let $v \in V$ and define $W^* \equiv \varphi(v)$. Since φ is surplus-maximizing, thus $W^* \in \mathcal{W}$.

For each $i \in N \setminus W^*$, since φ is surplus-maximizing, we have

$$\begin{aligned} \sigma_{(0, v_{-i})}(W^*) &= \sigma_v(W^*) \\ &= \max_{W \subseteq N \setminus \{i\}} \sigma_v(W) \\ &= \max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W). \end{aligned}$$

Altogether, for each $i \in N \setminus W^*$, $\tau_i(v) = 0$.

Let $i \in W^*$. By *inclusion cost coverage*, $W^* \setminus \{i\} \in \mathcal{W}$; thus

$$\begin{aligned} \max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W) &\geq \sigma_{(0, v_{-i})}(W^* \setminus \{i\}) \\ &= \sum_{W^* \setminus \{i\}} v_j - C(W^* \setminus \{i\}) \\ &= (\sigma_v(W^*) + C(W^*) - v_i) - C(W^* \setminus \{i\}) \\ &= ([\max_{W \subseteq N} \sigma_v(W)] + C(W^*) - v_i - C(W^* \setminus \{i\})). \end{aligned}$$

As $\varphi_i(v) = 1$, thus

$$\begin{aligned} \tau_i(v) &= ([\max_{W \subseteq N} \sigma_v(W)] - v_i) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)] \\ &\leq ([\max_{W \subseteq N} \sigma_v(W)] - v_i) - ([\max_{W \subseteq N} \sigma_v(W)] + C(W^*) - v_i - C(W^* \setminus \{i\})) \\ &= C(W^* \setminus \{i\}) - C(W^*) \\ &= -\mathcal{I}_i(W^*). \end{aligned}$$

Altogether, for each $i \in W^*$, $\tau_i(v) \leq -\mathcal{I}_i(W^*)$.

If $W^* = \emptyset$, it is immediate that $\sum \tau_i(v) = 0 = -C(\varphi(v))$, so (τ, φ) does not run a deficit at v . If W^* is nonempty, then since $W^* \in \mathcal{W}$, thus by *inclusion cost*

coverage, $\sum_{W^*} \tau_i(W^*) \geq C(W^*)$, so

$$\begin{aligned}
\sum \tau_i(v) &= \sum_{W^*} \tau_i(v) + \sum_{N \setminus W^*} \tau_i(v) \\
&= \sum_{W^*} \tau_i(v) \\
&\leq \sum_{W^*} (-\tau_i(W^*)) \\
&\leq -C(W^*) \\
&= -C(\varphi(v)),
\end{aligned}$$

so (τ, φ) does not run a deficit at v . Since $v \in V$ was arbitrary, we are done. ■

Our fourth lemma states that *inclusion cost coverage* is necessary for the existence of *autonomous* mechanisms. We proceed by contradiction, assuming that *inclusion cost coverage* is violated but there is an *autonomous* mechanism. Then (i) there is a group with finite cost W^* where *inclusion cost coverage* is violated, and (ii) each exclusion pivot mechanism is *autonomous*. For each sufficiently high x , we introduce a profile v^x where members of W^* have valuation x and the others have valuation $-nx$; the proof then consists of four steps.

First, we show that at any profile v^x , nobody outside of W^* can win: such an agent's valuation is so low that it would be better to select the empty set. Second, we show that each agent in W^* must win: such an agent's valuation is so high that it offsets any cost savings associated with his exclusion.

For each agent i in W^* , we then introduce the group W_i , which (i) includes only members of W^* and does not include i , (ii) has finite cost, (iii) among such groups is largest, and (iv) among such groups has smallest cost. Third, we show that at any profile v^x , the peers of i maximize their surplus without i at W_i ; the arguments are similar to those in the first two steps.

To conclude, we consider a profile v^x where, by the first two steps, W^* wins. We first use the structure of the exclusion pivot transfer policy to show that each loser pays nothing and each winner i pays an amount involving the costs of W^* and W_i . If for each winner i , $W_i = W^* \setminus \{i\}$, then each winner pays precisely his inclusion cost, which does not cover costs, contradicting *no-deficit*. If instead for some winner i , $W_i \neq W^* \setminus \{i\}$, then the net transfer at v^x is at least the sum of x and a fixed term; but then as x was any sufficiently high number, there is some

profile v^{x^*} where the cost of W^* is not covered, again contradicting *no-deficit*.

• **LEMMA 1.4:** Fix an environment. If the cost function violates *inclusion cost coverage*, then there is no *autonomous* mechanism.

PROOF: Fix an environment and assume, by way of contradiction, that the cost function violates *inclusion cost coverage* and there is an *autonomous* mechanism. Let (τ, φ) be an exclusion pivot mechanism. By [Lemma 1.2](#), (τ, φ) is *autonomous*.

Since C violates *inclusion cost coverage*, thus there is nonempty $W^* \in \mathcal{W}$ such that either

- (i) there is $i \in W^*$ such that $W^* \setminus \{i\} \notin \mathcal{W}$, or
- (ii) $\sum_{i \in W^*} \tau_i(W^*) < C(W^*)$.

Necessarily, $|W^*| \geq 2$.

Define the *maximum absolute finite cost*, $\kappa \in \mathbb{R}_+$, by:

$$\kappa \equiv \max_{W \in \mathcal{W}} |C(W)|.$$

Since $\emptyset \in \mathcal{W}$, thus κ is well-defined. For each $x \geq 2\kappa + 1$, define $v^x \in V$ by:

$$v_i^x \equiv \begin{cases} x, & i \in W^*, \\ -nx, & \text{else.} \end{cases}$$

◦ **STEP 1:** For each $x \geq 2\kappa + 1$ and each $i \in N \setminus W^*$, $i \notin \varphi(v^x)$.

If $N \setminus W^* = \emptyset$ we are done, so assume $N \setminus W^* \neq \emptyset$. Let $x \geq 2\kappa + 1$, let $i \in N \setminus W^*$, and let $W' \subseteq N$ such that $i \in W'$. Then

$$\begin{aligned} \sigma_{v^x}(W') &= \sum_{W' \cap W^*} v_j^x + \sum_{W' \setminus W^*} v_j^x - C(W') \\ &\leq (n-1)x + v_i^x + \kappa \\ &= (n-1)x - nx + \kappa \\ &= \kappa - x \\ &< 0 \\ &= \sigma_{v^x}(\emptyset), \end{aligned}$$

so since φ is surplus-maximizing, thus $\varphi(v^x) \neq W'$. Since W' including i was

arbitrary, thus $i \notin \varphi(v^x)$. Since $x \geq 2\kappa + 1$ and $i \in N \setminus W^*$ were arbitrary, we are done. $\square$

◦ STEP 2: For each $x \geq 2\kappa + 1$, $\varphi(v^x) = W^*$.

Let $x \geq 2\kappa + 1$. By Step 1, $\varphi(v^x) \subseteq W^*$. Let $W' \subsetneq W^*$. Then

$$\begin{aligned} \sigma_{v^x}(W^*) - \sigma_{v^x}(W') &= \left(\sum_{W^*} v_j^x - C(W^*) \right) - \left(\sum_{W'} v_j^x - C(W') \right) \\ &\geq (|W^*|x - \kappa) - ((|W^*| - 1)x + \kappa) \\ &= x - 2\kappa \\ &> 0. \end{aligned}$$

Since $W' \subsetneq W^*$ was arbitrary, and since φ is surplus-maximizing, thus $\varphi(v^x) = W^*$. $\square$

Before proceeding to the next step, we first introduce a family of sets $\{W_i\}_{i \in W^*}$. In particular, for each $i \in W^*$, let $W_i \subseteq W^* \setminus \{i\}$ be such that:

- $W_i \in \mathcal{W}$,
- for each $W \subseteq W^* \setminus \{i\}$ such that $W \in \mathcal{W}$, $|W_i| \geq |W|$, and
- for each $W \subseteq W^* \setminus \{i\}$ such that $W \in \mathcal{W}$ and $|W_i| = |W|$, $C(W_i) \leq C(W)$.

Thus W_i is one of the largest subsets of $W^* \setminus \{i\}$ with finite cost, and among these sets its cost is smallest. Since $\emptyset \subseteq W^* \setminus \{i\}$ and $\emptyset \in \mathcal{W}$, thus we can indeed select such a set to be W_i .

◦ STEP 3: For each $x \geq 2\kappa + 1$ and each $i \in W^*$, $[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i}^x)}(W)] = \sigma_{(0, v_{-i}^x)}(W_i)$.

Let $x \geq 2\kappa + 1$ and let $i \in W^*$. Define $\sigma^* \equiv [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i}^x)}(W)]$, and let $W_i^* \subseteq N \setminus \{i\}$ such that $\sigma_{(0, v_{-i}^x)}(W_i^*) = \sigma^*$. Since $\sigma^* \geq \sigma_{(0, v_{-i}^x)}(\emptyset) = 0$, thus $W_i^* \in \mathcal{W}$.

We claim that $W_i^* \subseteq W^*$. Indeed, assume, by way of contradiction, there is

$j \in W_i^* \setminus W^*$. Then as $i \notin W_i^*$, we have

$$\begin{aligned}
\sigma_{(0, v_{-i}^x)}(W_i^*) &= |W_i^* \cap W^*|x + (|W_i^* \setminus W^*|)(-nx) - C(W_i^*) \\
&\leq (n-1)x + (-nx) + \kappa \\
&= \kappa - x \\
&< 0 \\
&= \sigma_{(0, v_{-i}^x)}(\emptyset),
\end{aligned}$$

contradicting that W_i^* maximizes surplus at $(0, v_{-i}^*)$ among subsets of $N \setminus \{i\}$. Thus $W_i^* \subseteq W^*$, as desired.

Since $i \notin W_i^*$, altogether we have that $W_i^* \subseteq W^* \setminus \{i\}$ and $W_i^* \in \mathcal{W}$. Define $\mathcal{W}_1 \equiv \{W \subseteq W^* \setminus \{i\} \mid W \in \mathcal{W}\}$. If there are $W, W' \in \mathcal{W}_1$ such that $|W| > |W'|$, then

$$\begin{aligned}
\sigma_{(0, v_{-i}^x)}(W) - \sigma_{(0, v_{-i}^x)}(W') &= (|W|x - C(W)) - (|W'|x - C(W')) \\
&= (|W| - |W'|)x + (C(W') - C(W)) \\
&\geq x - 2\kappa \\
&> 0;
\end{aligned}$$

thus W_i^* is a set with highest cardinality in $\mathcal{W}_1$.

Define $\mathcal{W}_2 \equiv \{W \in \mathcal{W}_1 \mid |W| = |W_i^*|\}$. Clearly, W_i^* is a set with smallest cost in $\mathcal{W}_2$.

Define $\mathcal{W}_3 \equiv \{W \in \mathcal{W}_2 \mid C(W) = C(W_i^*)\}$. By construction, $W_i, W_i^* \in \mathcal{W}_3$. Clearly, $\sigma^* = \sigma_{(0, v_{-i}^x)}(W_i^*) = \sigma_{(0, v_{-i}^x)}(W_i)$, as desired. $\square$

◦ STEP 4: Conclude.

Let $x \geq 2\kappa + 1$. For each $i \in N \setminus W^*$, by the fact that φ is surplus-maximizing and Step 2,

$$\begin{aligned}
\tau_i(v^x) &= \left(\max_{W \subseteq N} \sigma_{v^x}(W) \right) - \varphi_i(v^x)v_i^x - \left[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i}^x)}(W) \right] \\
&= (\sigma_{v^x}(W^*) - 0) - \sigma_{v^x}(W^*) \\
&= 0.
\end{aligned}$$

For each $i \in W^*$, by Step 2, the fact that φ is surplus-maximizing, and Step 3,

$$\begin{aligned}
\tau_i(v^x) &= ([\max_{W \subseteq N} \sigma_{v^x}(W)] - \varphi_i(v^x)v_i^x) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i}^x)}(W)] \\
&= ([\sigma_{v^x}(W^*)] - x) - [\sigma_{(0, v_{-i}^x)}(W_i)] \\
&= ([|W^*|x - C(W^*)] - x) - [|W_i|x - C(W_i)] \\
&= (|W^*| - |W_i| - 1)x + (C(W_i) - C(W^*)).
\end{aligned}$$

Thus

$$\sum \tau_i(v^x) = \left[\sum_{W^*} (|W^*| - |W_i| - 1)x \right] + \left[\sum_{W^*} C(W_i) - C(W^*) \right].$$

For each $i \in W^*$, since $W_i \subseteq W^* \setminus \{i\}$, thus $|W_i| \leq |W^*| - 1$.

To conclude, we consider two cases: (i) for each $i \in W^*$, $|W_i| = |W^*| - 1$, and (ii) there is $i \in W^*$ such that $|W_i| < |W^*| - 1$. Equivalently, these two cases are (i) for each $i \in W^*$, $W^* \setminus \{i\}$ has finite cost, and (ii) there is $i \in W^*$ such that $W^* \setminus \{i\}$ has infinite cost.

First, assume that for each $i \in W^*$, $|W_i| = |W^*| - 1$. Then for each $i \in W^*$, $W_i = W^* \setminus \{i\}$ and thus $W^* \setminus \{i\} \in \mathcal{W}$. Then $\sum_{i \in W^*} \mathcal{I}_i(W^*) < C(W^*)$, so by construction of W^* and Step 2,

$$\begin{aligned}
\sum \tau_i(v^x) &= \left[\sum_{W^*} C(W_i) - C(W^*) \right] \\
&= \left[\sum_{W^*} C(W^* \setminus \{i\}) - C(W^*) \right] \\
&= \sum_{W^*} (-\mathcal{I}_i(W^*)) \\
&> -C(W^*) \\
&= -C(\varphi(v^x)),
\end{aligned}$$

contradicting *no-deficit*.

Second, assume there is $i \in W^*$ such that $|W_i| < |W^*| - 1$. Then

$$\sum \tau_i(v^x) \geq x + \left[\sum_{W^*} C(W_i) - C(W^*) \right].$$

Since $x \geq 2\kappa + 1$ was arbitrary, thus using Step 2, there is a sufficiently high

$x^* \geq 2\kappa + 1$ such that

$$\begin{aligned}\sum \tau_i(v^{x^*}) &> -C(W^*) \\ &= -C(\varphi(v^{x^*})),\end{aligned}$$

contradicting *no-deficit*. ■

To conclude, [Theorem 1](#) is a direct corollary of our previous lemmas:

- **THEOREM 1 (REPEATED):** Fix an environment. The following are equivalent:
 - there is a mechanism that satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*;
 - each exclusion pivot mechanism satisfies *production efficiency*, *strategy-proofness*, *no-deficit*, and *voluntary participation*; and
 - the cost function satisfies *inclusion cost coverage*.

PROOF: The first item implies the third by [Lemma 1.4](#), the third implies the second by [Lemma 1.3](#), and the second implies the first trivially. ■

Appendix 2

In this appendix, we prove [Theorem 2](#). We begin by proving that if costs are not *convex*, then there are no *equitable* mechanisms. We proceed by contradiction, assuming that costs are not *convex* but there is an *equitable* mechanism; the proof then consists of four steps.

First, we use the lack of *convexity* to identify an agent i and a pair of groups W_-, W_+ that together satisfy four properties. These properties imply that at certain profiles, another agent can determine whether $W_- \setminus \{i\}$ wins or W_+ wins, and thus determine whether or not i wins. In particular, these four properties are (i) i belongs to W_- , which is contained in W_+ ; (ii) the inclusion cost for i in W_+ is smaller than his inclusion cost in W_- ; (iii) $W_- \setminus \{i\}$ certainly has finite cost (and by the previous property, W_+ does as well); and (iv) among groups that contain $W_- \setminus \{i\}$ and that are contained in W_+ , the only that might have finite cost are $W_- \setminus \{i\}$, W_- , $W_+ \setminus \{i\}$, and W_+ .

Next, we take the other agent j to be anybody in W_+ but not W_- . In the second step, we carefully define (i) high and low valuations for i , and (ii) high and low valuations for j . We then use these to construct four profiles: v^{++} , v^{+-} , v^{-+} , and v^{--} . At profiles where the first superscript is $+$, i has his high valuation; at profiles where the first superscript is $-$, i has his low valuation; the second superscript plays the same role for j . The valuations of agents in $N \setminus \{i, j\}$ are fixed so that across these four profiles, *production efficiency* implies that the winning group must contain $W_- \setminus \{i\}$ and be contained in W_+ . Both valuations of i are between his inclusion cost in W_+ and his inclusion cost in W_- , while the valuations of j allow him to demand inclusion or exclusion across these profiles.

In the third step, we show that across the four profiles, W_+ wins whenever j has his high valuation and $W_- \setminus \{i\}$ wins whenever j has his low valuation. In the final step, we conclude by deriving a contradiction: j must receive the same transfer at both profiles where he wins and the same transfer at both profiles where he loses, which is impossible for any Groves mechanism.

• **LEMMA 2.1:** Fix an environment. If the cost function violates *convexity*, then there are no *equitable* mechanisms.

PROOF: Let C be a cost function that violates *convexity*, and assume, by way of contradiction, that (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

◦ **STEP 1:** There are $i \in N$ and $W_-, W_+ \subseteq N$ such that

- (i) $\{i\} \subseteq W_- \subseteq W_+$;
- (ii) $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$;
- (iii) $W_- \setminus \{i\} \in \mathcal{W}$; and
- (iv) for each $W \in 2^N \setminus \{W_- \setminus \{i\}, W_-, W_+ \setminus \{i\}, W_+\}$ such that $W_- \setminus \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$.

Since C violates *convexity*, there are $k \in N$ and $W'_-, W'_+ \subseteq N$ such that (i) $\{k\} \subseteq W'_- \subseteq W'_+$, and (ii) $\mathcal{I}_k(W'_-) > \mathcal{I}_k(W'_+)$.

We first claim that there are $i \in N$ and $W_-^* \subseteq W'_-$ such that (i) $\{i\} \subseteq W_-^* \subseteq W'_+$, (ii) $\mathcal{I}_i(W_-^*) > \mathcal{I}_i(W'_+)$, and (iii) $W_-^* \setminus \{i\} \in \mathcal{W}$. Indeed, if $W'_- \setminus \{k\} \in \mathcal{W}$, simply define $i \equiv k$ and $W_-^* \equiv W'_-$. If $W'_- \setminus \{k\} \notin \mathcal{W}$, then because $\mathcal{I}_k(W'_-) > \mathcal{I}_k(W'_+) \geq -\infty$, necessarily $W'_- \notin \mathcal{W}$. In this case, define $W_0 \equiv W'_-$, and for each $t \in \{1, 2, \dots, |W'_-|\}$, define

- $i_t \equiv \min W_{t-1}$, and
- $W_t \equiv W_{t-1} \setminus \{i_t\}$.

It is straightforward to verify that $W_{|W'_-|} = \emptyset$, so $C(W_{|W'_-|}) = 0$. Thus there is $t \in \{1, 2, \dots, |W'_-|\}$ such that $W_t \in \mathcal{W}$ and $W_{t-1} \notin \mathcal{W}$. Define $i \equiv i_t$ and define $W_-^* \equiv W_{t-1}$; then (i) $\{i\} \subseteq W_-^* \subseteq W'_- \subseteq W'_+$, and (iii) $W_-^* \setminus \{i\} = W_t \in \mathcal{W}$. Moreover, $W_-^* = W_{t-1} \notin \mathcal{W}$, so $\mathcal{I}_i(W_-^*) = \infty$. Since $\{i\} \subseteq W'_+$, thus we can define $\mathcal{I}_i(W'_+)$; and since $\mathcal{I}_k(W'_+) < \mathcal{I}_k(W'_-) \leq \infty$, thus $W'_+ \in \mathcal{W}$, so $\mathcal{I}_i(W'_+) < \infty$; altogether, then, we have (ii) $\mathcal{I}_i(W'_+) < \infty = \mathcal{I}_i(W_-^*)$. This completes the proof of the claim.

Define

$$\mathcal{W}_+ \equiv \{W \subseteq N \mid \{i\} \subseteq W_-^* \subseteq W \subseteq W'_+ \text{ and } \mathcal{I}_i(W_-^*) > \mathcal{I}_i(W)\},$$

which is nonempty as $W'_+ \in \mathcal{W}_+$. Let W_+ be a set that is minimal in $\mathcal{W}_+$ with respect to set inclusion.

Define

$$\mathcal{W}_- \equiv \{W \subseteq N \mid \{i\} \subseteq W_-^* \subseteq W \subseteq W_+, \mathcal{I}_i(W) > \mathcal{I}_i(W_+), \text{ and } W \setminus \{i\} \in \mathcal{W}\},$$

which is nonempty as $W_-^* \in \mathcal{W}_-$. Let W_- be a set that is maximal in $\mathcal{W}_-$ with respect to set inclusion.

By construction, we have that (i) $\{i\} \subseteq W_- \subseteq W_+$, (ii) $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, and (iii) $W_- \setminus \{i\} \in \mathcal{W}$. To complete the proof of Step 1, let $W \in 2^N$ such that $W_- \subsetneq W \subsetneq W_+$. Then we cannot have $\mathcal{I}_i(W) \leq \mathcal{I}_i(W_+)$, else $\mathcal{I}_i(W) < \mathcal{I}_i(W_-^*)$, so $W \in \mathcal{W}_+$, contradicting the minimality of W_+ in $\mathcal{W}_+$. Thus $\mathcal{I}_i(W) > \mathcal{I}_i(W_+)$, so we must have $W \setminus \{i\} \notin \mathcal{W}$; else we have $W \in \mathcal{W}_-$, contradicting the maximality of W_- in $\mathcal{W}_-$. As $W \setminus \{i\} \notin \mathcal{W}$, we must have $W \notin \mathcal{W}$; else $\mathcal{I}_i(W) = -\infty \leq \mathcal{I}_i(W_+) < \mathcal{I}_i(W_-^*)$, so $W \in \mathcal{W}_+$, again contradicting the minimality of W_+ in $\mathcal{W}_+$. Altogether, then, $W \setminus \{i\} \notin \mathcal{W}$ and $W \notin \mathcal{W}$. As $W \subseteq N$ such that $W_- \subsetneq W \subsetneq W_+$ was arbitrary, thus we have established (iv) for each $W \in 2^N \setminus \{W_- \setminus \{i\}, W_-, W_+ \setminus \{i\}, W_+\}$ such that $W_- \setminus \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$, as desired. $\square$

◦ STEP 2: Define $j \in N$, define $\kappa \in \mathbb{R}_+$, define $v_i^+, v_i^-, v_j^+, v_j^-, x^+, x^- \in \mathbb{R}$ such that $x^+ > 0 > x^-$, and define $v^{++}, v^{+-}, v^{-+}, v^{--} \in V$.

Because $W_- \subseteq W_+$ and $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, thus $W_- \subsetneq W_+$. Define $j \equiv$

$\min W_+ \setminus W_-$. Define the *maximum absolute finite cost*, $\kappa \in \mathbb{R}_+$, by

$$\kappa = \max_{W \in \mathcal{W}} |C(W)|.$$

As $\mathcal{I}_i(W_-) > \mathcal{I}_i(W_+)$, choose $v_i^+, v_i^- \in \mathbb{R}$ such that $\mathcal{I}_i(W_-) > v_i^+ > v_i^- > \mathcal{I}_i(W_+)$.

Define $v_j^+, v_j^-, x^+, x^- \in \mathbb{R}$ such that $x^+ > 0 > x^-$ by

$$\begin{aligned} v_j^+ &\equiv -v_i^- + 2\kappa + 1, \\ v_j^- &\equiv -v_i^+ - 2\kappa - 1, \\ x^+ &\equiv \max\{v_i^+, 0\} + \max\{v_j^+, 0\} + 2\kappa + 1, \text{ and} \\ x^- &\equiv \min\{v_i^-, 0\} + \min\{v_j^-, 0\} - 2\kappa - 1. \end{aligned}$$

Finally, define $v^{++}, v^{+-}, v^{-+}, v^{--} \in V$ by:

- (i) for each $v \in \{v^{++}, v^{+-}\}$, $v_i \equiv v_i^+$,
- (ii) for each $v \in \{v^{-+}, v^{--}\}$, $v_i \equiv v_i^-$,
- (iii) for each $v \in \{v^{++}, v^{-+}\}$, $v_j \equiv v_j^+$,
- (iv) for each $v \in \{v^{+-}, v^{--}\}$, $v_j \equiv v_j^-$, and
- (v) for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$ and each $k \in N \setminus \{i, j\}$, we have

$$v_k \equiv \begin{cases} x^+, & k \in W_- \setminus \{i\}, \\ 0, & k \in (W_+ \setminus W_-) \setminus \{j\}, \\ x^-, & k \in N \setminus W_+. \quad \square \end{cases}$$

◦ **STEP 3:** For each $v \in \{v^{++}, v^{+-}\}$, $\varphi(v) = W_+$, and for each $v \in \{v^{-+}, v^{--}\}$, $\varphi(v) = W_- \setminus \{i\}$.

This proof of this step consists of two claims, then a conclusion. First, we claim that for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$, $W_- \setminus \{i\} \subseteq \varphi(v) \subseteq W_+$. Indeed, let $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$. To see that $W_- \setminus \{i\} \subseteq \varphi(v)$, let $W \subseteq N$ such that

$W_- \setminus \{i\} \not\subseteq W$. Then there is $k^* \in (W_- \setminus \{i\}) \setminus W$, so

$$\begin{aligned}
\sigma_v(W_- \setminus \{i\}) - \sigma_v(W) &= \left[\sum_{W_- \setminus \{i\}} v_k - C(W_- \setminus \{i\}) \right] - \left[\sum_W v_k - C(W) \right] \\
&\geq [|W_- \setminus \{i\}|x^+ - \kappa] \\
&\quad - [(|W_- \setminus \{i\}| - 1)x^+ + \max\{v_i^+, 0\} + \max\{v_j^+, 0\} + \kappa] \\
&= x^+ - 2\kappa - \max\{v_i^+, 0\} - \max\{v_j^+, 0\} \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing* we have $\varphi(v) \neq W$. Since $W \subseteq N$ such that $W_- \setminus \{i\} \not\subseteq W$ was arbitrary, thus $W_- \setminus \{i\} \subseteq \varphi(v)$, as desired. To see that $\varphi(v) \subseteq W_+$, let $W \subseteq N$ such that $W \not\subseteq W_+$. Then there is $k^* \in W \setminus W_+$, so

$$\begin{aligned}
\sigma_v(W_+) - \sigma_v(W) &= \left[\sum_{W_+} v_k - C(W_+) \right] - \left[\sum_W v_k - C(W) \right] \\
&\geq [(|W_-| - 1)x^+ + v_i + v_j - \kappa] \\
&\quad - [x^- + (|W_-| - 1)x^+ + \max\{v_i, 0\} + \max\{v_j, 0\} + \kappa] \\
&= -x^- + (v_i - \max\{v_i, 0\}) + (v_j - \max\{v_j, 0\}) - 2\kappa \\
&= -x^- + \min\{v_i, 0\} + \min\{v_j, 0\} - 2\kappa \\
&\geq -x^- + \min\{v_i^-, 0\} + \min\{v_j^-, 0\} - 2\kappa \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing* we have $\varphi(v) \neq W$. Since $W \subseteq N$ such that $W \not\subseteq W_+$ was arbitrary, thus $\varphi(v) \subseteq W_+$, as desired. This completes the proof of our first claim.

Second, we claim that for each $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$, $\varphi(v) \in \{W_- \setminus \{i\}, W_+\}$. Indeed, let $v \in \{v^{++}, v^{+-}, v^{-+}, v^{--}\}$. By the first claim, $W_- \setminus \{i\} \subseteq \varphi(v) \subseteq W_+$. By Step 1, for each $W \in 2^N \setminus \{W_- \setminus \{i\}, W_-, W_+ \setminus \{i\}, W_+\}$ such that $W_- \setminus \{i\} \subseteq W \subseteq W_+$, we have $W \notin \mathcal{W}$, so $\sigma_v(W) = -\infty < 0 = \sigma_v(\emptyset)$, so as φ is *surplus-maximizing*, we have $\varphi(v) \neq W$. Altogether, then, $\varphi(v) \in \{W_- \setminus \{i\}, W_+ \setminus \{i\}, W_+\}$. Because $x_i(W_-) > v_i > x_i(W_+)$, it is straightforward to verify that $\sigma_v(W_- \setminus \{i\}) > \sigma_v(W_-)$ and $\sigma_v(W_+) > \sigma_v(W_+ \setminus \{i\})$; thus as φ is *surplus-maximizing*, we have $\varphi(v) \notin \{W_-, W_+ \setminus \{i\}\}$. This completes the proof of our second claim.

To conclude, first let $v \in \{v^{++}, v^{-+}\}$. Then

$$\begin{aligned}
\sigma_v(W_+) - \sigma_v(W_- \setminus \{i\}) &= \left[\sum_{W_+} v_k - C(W_+) \right] - \left[\sum_{W_- \setminus \{i\}} v_k - C(W_- \setminus \{i\}) \right] \\
&= \left[\sum_{W_- \setminus \{i\}} v_k + v_i + v_j^+ - C(W_+) \right] - \left[\sum_{W_- \setminus \{i\}} v_k - C(W_- \setminus \{i\}) \right] \\
&\geq \left[\sum_{W_- \setminus \{i\}} v_k + v_i + v_j^+ - \kappa \right] - \left[\sum_{W_- \setminus \{i\}} v_k + \kappa \right] \\
&= v_i + v_j^+ - 2\kappa \\
&\geq v_i^- + v_j^+ - 2\kappa \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing*, $\varphi(v) \neq W_- \setminus \{i\}$. By the second claim, $\varphi(v) = W_+$. Finally, let $v \in \{v^{+-}, v^{--}\}$. Then

$$\begin{aligned}
\sigma_v(W_- \setminus \{i\}) - \sigma_v(W_+) &= \left[\sum_{W_- \setminus \{i\}} v_k - C(W_- \setminus \{i\}) \right] - \left[\sum_{W_+} v_k - C(W_+) \right] \\
&= \left[\sum_{W_- \setminus \{i\}} v_k - C(W_- \setminus \{i\}) \right] - \left[\sum_{W_- \setminus \{i\}} v_k + v_i + v_j^- - C(W_+) \right] \\
&\geq \left[\sum_{W_- \setminus \{i\}} v_k - \kappa \right] - \left[\sum_{W_- \setminus \{i\}} v_k + v_i + v_j^- + \kappa \right] \\
&= -2\kappa - v_i - v_j^- \\
&\geq -2\kappa - v_i^+ - v_j^- \\
&> 0,
\end{aligned}$$

so as φ is *surplus-maximizing*, $\varphi(v) \neq W_+$. By the second claim, $\varphi(v) = W_- \setminus \{i\}$. This completes the proof of the step. $\square$

◦ STEP 4: Conclude.

By Step 3, we have that both i and j win at v^{++} and v^{-+} , and we have that both i and j lose at v^{+-} and v^{--} .

By *no-envy*, $\tau_j(v^{++}) = \tau_i(v^{++})$. By *strategy-proofness*, $\tau_i(v^{++}) = \tau_i(v^{-+})$. By *no-envy*, $\tau_i(v^{-+}) = \tau_j(v^{-+})$. Altogether, $\tau_j(v^{++}) = \tau_j(v^{-+})$.

By *no-envy*, $\tau_j(v^{+-}) = \tau_i(v^{+-})$. By *strategy-proofness*, $\tau_i(v^{+-}) = \tau_i(v^{--})$. By *no-envy*, $\tau_i(v^{--}) = \tau_j(v^{--})$. Altogether, $\tau_j(v^{+-}) = \tau_j(v^{--})$.

As $v_i^+ > v_i^-$, thus by Step 3 and [Theorem H](#), we have

$$\begin{aligned}\tau_j(v^{++}) - \tau_j(v^{+-}) &= \sigma_{v^{++}}(W_+) - v_j^+ - \sigma_{v^{+-}}(W_- \setminus \{i\}) \\ &> \sigma_{v^{--}}(W_+) - v_j^+ - \sigma_{v^{--}}(W_- \setminus \{i\}) \\ &= \tau_j(v^{-+}) - \tau_j(v^{--}).\end{aligned}$$

But then

$$\begin{aligned}0 &= \tau_j(v^{++}) - \tau_j(v^{-+}) \\ &> \tau_j(v^{+-}) - \tau_j(v^{--}) \\ &= 0,\end{aligned}$$

contradicting that $0 = 0$. ■

Our second impossibility lemma in this appendix states that if costs are *convex* but not *symmetric*, then there are no *equitable* mechanisms. In the proof, we first use the absence of *symmetry* to identify a group W^* and a pair of agents i, j outside of that group such that the cost of $W^* \cup \{i\}$ is less than the cost of $W^* \cup \{j\}$. We then use *convexity* to construct a particular non-degenerate interval from which we pick a low valuation for i and a high valuation for j ; we complete the profile v^* by assigning a large positive valuation to members of W^* and a large negative valuation to the others. By *production efficiency*, $W^* \cup \{i\}$ must win at this profile, but *no-envy* then implies that the valuation of i is at least the valuation of j , which is not true.

• **LEMMA 2.2:** Fix an environment. If the cost function satisfies *convexity* but not *symmetry*, then there are no *equitable* mechanisms.

PROOF: Fix an environment whose cost function satisfies *convexity* but not *symmetry*, and assume, by way of contradiction, that (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *no-envy*.

We claim there are $W^* \subseteq N$ and $i, j \in N \setminus W^*$ such that $C(W^* \cup \{i\}) < C(W^* \cup \{j\})$. Indeed, since C is not *symmetric*, there are $W, W' \subseteq N$ such that $|W| = |W'|$ and $C(W) < C(W')$. Define $W_0 \equiv W$, and for each $t \in \{1, 2, \dots, |W \setminus W'|\}$, define

- $i_t \equiv \min(W_{t-1} \setminus W')$,
- $j_t \equiv \min(W' \setminus W_{t-1})$, and

- $W_t \equiv (W_{t-1} \setminus \{i_t\}) \cup \{j_t\}$.

It is straightforward to verify that $W_{|W \setminus W'|} = W'$. Thus there is $t \in \{1, 2, \dots, |W \setminus W'|\}$ such that $C(W_{t-1}) < C(W_t)$. Define $W^* \equiv W_t \cap W_{t-1}$, define $i \equiv i_t$, and define $j \equiv j_t$; then $W^* \subseteq N$ and $i, j \in N \setminus W^*$ are as desired.

Since $W^* \cup \{i\} \in \mathcal{W}$, thus $W^* \in \mathcal{W}$; else there is $k \in W^*$ such that $\mathcal{I}_k(W^*) = \infty > \mathcal{I}_k(W^* \cup \{i\})$, contradicting *convexity*. Then $\mathcal{I}_j(W^* \cup \{j\}) \neq -\infty$, so we can define the *maximum absolute finite cost* $\kappa \in \mathbb{R}_+$, the *upper bound* $v^{max} \in \mathbb{R}$, and the *lower bound* $v^{min} \in \mathbb{R}$, by

$$\begin{aligned} \kappa &\equiv \max_{W \in \mathcal{W}} |C(W)|, \\ v^{max} &\equiv \min\{2\kappa + 1, \mathcal{I}_j(W^* \cup \{j\})\}, \text{ and} \\ v^{min} &\equiv \mathcal{I}_i(W^* \cup \{i\}). \end{aligned}$$

We claim $v^{max} > v^{min}$. Indeed, since $W^*, W^* \cup \{i\} \in \mathcal{W}$, thus it cannot be that $\mathcal{I}_i(W^* \cup \{i\}) > 2\kappa$, else

$$\begin{aligned} C(W^* \cup \{i\}) &= C(W^*) + \mathcal{I}_i(W^* \cup \{i\}) \\ &> -\kappa + 2\kappa, \end{aligned}$$

contradicting the construction of κ as the maximum absolute finite cost. Then $2\kappa + 1 > 2\kappa \geq \mathcal{I}_i(W^* \cup \{i\})$. Moreover, since $C(W^* \cup \{j\}) > C(W^* \cup \{i\})$ and $W^* \in \mathcal{W}$, thus $\mathcal{I}_j(W^* \cup \{j\}) > \mathcal{I}_i(W^* \cup \{i\})$. Altogether, then, $v^{max} > v^{min}$, as desired.

Let $v^h, v^l \in \mathbb{R}$ such that $v^{max} > v^h > v^l > v^{min}$, and define $v^* \in V$ such that for each $k \in N$,

$$v_k^* \equiv \begin{cases} v^l, & k = i, \\ v^h, & k = j, \\ 6\kappa + 3, & k \in W^*, \\ -(6\kappa + 3), & \text{else.} \end{cases}$$

We first claim that $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$. Indeed, if there were $k^* \in$

$W^* \setminus \varphi(v^*)$, then

$$\begin{aligned}
\sigma_{v^*}(W^*) - \sigma_{v^*}(\varphi(v^*)) &= \left(\sum_{W^*} v_k - C(W^*) \right) - \left(\sum_{\varphi(v^*)} v_k - C(\varphi(v^*)) \right) \\
&\geq (|W^*|(6\kappa + 3) - \kappa) \\
&\quad - (|W^* - 1|(6\kappa + 3) + \max\{v^l, 0\} + \max\{v^h, 0\} + \kappa) \\
&= (6\kappa + 3) - 2\kappa - \max\{v^l, 0\} - \max\{v^h, 0\} \\
&> (6\kappa + 3) - 2\kappa - (2\kappa + 1) - (2\kappa + 1) \\
&> 0,
\end{aligned}$$

contradicting that φ is *surplus-maximizing*. If there were $k^* \in \varphi(v^*) \setminus (W^* \cup \{i, j\})$, then

$$\begin{aligned}
\sigma_{v^*}(W^*) - \sigma_{v^*}(\varphi(v^*)) &= \left(\sum_{W^*} v_k - C(W^*) \right) - \left(\sum_{\varphi(v^*)} v_k - C(\varphi(v^*)) \right) \\
&\geq (|W^*|(6\kappa + 3) - \kappa) \\
&\quad - (|W^*|(6\kappa + 3) + \max\{v^l, 0\} + \max\{v^h, 0\} - (6\kappa + 3) + \kappa) \\
&= (6\kappa + 3) - 2\kappa - \max\{v^l, 0\} - \max\{v^h, 0\} \\
&> (6\kappa + 3) - 2\kappa - (2\kappa + 1) - (2\kappa + 1) \\
&> 0,
\end{aligned}$$

contradicting that φ is *surplus-maximizing*. Thus $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$, as desired.

Next, we claim $\varphi(v^*) = W^* \cup \{i\}$. Indeed, if $\varphi(v) = W^* \cup \{i, j\}$, then by *production efficiency*, $W^* \cup \{i, j\} \in \mathcal{W}$. Since $W^* \cup \{i\} \in \mathcal{W}$, thus by *convexity*

$$\begin{aligned}
\sigma_{v^*}(W^* \cup \{i\}) - \sigma_{v^*}(W^* \cup \{i, j\}) &= -v_j^* + \mathcal{I}_j(W^* \cup \{i, j\}) \\
&\geq -v^h + \mathcal{I}_j(W^* \cup \{j\}) \\
&> 0,
\end{aligned}$$

contradicting that φ is *surplus-maximizing*. Similarly, if $\varphi(v) = W^* \cup \{j\}$, then by *production efficiency*, $W^* \cup \{j\} \in \mathcal{W}$. Since $W^* \in \mathcal{W}$, thus

$$\begin{aligned}
\sigma_{v^*}(W^*) - \sigma_{v^*}(W^* \cup \{j\}) &= -v_j^* + \mathcal{I}_j(W^* \cup \{j\}) \\
&= -v^h + \mathcal{I}_j(W^* \cup \{j\}) \\
&> 0,
\end{aligned}$$

contradicting that φ is *surplus-maximizing*. Finally, if $\varphi(v) = W^*$, then since $W^*, W^* \cup \{i\} \in \mathcal{W}$, thus

$$\begin{aligned}\sigma_{v^*}(W^* \cup \{i\}) - \sigma_{v^*}(W^*) &= v_i^* - \mathcal{I}_i(W^* \cup \{i\}) \\ &= v^l - \mathcal{I}_i(W^* \cup \{i\}) \\ &> 0,\end{aligned}$$

contradicting that φ is *surplus-maximizing*. As $W^* \subseteq \varphi(v^*) \subseteq W^* \cup \{i, j\}$, altogether, then, $\varphi(v^*) = W^* \cup \{i\}$, as desired.

To conclude, by *no-envy*, $\tau_i(v^*) + v^h \leq \tau_j(v^*)$ and $\tau_j(v^*) \leq \tau_i(v^*) + v^l$. But then

$$\begin{aligned}\tau_i(v^*) &\leq \tau_j(v^*) - v^h \\ &\leq \tau_i(v^*) + v^l - v^h,\end{aligned}$$

so $v^h \leq v^l$, contradicting that $v^h > v^l$. ■

Finally, we prove that if costs are *convex* and *symmetric*, then each exclusion pivot mechanism is *equitable*. We begin by taking an arbitrary valuation profile and using it to construct a reported demand curve, which is used alongside the fixed supply curve given by the marginal costs. By *production efficiency*, winners must be served so long as demand exceeds supply, and may be served when the two are equal; we define $\check{q}$ to be the minimum efficient quantity and $\hat{q}$ to be the maximum efficient quantity. Similarly, for a given agent i , we can define a demand curve with i removed and thus $\hat{q}_i$, the maximum efficient quantity given i is removed.

The key observation is that for all winners, $\hat{q}$ and $\hat{q}_i$ share the same relationship. Indeed, if the $(\hat{q} + 1)$ th-highest valuation is at least the $\hat{q}$ th-highest marginal cost, then for each winner we have that $\hat{q}_i = \hat{q}$; otherwise, for each winner we have that $\hat{q}_i = \hat{q} - 1$. In the first case, any winner can be replaced when he is removed, in the sense that it is still efficient to serve $\hat{q}$ agents; in the second case, no winner can be replaced.

Finally, we use the structure of the exclusion pivot transfer policy to prove that whether or not winners are replaceable, and no matter how many winners between $\check{q}$ and $\hat{q}$ are selected, we have that (i) no loser envies a loser, (ii) no winner

envies a loser, (iii) no winner envies a winner, and (iv) no loser envies a winner.

• **LEMMA 2.3:** Fix an environment. If the cost function is *convex* and *symmetric*, then each exclusion pivot mechanism satisfies *no-envy*.

PROOF: Fix an environment whose cost function is *convex* and *symmetric*, and let (τ, φ) be an exclusion pivot mechanism. Since the cost function is symmetric, we abuse notation and write $C : \{0, 1, 2, \dots, n\} \rightarrow \mathbb{R} \cup \{\infty\}$ so that $C(q)$ denotes the cost of producing any q winners, and write $\mathcal{Q} \subseteq \{0, 1, \dots, n\}$ to denote the quantities of winners that can be produced for finite cost. By [Lemma 1.1](#), (τ, φ) satisfies *production efficiency*, *strategy-proofness*, and *voluntary participation*. We want to show that (τ, φ) satisfies *no-envy*; thus let $v \in V$. If $n = 1$ we are done, so assume $n \geq 2$.

To avoid confusion, we refer to agents as members of N and nonzero quantities as members of $\{1, 2, \dots, n\}$, even though these are the same set. We also introduce a ranking of the agents to be used with the nonzero quantity notation; (i) label the agents $A_1, A_2, \dots, A_n$ such that $v_{A_1} \geq v_{A_2} \geq \dots \geq v_{A_n}$, and (ii) for each $i \in N$, define $q_i \in \{1, 2, \dots, n\}$ to be the solution to $i = A_{q_i}$. Depending on the argument, we will sometimes refer to an agent as i and other times as A_q , but never as q (which is reserved for quantities).

For each $q \in \{1, 2, \dots, n\}$, define

$$S_q \equiv \begin{cases} C(q) - C(q-1), & q \in \mathcal{Q}, \\ \infty, & \text{else,} \end{cases}$$

$$D_q \equiv v_{A_q}.$$

We refer to $(S_q)_{q \in \{1, 2, \dots, n\}}$ as the *supply curve* and $(D_q)_{q \in \{1, 2, \dots, n\}}$ as the *demand curve*. Notice that S_q is the marginal cost of including a q th winner, so by *convexity*, we have $S_1 \leq S_2 \leq \dots \leq S_n$. By construction, $D_1 \geq D_2 \geq \dots \geq D_n$.

For each $q \in \{1, 2, \dots, n-1\}$, define

$$D_{q|i} \equiv \begin{cases} D_q, & q < q_i, \\ D_{q+1}, & q \geq q_i, \end{cases}$$

We refer to $(D_{q|i})_{q \in \{1, 2, \dots, n-1\}}$ as the *demand curve given i is removed*. By construction, $D_{1|i} \geq D_{2|i} \geq \dots \geq D_{n|i}$.

Finally, define

$$\begin{aligned}\check{q} &\equiv \max(\{q \in \{1, 2, \dots, n\} | D_q > S_q\} \cup \{0\}), \\ \hat{q} &\equiv \max(\{q \in \{1, 2, \dots, n\} | D_q \geq S_q\} \cup \{0\}), \text{ and} \\ \text{for each } i \in N, \hat{q}_i &\equiv \max(\{q \in \{1, 2, \dots, n-1\} | D_{q|i} \geq S_q\} \cup \{0\}).\end{aligned}$$

We refer to $\check{q}$ as the *minimum efficient quantity*, $\hat{q}$ as the *maximum efficient quantity*, and $\hat{q}_i$ as the *maximum efficient quantity given i is removed*; these terms will be justified by the proof.

◦ STEP 1: We have $|\varphi(v)| \in \{\check{q}, \check{q} + 1, \dots, \hat{q}\}$ and $[\max_{W \subseteq N} \sigma_v(W)] = \sum_{q=1}^{\hat{q}} (D_q - S_q)$.

For each $q \in \{0, 1, \dots, n\}$, define $\sigma^*(q) \equiv \sigma_v(\{A_{q'} \in N | q' \leq q\})$ to be the surplus of serving the q highest-ranked agents at v . Then for each $q \in \{0, 1, \dots, n\}$, we have

$$\sigma^*(q) = \begin{cases} \sum_{q'=1}^q (D_{q'} - S_{q'}), & q \in \mathcal{Q}, \\ -\infty, & \text{else.} \end{cases}$$

Since D_q is non-increasing in q and S_q is non-decreasing in q , thus over $\mathcal{Q}$ we have that $(D_q - S_q)$ is non-increasing in q . Then σ^* is maximized with any finite sum that includes all terms $(D_q - S_q)$ that are positive and none that are negative, so its maximizers are $\{\check{q}, \check{q} + 1, \dots, \hat{q}\}$.

For each $W \subseteq N$ such that $|W| \notin \{\check{q}, \check{q}+1, \dots, \hat{q}\}$, we have $\sigma_v(\{A_1, A_2, \dots, A_{\hat{q}}\}) = \sigma^*(\hat{q}) > \sigma^*(|W|) \geq \sigma_v(W)$, so by *production efficiency*, $\varphi(v) \neq W$; thus $|\varphi(v)| \in \{\check{q}, \check{q} + 1, \dots, \hat{q}\}$, as desired. For each $W \subseteq N$, we have $\sigma_v(\{A_1, A_2, \dots, A_{\hat{q}}\}) = \sigma^*(\hat{q}) \geq \sigma^*(|W|) \geq \sigma_v(W)$; thus $[\max_{W \subseteq N} \sigma_v(W)] = \sigma^*(\hat{q}) = \sum_{q=1}^{\hat{q}} (D_q - S_q)$, as desired. $\square$

◦ STEP 2: For each $i \in N$, $[\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)] = \sum_{q=1}^{\hat{q}_i} (D_{q|i} - S_q)$.

The argument is analogous to that in the previous step. $\square$

◦ STEP 3: For each $i \in \varphi(v)$, we have (i) for each $j \in N \setminus \varphi(v)$, $v_i \geq v_j$; and (ii) $v_i \geq D_{\hat{q}}$.

Let $i \in \varphi(v)$. For each $j \in N \setminus \varphi(v)$, we must have $v_i \geq v_j$; else $\sigma_v((\varphi(v) \setminus \{i\}) \cup$

$\{j\}) > \sigma_v(\varphi(v))$, contradicting *production efficiency*. Thus v_i is among the $|\varphi(v)|$ highest valuations, so by Step 1, v_i is among the $\hat{q}$ highest valuations; thus $v_i \geq D_{\hat{q}}$. $\square$

◦ STEP 4: If $\hat{q} = n$, then for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$.

Assume $\hat{q} = n$ and let $i \in \varphi(v)$. Since $D_{n-1} \geq D_n \geq S_n \geq S_{n-1}$, it follows immediately that $D_{n-1|i} \geq S_{n-1}$, so $\hat{q}_i = n - 1 = \hat{q} - 1$, as desired. $\square$

◦ STEP 5: If $\hat{q} \in \{1, 2, \dots, n - 1\}$, then (i) $D_{\hat{q}+1} \geq S_{\hat{q}}$ implies for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$; and (ii) $D_{\hat{q}+1} < S_{\hat{q}}$ implies for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$.

Assume $\hat{q} \in \{1, 2, \dots, n - 1\}$ and let $i \in \varphi(v)$. Recall that $i = A_{q_i}$, while $\hat{q}_i$ is the maximum efficient quantity given i is removed. We consider three cases.

CASE 1: If $\hat{q} < q_i$, then

- for each $q \in \{1, 2, \dots, \hat{q}\}$, $D_{q|i} = D_q \geq S_q$,
- for each $q \in \{\hat{q} + 1, \hat{q} + 2, \dots, q_i - 1\}$, $D_{q|i} = D_q < S_q$, and
- for each $q \in \{q_i, q_i + 1, \dots, n - 1\}$, $D_{q|i} = D_{q+1} \leq D_q < S_q$,

so $\hat{q}_i = \hat{q}$.

CASE 2: If $q_i \leq \hat{q}$ and $D_{\hat{q}+1} \geq S_{\hat{q}}$, then

- for each $q \in \{1, 2, \dots, q_i - 1\}$, $D_{q|i} = D_q \geq S_q$,
- for each $q \in \{q_i, q_i + 1, \dots, \hat{q} - 1\}$, $D_{q|i} = D_{q+1} \geq S_{q+1} \geq S_q$,
- $D_{\hat{q}|i} = D_{\hat{q}+1} \geq S_{\hat{q}}$, and
- for each $q \in \{\hat{q} + 1, \hat{q} + 2, \dots, n - 1\}$, $D_{q|i} = D_{q+1} \leq D_q < S_q$,

so $\hat{q}_i = \hat{q}$.

CASE 3: If $q_i \leq \hat{q}$ and $D_{\hat{q}+1} < S_{\hat{q}}$, then we have the same bullets as in Case 2 except that the third is changed to $D_{\hat{q}|i} = D_{\hat{q}+1} < S_{\hat{q}}$, so $\hat{q}_i = \hat{q} - 1$.

If $D_{\hat{q}+1} \geq S_{\hat{q}}$, then we are in Case 1 or Case 2, so $\hat{q}_i = \hat{q}$. If $D_{\hat{q}+1} < S_{\hat{q}}$, then since $i \in \varphi(v)$, thus by Step 3 we have $v_i \geq D_{\hat{q}} \geq S_{\hat{q}} > D_{\hat{q}+1}$; then $q_i \leq \hat{q}$, so we are in Case 3 and $\hat{q}_i = \hat{q} - 1$. Since $i \in \varphi(v)$ was arbitrary, we are done. $\square$

◦ STEP 6: For each $i \in \varphi(v)$, $\sum_{q=1}^{\hat{q}_i} D_{q|i} = \sum_{q=1}^{\hat{q}_i+1} D_q - v_i$.

Let $i \in \varphi(v)$, and recall $v_i = D_{q_i}$. If $q_i \leq \hat{q}_i$, then

$$\begin{aligned} \sum_{q=1}^{\hat{q}_i} D_{q|i} &= \sum_{q=1}^{q_i-1} D_{q|i} + \sum_{q=q_i}^{\hat{q}_i} D_{q|i} + [D_{q_i} - v_i] \\ &= \sum_{q=1}^{q_i-1} D_q + \sum_{q=q_i}^{\hat{q}_i} D_{q+1} + [D_{q_i} - v_i] \\ &= \sum_{q=1}^{\hat{q}_i+1} D_q - v_i, \end{aligned}$$

as desired.

If $q_i > \hat{q}_i$, then $q_i \geq \hat{q}_i + 1$, so $v_i = D_{q_i} \leq D_{\hat{q}_i+1}$. Moreover, since $i \in \varphi(v)$, thus by Step 1, $\hat{q} \neq 0$; so by Step 4 and Step 5, $\hat{q} \leq \hat{q}_i + 1$; so by Step 3, $v_i \geq D_{\hat{q}} \geq D_{\hat{q}_i+1}$. Then $v_i = D_{\hat{q}_i+1}$, so

$$\begin{aligned} \sum_{q=1}^{\hat{q}_i} D_{q|i} &= \sum_{q=1}^{\hat{q}_i} D_q + [D_{\hat{q}_i+1} - v_i] \\ &= \sum_{q=1}^{\hat{q}_i+1} D_q - v_i, \end{aligned}$$

as desired. $\square$

◦ STEP 7: Conclude.

Since (τ, φ) is an exclusion pivot mechanism, thus for each $i \in N$, we have

$$\tau_i(v) = ([\max_{W \subseteq N} \sigma_v(W)] - \varphi_i(v)v_i) - [\max_{W \subseteq N \setminus \{i\}} \sigma_{(0, v_{-i})}(W)].$$

For each $i \notin \varphi(v)$, clearly $\varphi(v)$ maximizes $\sigma_{(0, v_{-i})}$ among subsets of $N \setminus \{i\}$, so

$$\begin{aligned}\tau_i(v) &= \sigma_v(\varphi(v)) - \sigma_{(0, v_{-i})}(\varphi(v)) \\ &= 0.\end{aligned}$$

Thus no loser envies another loser. Moreover, since (τ, φ) satisfies *voluntary participation*, no winner envies a loser. If $\hat{q} = 0$, then by Step 1 everybody is a loser and we are done; thus let us assume $\hat{q} \in \{1, 2, \dots, n\}$. By Step 4 and Step 5, either (i) for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$; or (ii) for each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$. We consider both cases.

CASE 1: For each $i \in \varphi(v)$, $\hat{q}_i = \hat{q} - 1$. Then by Step 1, Step 2, and Step 6, for each $i \in \varphi(v)$, we have

$$\begin{aligned}\tau_i(v) &= ([\sum_{q=1}^{\hat{q}} D_q] - C(\hat{q}) - v_i) - ([\sum_{q=1}^{\hat{q}_i} D_{q|i}] - C(\hat{q}_i)) \\ &= ([\sum_{q=1}^{\hat{q}_i+1} D_q] - C(\hat{q}_i + 1) - v_i) - ([\sum_{q=1}^{\hat{q}_i+1} D_q] - v_i - C(\hat{q}_i)) \\ &= -S_{\hat{q}_i+1} \\ &= -S_{\hat{q}}.\end{aligned}$$

In this case, no winner envies another winner.

CASE 2: For each $i \in \varphi(v)$, $\hat{q}_i = \hat{q}$. Then by Step 1, Step 2, and Step 6, for each $i \in \varphi(v)$, we have

$$\begin{aligned}\tau_i(v) &= ([\sum_{q=1}^{\hat{q}} D_q] - C(\hat{q}) - v_i) - ([\sum_{q=1}^{\hat{q}_i} D_{q|i}] - C(\hat{q}_i)) \\ &= ([\sum_{q=1}^{\hat{q}_i} D_q] - C(\hat{q}_i) - v_i) - ([\sum_{q=1}^{\hat{q}_i+1} D_q] - v_i - C(\hat{q}_i)) \\ &= -D_{\hat{q}_i+1} \\ &= -D_{\hat{q}+1}.\end{aligned}$$

In this case, no winner envies another winner.

Assume, by way of contradiction, that a loser i envies a winner. If $\hat{q} = n$, then

by Step 4 we are in Case 1; so $v_i > S_n$, so $\sigma_v(|\varphi(v)| \cup \{i\}) > \sigma_v(\varphi(v))$, contradicting *production efficiency*. If $\hat{q} \in \{1, 2, \dots, n-1\}$, then by Step 5 and the above cases, either $D_{\hat{q}+1} \geq S_{\hat{q}}$ and $v_i > D_{\hat{q}+1}$ or $D_{\hat{q}+1} < S_{\hat{q}}$ and $v_i > S_{\hat{q}}$. In both cases, we have $v_i > D_{\hat{q}+1}$ and $v_i > S_{\hat{q}}$. Define

$$q^* \equiv \max\{q \in \{1, 2, \dots, n\} | D_{q^*} = v_i\}.$$

Since $D_{q^*} = v_i > D_{\hat{q}+1}$, thus $q^* \leq \hat{q}$, so $D_{q^*} = v_i > S_{\hat{q}} \geq S_{q^*}$. Then $q^* \leq \check{q}$, so by Step 1 there must be at least q^* winners, so by Step 3 i must be a winner, contradicting that i is a loser. ■

To conclude, [Theorem 2](#) is a direct corollary of our previous lemmas:

- THEOREM 2 (REPEATED): Fix an environment. The following are equivalent:
 - there is a mechanism that satisfies *production efficiency*, *strategy-proofness*, and *no-envy*;
 - each exclusion pivot mechanism satisfies *production efficiency*, *strategy-proofness*, and *no-envy*; and
 - the cost function satisfies *convexity* and *symmetry*.

PROOF: The first item implies the third by [Lemma 2.1](#) and [Lemma 2.2](#), the third implies the second by [Lemma 2.3](#), and the second implies the first trivially. ■

Appendix 3

In this appendix, we establish seven propositions providing logical relationships between *inclusion cost coverage* and other standard properties of cost functions. In addition to the properties introduced in [Section 3.1](#), we consider the following:

DEFINITION: Fix an environment. The cost function satisfies

- *monotonicity* if and only if for each pair $W, W' \subseteq N$ such that $W \subseteq W'$, we have $C(W') \geq C(W)$; and
- *superadditivity* if and only if for each pair $W, W' \subseteq N$ such that $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$, we have (i) $W, W' \in \mathcal{W}$, and (ii) $C(W \cup W') \geq C(W) + C(W')$.

Finally, if the cost function is *symmetric*, then for each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. In this case, the cost function satisfies

- *non-decreasing average costs* if and only if for each pair $w, w' \in \{1, 2, \dots, n\}$ such that $w' > w$,
 - (i) $C(w) = \infty$ implies $C(w') = \infty$, and
 - (ii) if $C(w), C(w') \in \mathbb{R}$, then $\frac{C(w)}{w} \leq \frac{C(w')}{w'}$.

In particular, as promised in [Section 3.1](#), we establish that (i) *convexity* implies *inclusion cost coverage* ([Proposition 1](#)); (ii) *inclusion cost coverage* does not imply *monotonicity* ([Proposition 2](#)); (iii) under *symmetry*, *inclusion cost coverage* is equivalent to *non-decreasing average costs* ([Proposition 3](#)); (iv) *symmetry* and *inclusion cost coverage* do not together imply *convexity* ([Proposition 4](#)); (v) *symmetry* and *inclusion cost coverage* together imply *superadditivity* ([Proposition 5](#)); (vi) *symmetry* and *superadditivity* do not together imply *inclusion cost coverage* ([Proposition 6](#)); and (vii) *monotonicity* and *inclusion cost coverage* do not together imply *superadditivity* ([Proposition 7](#)).

• **PROPOSITION 1:** Fix a set of agents. If a cost function satisfies *convexity*, then it satisfies *inclusion cost coverage*.

PROOF: Let C be *convex* and let $W \in \mathcal{W}$ be nonempty. Define $w \equiv |W|$, and re-index the agents such that $W = \{1, 2, \dots, w\}$. Define $W_0 \equiv \emptyset$, and for each $i \in W$, define $W_i \equiv \{1, 2, \dots, i\}$. For each $i \in W$, by *convexity* we have $\mathcal{I}_i(W) \geq \mathcal{I}_i(W_i)$; thus as $W \in \mathcal{W}$, we must have $W_i \in \mathcal{W}$.

For each $i \in W$, since $W_i \in \mathcal{W}$ and $W_{i-1} \in \mathcal{W}$, thus $\mathcal{I}_i(W_i) \in \mathbb{R}$, so by *convexity* we have $\mathcal{I}_i(W) \neq -\infty$, so as $W \in \mathcal{W}$ we have $\mathcal{I}_i(W) \in \mathbb{R}$. Thus by *convexity*,

$$\begin{aligned}
 C(W) &= [C(W_w) - C(W_{w-1})] + [C(W_{w-1}) - C(W_{w-2})] + \dots + [C(W_1) - C(W_0)] \\
 &= \mathcal{I}_w(W_w) + \dots + \mathcal{I}_1(W_1) \\
 &\leq \mathcal{I}_w(W) + \dots + \mathcal{I}_1(W) \\
 &= \sum_{i \in W} \mathcal{I}_i(W).
 \end{aligned}$$

Altogether, since W was an arbitrary nonempty member of $\mathcal{W}$, thus C satisfies

inclusion cost coverage, as desired. ■

- **PROPOSITION 2:** A cost function may satisfy *inclusion cost coverage* but not *monotonicity*.

PROOF: The proof is by example. Let $N = \{1, 2, 3\}$, and define C as follows:

$$C(W) = \begin{cases} 1, & W = N, \\ 2, & W = \{1, 2\}, \\ 0, & \text{else.} \end{cases}$$

It is straightforward to verify C that satisfies *inclusion cost coverage* but not *monotonicity*. ■

- **PROPOSITION 3:** Fix a set of agents. If a cost function satisfies *symmetry*, then it satisfies *inclusion cost coverage* if and only if it satisfies *non-decreasing average costs*.

PROOF: Let C satisfy *symmetry*. For each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. We establish both implications:

[$\Rightarrow$] Assume C satisfies *inclusion cost coverage*, and let $w, w' \in \{1, 2, \dots, n\}$ such that $w' > w$. Let $W, W' \in 2^N$ such that $|W| = w$ and $|W'| = w'$, and re-index the agents such that $W' \setminus W = \{1, 2, \dots, w' - w\}$. Define $W_0 \equiv W$, and for each $i \in W' \setminus W$, define $W_i \equiv W_{i-1} \cup \{i\}$.

If $C(w) = \infty$, then $W = W_0 \notin \mathcal{W}$. By *inclusion cost coverage*, for each $i \in W' \setminus W$, $W_{i-1} \notin \mathcal{W}$ implies $W_i \notin \mathcal{W}$; thus $W' = W_{w'-w} \notin \mathcal{W}$, so $C(w') = \infty$.

If $C(w), C(w') \in \mathbb{R}$, then $W' - W_{w'-w} \in \mathcal{W}$. By *inclusion cost coverage*, for each $i \in W' \setminus W$, $W_i \in \mathcal{W}$ implies $W_{i-1} \in \mathcal{W}$; thus by *inclusion cost coverage* and *symmetry*, for each $i \in W' \setminus W$, we have

$$\begin{aligned} C(w+i) &= C(W_i) \\ &\leq |W_i| [C(W_i) - C(W_{i-1})] \\ &= (w+i)C(w+i) - (w+i)C(w+i-1), \end{aligned}$$

or $\frac{C(w+i-1)}{w+i-1} \leq \frac{C(w+i)}{w+i}$. Altogether, then, $\frac{C(w)}{w} \leq \frac{C(w')}{w'}$. Since $w, w' \in$

$\{1, 2, \dots, n\}$ such that $w' > w$ were arbitrary, thus C satisfies *non-decreasing average costs*, as desired.

[$\Leftarrow$] Assume C satisfies *non-decreasing average costs*. Let $W \in \mathcal{W}$ be nonempty and define $w \equiv |W|$. Then $C(w) \neq \infty$. By *non-decreasing average costs*, $C(w-1) \neq \infty$, so for each $i \in W$, $W \setminus \{i\} \in \mathcal{W}$. Moreover, by *non-decreasing average costs*, $\frac{C(w-1)}{w-1} \leq \frac{C(w)}{w}$, so $wC(w-1) \leq wC(w) - C(w)$, so $C(w) - C(w-1) \geq \frac{C(w)}{w}$. Thus for each $i \in W$, we have $x_i(W) \geq \frac{C(W)}{|W|}$, so $\sum_W x_i(W) \geq C(W)$. Since W was an arbitrary nonempty member of $\mathcal{W}$, thus C satisfies *inclusion cost coverage*, as desired. ■

• PROPOSITION 4: A cost function may satisfy *symmetry* and *inclusion cost coverage* but not *convexity*.

PROOF: This is established by the first example of a cost function in the proof sketch of [Theorem 2](#). ■

• PROPOSITION 5: Fix a set of agents. If a cost function satisfies *symmetry* and *inclusion cost coverage*, then it satisfies *superadditivity*.

PROOF: Let C satisfy *symmetry* and *inclusion cost coverage*. For each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C(w)$ to denote the cost of producing any w winners. By [Proposition 3](#), C satisfies *non-decreasing average costs*.

Let $W, W' \subseteq N$ such that $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$. Define $w \equiv |W|$ and $w' \equiv |W'|$. Then $C(w+w') \neq \infty$, so by *non-decreasing average costs*, we have $C(w) \neq \infty$ and $C(w') \neq \infty$, so $W \in \mathcal{W}$ and $W' \in \mathcal{W}$. If $w = 0$ or $w' = 0$, then $C(W \cup W') = C(w+w') = C(w) + C(w') = C(W) + C(W')$, as desired. If $w, w' \in \{1, 2, \dots, n\}$, then by *non-decreasing average costs*,

$$\begin{aligned} C(W) + C(W') &= C(w) + C(w') \\ &= w \frac{C(w)}{w} + w' \frac{C(w')}{w'} \\ &\leq w \frac{C(w+w')}{w+w'} + w' \frac{C(w+w')}{w+w'} \\ &= C(w+w') \\ &= C(W \cup W'), \end{aligned}$$

as desired. Since $W, W' \subseteq N$ with $W \cap W' = \emptyset$ and $W \cup W' \in \mathcal{W}$ were arbitrary, thus C satisfies *superadditivity*, as desired. ■

• **PROPOSITION 6:** Fix a set of agents. A cost function may satisfy *symmetry* and *superadditivity* but not *inclusion cost coverage*.

PROOF: The proof is by example. Let $N = \{1, 2, 3\}$, and define C as follows:

$$C(W) = \begin{cases} 0, & |W| = 0, \\ 1, & |W| = 1, \\ 4, & |W| = 2, \\ 5, & |W| = 3. \end{cases}$$

It is straightforward to verify that C satisfies *symmetry* and *superadditivity* but not *inclusion cost coverage*. ■

• **PROPOSITION 7:** Fix a set of agents. A cost function may satisfy *monotonicity* and *inclusion cost coverage*, but not *superadditivity*.

PROOF: The proof is by example. Let $N = \{1, 2, 3, 4\}$, and define C as follows:

$$C(W) = \begin{cases} 4, & W = N, \\ 3, & |W| = 3, W = \{1, 2\}, \text{ or } W = \{3, 4\}, \\ 0, & \text{else.} \end{cases}$$

It is straightforward to verify that C satisfies *monotonicity* and *inclusion cost coverage*, but $C(\{1, 2, 3, 4\}) < C(\{1, 2\}) + C(\{3, 4\})$ and thus C does not satisfy *superadditivity*. ■

Appendix 4

In the rest of the paper, we assume that (i) each agent's set of admissible valuations is $\mathbb{R}$, and (ii) each group may be associated with either a cost or a revenue. For many economic applications, however, it is common knowledge that inclusion is a good that is costly to provide; thus in this appendix, we provide analogues to our main results that apply when (i) each agent's set of admissible valuations is $\mathbb{R}_+$,

and (ii) each group is necessarily associated with a non-negative cost.¹³

We prove these analogous results using our main results, and the key insight of the proof is that the following are equivalent: (i) there is a desirable mechanism defined for non-negative valuations, and (ii) after reducing costs to reflect that any group can include outsiders if doing so will reduce costs, there is a desirable mechanism defined for all valuations. To state our results and arguments formally, we require the following definitions:

DEFINITION: Fix an environment with cost function C . We say that C is *non-negative* if and only if for each $W \subseteq N$, $C(W) \in \mathbb{R}_+ \cup \{\infty\}$. In this case, we let $C^\supseteq$ denote the cost function such that for each $W \subseteq N$,

$$C^\supseteq(W) = \min_{\{W' \subseteq N \mid W' \supseteq W\}} C(W').$$

Note that since C is non-negative, thus $C^\supseteq(\emptyset) = 0$, so $C^\supseteq$ is indeed a cost function.

Throughout this appendix, we use the prefixes $\mathbb{R}$ - and $\mathbb{R}_+$ - to specify the preference domain for environments and mechanisms; we use the prefixes C - and $C^\supseteq$ - to specify the cost function for costs, deficits, and surpluses; and we use the prefixes $(\mathbb{R}, C)$ -, $(\mathbb{R}_+, C)$ -, $(\mathbb{R}, C^\supseteq)$ -, and $(\mathbb{R}_+, C^\supseteq)$ - to specify both the preference domain and cost function for standard pivot mechanisms, exclusion pivot mechanisms, autonomy, equity, and all axioms (including those that do not depend on the cost function). For example, an $\mathbb{R}$ -mechanism may satisfy $(\mathbb{R}, C^\supseteq)$ -*no-deficit* but not $(\mathbb{R}, C)$ -*no-deficit*. We omit the obvious formal definitions.

We can now state the analogues to our main results for the case of non-negative valuations:

THEOREM 3: Fix an $\mathbb{R}_+$ -environment with a non-negative cost function C . The following are equivalent:

- there is an $\mathbb{R}_+$ -mechanism that satisfies $(\mathbb{R}_+, C)$ -*production efficiency*, $(\mathbb{R}_+, C)$ -*strategy-proofness*, $(\mathbb{R}_+, C)$ -*no-deficit*, and $(\mathbb{R}_+, C)$ -*voluntary participation*;
- each $(\mathbb{R}_+, C)$ -standard pivot mechanism satisfies $(\mathbb{R}_+, C)$ -*production efficiency*, $(\mathbb{R}_+, C)$ -*strategy-proofness*, $(\mathbb{R}_+, C)$ -*no-deficit*, and $(\mathbb{R}_+, C)$ -*voluntary participation*; and

¹³We thank an anonymous referee for suggesting that we investigate the robustness of our results to smaller preference domains.

- the cost function $C^\supseteq$ satisfies *inclusion cost coverage*.

THEOREM 4: Fix an $\mathbb{R}_+$ -environment with a non-negative cost function C . The following are equivalent:

- there is an $\mathbb{R}_+$ -mechanism that satisfies $(\mathbb{R}_+, C)$ -*production efficiency*, $(\mathbb{R}_+, C)$ -*strategy-proofness*, and $(\mathbb{R}_+, C)$ -*no-envy*;
- each $(\mathbb{R}_+, C)$ -standard pivot mechanism satisfies $(\mathbb{R}_+, C)$ -*production efficiency*, $(\mathbb{R}_+, C)$ -*strategy-proofness*, and $(\mathbb{R}_+, C)$ -*no-envy*; and
- the cost function $C^\supseteq$ satisfies *convexity* and *symmetry*.

Before proceeding, we remark that for both sets of axioms, disallowing negative valuations strictly expands the class of cost functions for which there are desirable Groves mechanisms. Indeed, if an $\mathbb{R}$ -mechanism satisfies either set of axioms, then its restriction to $\mathbb{R}_+^N$ does as well, so the class can only expand. Moreover, let $N = \{1, 2\}$, and define C as follows:

$$C(W) = \begin{cases} 2, & W = \{2\}, \\ 1, & W = \{1\}, \\ 0, & \text{else.} \end{cases}$$

It is easy to verify that C violates *inclusion cost coverage*, *convexity*, and *symmetry*, while $C^\supseteq$ satisfies all three properties; thus no $\mathbb{R}$ -mechanism satisfies either set of axioms, while there are $\mathbb{R}_+$ -mechanisms that satisfy all axioms. Indeed, the constant mechanism where everybody always wins and pays nothing is an $\mathbb{R}_+$ -mechanism that satisfies all axioms.

PROOF: Fix an $\mathbb{R}_+$ -environment with a non-negative cost function C . Observe that by construction, $C^\supseteq$ is monotonic. We prove both [Theorem 3](#) and [Theorem 4](#) together.

◦ STEP 1: If there is an $\mathbb{R}_+$ -mechanism that is $(\mathbb{R}_+, C)$ -autonomous, then there is an $\mathbb{R}$ -mechanism that is $(\mathbb{R}, C^\supseteq)$ -autonomous. Also, if there is an $\mathbb{R}_+$ -mechanism that is $(\mathbb{R}_+, C)$ -equitable, then there is an $\mathbb{R}$ -mechanism that is $(\mathbb{R}, C^\supseteq)$ -equitable.

Let (τ, φ) be an $\mathbb{R}_+$ -mechanism. Let (τ', φ') be the $\mathbb{R}_+$ mechanism constructed from (τ, φ) as follows: (i) $\tau' \equiv \tau$, and (ii) for each $v \in \mathbb{R}_+^N$, $\varphi'(v) \equiv \{i \in \varphi(v) | v_i >$

0}. Finally, for each $v \in \mathbb{R}^N$, define $v^+ \in \mathbb{R}_+^N$ by

$$v_i^+ \equiv \begin{cases} v_i, & v_i \geq 0, \\ 0, & \text{else,} \end{cases}$$

and define $(\tau^\supset, \varphi^\supset)$ to be the $\mathbb{R}$ -mechanism such that for each $v \in \mathbb{R}^N$, $\tau^\supset(v) = \tau'(v^+)$ and $\varphi^\supset(v) = \varphi'(v^+)$.

To complete Step 1, it suffices to prove that for each of the five axioms α that we used in our theorems, if (τ, φ) satisfies $(\mathbb{R}_+, C)$ - α , then $(\tau^\supset, \varphi^\supset)$ satisfies $(\mathbb{R}, C^\supset)$ - α ; thus let α be an axiom that satisfies this hypothesis.

First, we claim that since (τ, φ) satisfies $(\mathbb{R}_+, C)$ - α , thus (τ, φ) satisfies $(\mathbb{R}_+, C^\supset)$ - α . Indeed, for *production efficiency*, notice that for each $v \in \mathbb{R}_+^N$, (i) for each $W \subseteq N$, the $C^\supset$ -surplus is at least as great as the C -surplus as no costs increase from C to $C^\supset$, and (ii) for each $W \subseteq N$, if the $C^\supset$ -surplus is greater than the C -surplus, then W has a superset W' such that $C^\supset(W) = C(W')$, so as valuations are non-negative the $C^\supset$ -surplus of W is at most the C -surplus of W' ; thus from C to $C^\supset$ no group's surplus decreases while the maximum surplus does not increase, so $\varphi(v)$ maximizes $C^\supset$ -surplus. For *no-deficit*, it suffices to observe that no costs increase from C to $C^\supset$. The other axioms (*strategy-proofness*, *voluntary participation*, and *no-envy*) do not involve costs.

Second, we claim that since (τ, φ) satisfies $(\mathbb{R}_+, C^\supset)$ - α , thus (τ', φ') satisfies $(\mathbb{R}_+, C^\supset)$ - α . For *production efficiency*, since $C^\supset$ is monotonic, thus removing agents with zero valuation can never reduce $C^\supset$ -surplus. For *strategy-proofness*, it suffices to observe that (i) if an agent has zero valuation, then he finds honesty under (τ', φ') to be equivalent to honesty under (τ, φ) , which is at least as desirable as misreporting under both mechanisms, and (ii) if an agent has positive valuation, then he finds honesty under both mechanisms to be at least as desirable as misreporting zero under (τ, φ) , which is better than misreporting zero under (τ', φ') . For *no-deficit*, notice that since $C^\supset$ is monotonic, thus $C^\supset$ -costs can only decrease while transfers are unchanged. For *voluntary participation*, it suffices to observe that (τ, φ) and (τ', φ') are Pareto-indifferent, as the only change is that winners with zero valuation become losers. For *no-envy*, notice that for each $v \in \mathbb{R}_+^N$, no agent's φ' bundle is valued more than his φ bundle by anybody as valuations are non-negative, while any agent whose bundle changes is indifferent between his φ' bundle and φ bundle as he has zero valuation.

To conclude, we claim that since (τ', φ') satisfies $(\mathbb{R}_+, C^\supset)$ - α , thus $(\tau^\supset, \varphi^\supset)$ satisfies $(\mathbb{R}, C^\supset)$ - α . For *production efficiency*, notice that for each $v \in \mathbb{R}^N$, the

$C^\supset$ -surplus of $\varphi'(v^+) = \varphi^\supset(v)$ is the same from v^+ to v as this group only includes agents whose valuations are positive and equal at both profiles, while the $C^\supset$ -surplus of other groups can only become lower from v^+ to v . For *strategy-proofness*, though the domain of admissible reports expands from (τ', φ') to $(\tau^\supset, \varphi^\supset)$, the strategic opportunities do not as each negative report is treated as a zero report. For *no-deficit*, the result is trivial. For *voluntary participation* and *no-envy*, notice that for each $v \in \mathbb{R}^N$, if an agent's preferences change from v^+ to v , then his valuation is negative at v and zero at v^+ ; in this case he loses and receives a non-negative transfer at v^+ according to (τ', φ') , so he loses and receives a non-negative transfer at v according to $(\tau^\supset, \varphi^\supset)$, and moreover he assigns less utility to each winning bundle from v^+ to v .

◦ STEP 2: If there is an $\mathbb{R}$ -mechanism that is $(\mathbb{R}, C^\supset)$ -autonomous, then there is an $\mathbb{R}_+$ -mechanism that is $(\mathbb{R}_+, C)$ -autonomous. Also, if there is an $\mathbb{R}$ -mechanism that is $(\mathbb{R}, C^\supset)$ -equitable, then there is an $\mathbb{R}_+$ -mechanism that is $(\mathbb{R}_+, C)$ -equitable.

Let $(\tau^\supset, \varphi^\supset)$ be an $(\mathbb{R}, C^\supset)$ -exclusion pivot mechanism such that for each $v \in \mathbb{R}^N$, $\varphi^\supset(v)$ has no strict superset that maximizes $C^\supset$ -surplus at v . Let (τ, φ) denote the restriction of $(\tau^\supset, \varphi^\supset)$ to $\mathbb{R}_+^N$.

By [Theorem 1](#), if there is an $(\mathbb{R}, C^\supset)$ -autonomous mechanism, then $(\tau^\supset, \varphi^\supset)$ is such a mechanism. By [Theorem 2](#), if there is an $(\mathbb{R}, C^\supset)$ -equitable mechanism, then $(\tau^\supset, \varphi^\supset)$ is such a mechanism. Altogether, then, to complete Step 2, it suffices to prove that for each of the five axioms α that we used in our theorems, if $(\tau^\supset, \varphi^\supset)$ satisfies $(\mathbb{R}, C^\supset)$ - α , then (τ, φ) satisfies $(\mathbb{R}_+, C)$ - α ; thus let α be an axiom that satisfies this hypothesis. As (τ, φ) is a restriction of $(\tau^\supset, \varphi^\supset)$, the argument is trivial if α is *strategy-proofness*, *voluntary participation*, or *no-envy*; thus assume that α is *production efficiency* or *no-deficit*.

We first claim that for each $v \in \mathbb{R}_+^N$, $C(\varphi(v)) = C^\supset(\varphi(v))$. Indeed, by definition, $\varphi(v) = \varphi^\supset(v)$ maximizes $C^\supset$ -surplus at v . We cannot have $C(\varphi(v)) \neq C^\supset(\varphi(v))$; else $\varphi(v)$ has a strict superset with smaller C -cost, so $\varphi(v)$ has a strict superset with equal $C^\supset$ -cost, so since at v all valuations are non-negative, thus $\varphi(v) = \varphi^\supset(v)$ has a strict superset that maximizes $C^\supset$ -surplus at v , contradicting that $(\tau^\supset, \varphi^\supset)$ always selects a group with no such strict superset.

If α is *production efficiency*, then it suffices to observe that from $C^\supset$ to C , the cost of $\varphi(v)$ does not increase while the costs of other groups can only increase. If α is *no-deficit*, it suffices to observe that from $C^\supset$ to C , the cost of $\varphi(v)$ does not increase.

◦ STEP 3: Conclude.

By Step 1 and Step 2, we have that (i) there is an $(\mathbb{R}_+, C)$ -autonomous mechanism if and only if there is an $(\mathbb{R}, C^\supseteq)$ -autonomous mechanism, and (ii) there is an $(\mathbb{R}_+, C)$ -equitable mechanism if and only if there is an $(\mathbb{R}, C^\supseteq)$ -equitable mechanism. Thus by Theorem 1 and Theorem 2, (i) there is an $(\mathbb{R}_+, C)$ -autonomous mechanism if and only if $C^\supseteq$ satisfies *inclusion cost coverage*, and (ii) there is an $(\mathbb{R}_+, C)$ -equitable mechanism if and only if $C^\supseteq$ satisfies *convexity* and *symmetry*.

To complete the proof of both theorems, fix a theorem whose proof we will complete by assuming one of the following: there is an $(\mathbb{R}_+, C)$ -autonomous mechanism or there is an $(\mathbb{R}_+, C)$ -equitable mechanism. Assume, by way of contradiction, that there are $(\mathbb{R}_+, C)$ -standard pivot mechanism (τ, φ) and axiom α such that α appears in the statement of the fixed theorem and (τ, φ) violates $(\mathbb{R}_+, C)$ - α . As (τ, φ) is an $(\mathbb{R}_+, C)$ -standard pivot mechanism, it follows from Holmström (1979)¹⁴ that (τ, φ) satisfies $(\mathbb{R}_+, C)$ -production efficiency and $(\mathbb{R}_+, C)$ -strategy-proofness; thus α must be *no-deficit*, *voluntary participation*, or *no-envy*.

For each $v \in \mathbb{R}^N$, define $v^+ \in \mathbb{R}_+^N$ as in Step 1, and use (τ, φ) to construct the $\mathbb{R}_+$ -mechanism (τ', φ') and the $\mathbb{R}$ -mechanism $(\tau^\supseteq, \varphi^\supseteq)$ as in Step 1. By the arguments in Step 1, we have that

- (τ, φ) satisfies $(\mathbb{R}_+, C^\supseteq)$ -production efficiency and $(\mathbb{R}_+, C^\supseteq)$ -strategy-proofness,
- (τ', φ') satisfies $(\mathbb{R}_+, C^\supseteq)$ -production efficiency and $(\mathbb{R}_+, C^\supseteq)$ -strategy-proofness, and
- $(\tau^\supseteq, \varphi^\supseteq)$ satisfies $(\mathbb{R}, C^\supseteq)$ -production efficiency and $(\mathbb{R}, C^\supseteq)$ -strategy-proofness.

To establish our contradiction and thus complete the proof of both theorems, we establish two claims and then conclude.

First, we claim that since (τ, φ) violates $(\mathbb{R}_+, C)$ - α , thus $(\tau^\supseteq, \varphi^\supseteq)$ violates $(\mathbb{R}, C^\supseteq)$ - α . To prove the claim, let $v \in \mathbb{R}_+^N$ be a profile where (τ, φ) violates $(\mathbb{R}_+, C)$ - α ; we consider three cases.

CASE 1: α is *no-deficit*. We first observe that (τ, φ) runs a $C^\supseteq$ -deficit at v . Indeed, if not, then $C^\supseteq(\varphi(v)) < C(\varphi(v))$. But then $\varphi(v)$ has a superset W^* with lower C -cost, so as valuations are non-negative at v , thus W^* has greater C -surplus than $\varphi(v)$ at v , contradicting that (τ, φ) satisfies $(\mathbb{R}_+, C)$ -production efficiency.

¹⁴In particular, this follows from a modified version of the Holmström (1979) theorem that allows for cost functions; see the discussion in Appendix 1 of the main paper.

It follows that (τ', φ') runs a $C^\succeq$ -deficit at v . Indeed, if not, then since transfers are not changed, thus $C^\succeq(\varphi'(v)) < C^\succeq(\varphi(v))$. But then since φ' is constructed from φ by removing zero valuation agents, thus $\varphi'(v)$ has greater $C^\succeq$ -surplus than $\varphi(v)$ at v , contradicting that (τ, φ) satisfies $(\mathbb{R}_+, C^\succeq)$ -production efficiency.

This directly implies that $(\tau^\succeq, \varphi^\succeq)$ runs a $C^\succeq$ -deficit at v .

CASE 2: α is *voluntary participation*. Since (τ, φ) and (τ', φ') are Pareto-indifferent, thus (τ', φ') has an involuntary assignment at v , which directly implies that $(\tau^\succeq, \varphi^\succeq)$ has an involuntary assignment at v .

CASE 3: α is *no-envy*. In this case, we have fixed the theorem we are proving by assuming there is an $(\mathbb{R}_+, C)$ -equitable mechanism, so as argued at the start of Step 3 we have that $C^\succeq$ satisfies *convexity* and *symmetry*. For each $w \in \{0, 1, \dots, n\}$, we abuse notation by writing $C^\succeq(w)$ to denote the $C^\succeq$ -cost of producing any w winners.

Assume, by way of contradiction, that (τ', φ') has no envy at v . To establish the contradiction, we first argue that (i) there is $i_0 \in \varphi(v) \setminus \varphi'(v)$ who has zero valuation and is envied under (τ, φ) , (ii) $C(\varphi(v)) = 0$, and (iii) there is $i^* \in N \setminus \varphi(v)$ with positive valuation; we then conclude.

First, since (τ, φ) has envy at v but (τ', φ') does not, there must be an agent whose assignment differs across these two mechanisms, so there must be an agent with zero valuation who belongs to $\varphi(v) \setminus \varphi'(v)$. As an agent with zero valuation is indifferent between each agent's pair of assignments from these mechanisms, such an agent cannot envy nobody under one mechanism and somebody under the other; thus an agent i_0 in $\varphi(v) \setminus \varphi'(v)$ with zero valuation must be envied under (τ, φ) but not (τ', φ') .

Second, define $w \equiv |\varphi(v)|$ and $w' \equiv |\varphi'(v)|$. Since $\varphi'(v) \subseteq \varphi(v)$ and $i_0 \in \varphi(v) \setminus \varphi'(v)$, thus $w > w'$. Since both (τ, φ) and (τ', φ') satisfy $(\mathbb{R}_+, C^\succeq)$ -production efficiency, and since $\varphi(v)$ and $\varphi'(v)$ only differ by zero valuation agents, necessarily $C^\succeq(w) = C^\succeq(w')$. Since $C^\succeq$ is non-negative and convex, this common $C^\succeq$ -cost must be zero, so there is $W_0 \subseteq N$ such that $W_0 \supseteq \varphi(v)$ and $C(W_0) = 0$. It follows that $C(\varphi(v)) = 0$; else as valuations are non-negative at v , thus W_0 has greater C -surplus than $\varphi(v)$ at v , contradicting that (τ, φ) satisfies $(\mathbb{R}_+, C)$ -production efficiency.

Third, we claim there is $i^* \in N \setminus \varphi(v)$ with positive valuation. Indeed, it follows from the definition of $(\mathbb{R}_+, C)$ -standard pivot mechanism that each agent with zero valuation (including the envied agent i_0) receives zero transfer under (τ, φ) . Since

there is an agent i_+ who envies i_0 , it must be that i_+ has positive valuation and either (i) loses under (τ, φ) , or (ii) wins with a negative transfer under (τ, φ) . If i_+ loses under (τ, φ) , then take $i^* = i_+$ and we are done; thus let us assume that i_+ wins and receives a negative transfer under (τ, φ) . Then by the definition of $(\mathbb{R}_+, C)$ -standard pivot mechanism, there must be $W^* \subseteq N$ such that the peers of i_+ achieve a greater C -surplus at W^* than $\varphi(v)$. Since valuations are non-negative and $C(\varphi(v)) = 0$, there must be $i^* \in W^* \setminus \varphi(v)$ with positive valuation, as desired.

To conclude, define $W' \equiv (\varphi(v) \setminus \{i_0\}) \cup \{i^*\}$. Since (i) $C^\supseteq(\varphi(v)) = 0$, (ii) $|\varphi(v)| = |W'|$, and (iii) $C^\supseteq$ is symmetric, thus $C^\supseteq(W') = 0$. But then W' has a superset W'_0 such that $C(W'_0) = 0$, so as valuations are non-negative the C -surplus of W'_0 is greater than that of $\varphi(v)$, contradicting that $\varphi(v)$ maximizes C -surplus.

Since (τ', φ') has envy at v , it follows directly that $(\tau^\supseteq, \varphi^\supseteq)$ has envy at v .

Thus in all three cases, we have that $(\tau^\supseteq, \varphi^\supseteq)$ violates $(\mathbb{R}, C^\supseteq)$ - α , as desired.

Second, we claim that $(\tau^\supseteq, \varphi^\supseteq)$ is an $(\mathbb{R}, C^\supseteq)$ -exclusion pivot mechanism. Indeed, for each $i \in N$ and each $v \in \mathbb{R}^N$,

$$\begin{aligned} \tau_i^\supseteq(v) &= \tau'_i(v^+) \\ &= \tau_i(v^+) \\ &= \left(\max_{W \in \mathcal{W}} \left[\sum_{j \in W} v_j^+ - C(W) \right] - \varphi_i(v^+) v_i^+ \right) - \max_{W \in \mathcal{W}} \left[\sum_{j \in W \setminus \{i\}} v_j^+ - C(W) \right]. \end{aligned}$$

As argued in Step 1, the maximum C -surplus at v^+ is equal to the maximum $C^\supseteq$ -surplus at v^+ , and the maximum C -surplus at $(0, v_{-i}^+)$ is equal to the maximum $C^\supseteq$ -surplus at $(0, v_{-i}^+)$; thus

$$\tau_i^\supseteq(v) = \left(\max_{W \in \mathcal{W}} \left[\sum_{j \in W} v_j^+ - C^\supseteq(W) \right] - \varphi_i(v^+) v_i^+ \right) - \max_{W \in \mathcal{W}} \left[\sum_{j \in W \setminus \{i\}} v_j^+ - C^\supseteq(W) \right].$$

Since $C^\supseteq$ is monotonic, thus including agent i cannot increase the $C^\supseteq$ -surplus of his peers, so

$$\tau_i^\supseteq(v) = \left(\max_{W \in \mathcal{W}} \left[\sum_{j \in W} v_j^+ - C^\supseteq(W) \right] - \varphi_i(v^+) v_i^+ \right) - \max_{W \subseteq N \setminus \{i\}} \left[\sum_{j \in W} v_j^+ - C^\supseteq(W) \right].$$

Finally, as argued in Step 1, we have that (i) the maximum $C^\supseteq$ -surplus does not change from v^+ to v , (ii) $\varphi_i(v^+) \neq \varphi'_i(v^+)$ implies $v_i^+ = 0$, (iii) the maximum $C^\supseteq$ -surplus does not change from $(0, v_{-i}^+)$ to $(0, v_{-i})$, (iv) $\varphi'_i(v^+) = \varphi_i^\supseteq(v)$, and

(v) $v_i^+ \neq v_i$ implies $\varphi'_i(v^+) = \varphi_i^\supset(v) = 0$; thus

$$\begin{aligned}\tau_i^\supset(v) &= \left(\max_{W \in \mathcal{W}} \left[\sum_{j \in W} v_j - C^\supset(W) \right] - \varphi'_i(v^+)v_i^+ \right) - \max_{W \subseteq N \setminus \{i\}} \left[\sum_{j \in W \setminus \{i\}} v_j - C^\supset(W) \right] \\ &= \left(\max_{W \in \mathcal{W}} \left[\sum_{j \in W} v_j - C^\supset(W) \right] - \varphi_i^\supset(v)v_i \right) - \max_{W \subseteq N \setminus \{i\}} \left[\sum_{j \in W \setminus \{i\}} v_j - C^\supset(W) \right].\end{aligned}$$

Since $i \in N$ and $v \in \mathbb{R}^N$ were arbitrary, and since $(\tau^\supset, \varphi^\supset)$ satisfies $(\mathbb{R}, C^\supset)$ -production efficiency, thus $(\tau^\supset, \varphi^\supset)$ is an $(\mathbb{R}, C^\supset)$ -exclusion pivot mechanism, as desired.

To conclude, by our previous two claims, $(\tau^\supset, \varphi^\supset)$ is an $(\mathbb{R}, C^\supset)$ -exclusion pivot mechanism that violates $(\mathbb{R}, C^\supset)$ - α , contradicting that there is no such $(\mathbb{R}, C^\supset)$ -exclusion pivot mechanism by Step 1 and either [Theorem 1](#) or [Theorem 2](#). ■

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