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# Clique games: a family of games with coincidence between the nucleolus and the Shapley value

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# Clique games: a family of games with coincidence between the nucleolus and the Shapley value*

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**Abstract**

We introduce a new family of cooperative games for which there is coincidence between the nucleolus and the Shapley value. These so-called clique games are such that players are divided into cliques, with the value created by a coalition linearly increasing with the number of agents belonging to the same clique. Agents can belong to multiple cliques, but for a pair of cliques, at most a single agent belong to their intersection. Finally, if two players do not belong to the same clique, there is at most one way to link the two players through a chain of players, with any two adjacent players in the chain belonging to a common clique.

We provide multiple examples for clique games, chief among them minimum cost spanning tree problems. This allows us to obtain new correspondence results between the nucleolus and the Shapley value, as well as other cost sharing methods for the minimum cost spanning tree problem.

**Keywords: nucleolus; Shapley value; clique; minimum cost spanning tree**

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# 1 Introduction

The Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969) are two well known values for cooperative games. The Shapley value is an average of the marginal contributions of a player, while the prenucleolus is the value that minimizes the dissatisfaction of the worst-off coalitions. The nucleolus differs from the prenucleolus by also requiring that the value be individually rational.

Coincidence between these two values is uncommon and, in general, difficult to check without computing both values. Recently, Yokote et al. (2017) provide a sufficient and necessary condition for this coincidence to hold, but it requires the computation of both the Shapley value and a parametric family of sets that mimic the computation of the (pre)nucleolus.[1] This characterization can be applied in order to identify the correspondence in some particular classes of games, such as airport games (Littlechild and Owen, 1973), bidder collusion games (Graham et al., 1990) and polluted river games (Ni and Wang, 2007). Csóka and Herings (2017) also find coincidence is some three-player games based on bankruptcy problems. As discussed by Kar et al. (2009), for general cooperative games we have coincidence if the game only has two players or if all players are symmetric within the normalized game. Some other games have also been proposed (Deng and Papadimitriou, 1994; van den Nouweland et al., 1996), all having in common that the value of a coalition is equal to the sum of the values created by the pairs composing that coalition. That family was extended by Kar et al. (2009), who show that the coincidence persists in games that satisfy the so-called PS property. Such games are such that the marginal contributions of player $i$ to $S$ and to its complement $N \setminus (S \cup \{i\})$ sums up to a player-specific constant.

In this paper, we present another family of games, called *clique games*, in which the Shapley value and the nucleolus coincide. The family can be described as follows: the set of players is divided into cliques that cover it. A coalition will create value when it contains many players belonging to the

[1] Additionally, the condition also requires to check whether the sets in this parametric family are balanced.

same clique, with the value increasing linearly with the number of agents in the same clique. Players may belong to more than one clique, but the intersection of two cliques contains at most a single player. Finally, if two players are not in the same clique, there exists at most one way to "connect" them through a chain of connected cliques.

The family of clique games has a non-empty intersection with PS-games, but some clique games are not PS-games, and some PS-games are not clique games.

Clique games are convex, and hence their respective Shapley values are the average of extreme points in the core. We thus obtain a link between three crucial concepts of cooperative game theory: the nucleolus, the core, and the Shapley value.

While the conditions for a game to be a clique game seem demanding, we provide three relevant examples. The first one has producers selling goods to buyers organized in exclusive territories. The second one is a job scheduling problem (Bahel and Trudeau, 2017) in which agents have jobs to be executed on machines that can only process one job at a time, with the jobs having fixed start and finish times.

Our third example is the one we mainly focus on: the minimum cost spanning tree (*mcst*) problem. First introduced by Bird (1976), this well-studied game has players connecting to a source through a network, with the cost of an edge being a fixed amount that is paid if the edge is used, regardless of the number of users of the edge. The game has always a non-empty core even though it is not convex. Moreover, the Shapley value is not always in the core (Dutta and Kar, 2004).

Nevertheless, Bergantiños and Vidal-Puga (2007a) and Trudeau (2012) propose Shapley value-based solutions that are in the core, by first modifying the costs of the edges. For any pair of nodes in the network, Bergantiños and Vidal-Puga (2007a) look at the paths between them and ranks them according to their most expensive edge. The edge between the pair of nodes is then assigned the cost of that cheapest most expensive edge, allowing to obtain the so-called irreducible *mcst* problem. The Shapley value of that game yields the folk solution. The solution proposed by Trudeau (2012) is similar, but looks at cycles instead of paths, yielding a cycle-complete *mcst* problem and the cycle-complete solution.

Bergantiños and Vidal-Puga (2007b) also provide another Shapley value-based definition of the folk solution, by defining a cost game assuming that any coalition can connect either to the source or to any other node.

We identify *mcst* problems that generate clique games. In particular, it turns out that if we consider elementary *mcst* problems (in which all edges have a cost of 0 or 1), which form a basis for all *mcst* problems, the subset of cycle-complete problems (which include irreducible problems) generates clique games. Our result on clique games then applies, yielding that the nucleolus coincides with the cycle-complete solution for cycle-complete problems and with the folk solution for irreducible problems.

We can extend the correspondence one step further: for all elementary *mcst* problems, the folk (cycle-complete) solution corresponds to the nucleolus and the permutation-weighted average of the extreme points of the core of the public (private) *mcst* game.

The paper is divided as follows: preliminary definitions are in Section 2. Section 3 describes and illustrates clique games. Section 4 contains the correspondence results. The application and extension of the results to *mcst* problems are described in Section 5.

# 2 Preliminaries

Let $N = \{1, \cdots, n\}$ be a set of agents. A *transferable utility game* (*TU game*, for short) is a pair $(N, v)$ where $v$ is a real-valued function defined on all subsets $S \subseteq N$ satisfying $v(\emptyset) = 0$. Given $i \in N$ and $S \subseteq N \setminus \{i\}$, the marginal contribution of agent $i$ to $S$ is defined as

$$\Delta_i^v(S) = v(S \cup \{i\}) - v(S).$$

A game is *convex* if $\Delta_i^v(S) \leq \Delta_i^v(T)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

A *value* is a function that associates with each TU game $(N, v)$ a payoff allocation $x \in \mathbb{R}^N$. Two well-known values for TU games are the Shapley value (Shapley, 1953) and the (pre)nucleolus (Schmeidler, 1969).

The *Shapley value* of the game $(N, v)$ is the payoff allocation $Sh(v)$ defined as

$$Sh_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \Delta_i^v(P_i(\pi))$$

for all $i \in N$, where $\Pi$ is the set of all orderings of $N$ and $P_i(\pi)$ is the set of predecessors of agent $i$ in $\pi$, i.e. $P_i(\pi) = \{j : \pi(j) < \pi(i)\}$.

The *excess* of a coalition $S$ in a TU game $(N, v)$ with respect to an allocation $x$ is defined as $e(S, x, v) = \sum_{i \in N} x_i - v(S)$. The vector $\theta(x)$ is constructed by rearranging the $2^n$ excesses in (weakly) increasing order. If $x, y \in \mathbb{R}^N$ are two allocations, then $\theta(x) >_L \theta(y)$ means that $\theta(x)$ is lexicographically larger than $\theta(y)$. As usual, we write $\theta(x) \geq_L \theta(y)$ to indicate that either $\theta(x) >_L \theta(y)$ or $x = y$.

The *nucleolus* of the game $(N, v)$ is the set

$$Nu(v) = \{x \in X : \theta(x) \geq_L \theta(y) \forall y \in X\}$$

where $X = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \forall i \in N\right\}$ is the set of individually rational allocations. When $X \neq \emptyset$, as it is the case for the TU games we study here, it is well-known that $Nu(v)$ is a singleton, whose unique element we denote, with some abuse of notation, also as $Nu(v)$.

By contrast, the *prenucleolus* of the game $(N, v)$ is the set

$$Pre(v) = \left\{x \in X^0 : \theta(x) \geq_L \theta(y) \forall y \in X^0\right\}$$

where $X^0 = \left\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)\right\}$ is the set of allocations. Whenever the prenucleolus is individually rational, which will be the case in all games that we consider, it coincides with the nucleolus. Therefore, from now on, we focus exclusively on the nucleolus.

The *core* is the set of allocations such that no coalition is assigned less than its stand-alone value. Formally,

$$Core(v) = \left\{x \in X^0 : \sum_{i \in S} x_i \geq v(S) \forall S \subset N\right\}.$$

When $Core(v) \neq \emptyset$, for each $\pi \in \Pi$, let $y^{\pi} \in Core(v)$ be the allocation that lexicographically maximizes the allocations with respect to the order given by the permutation. The *permutation-weighted average of extreme points of the core* is the average of these allocations:

$$\bar{y}(v) = \sum_{\pi \in \Pi(N)} \frac{1}{n!} y^{\pi}(v).$$

If the game is convex, $\bar{y}$ is the Shapley value. It is also closely related to the selective value (Vidal-Puga, 2004) and the Alexia value (Tijs, 2005), the permutation-weighted average of leximals. All of these values coincide for the minimum cost spanning tree problem studied in Section 5.

On some occasions, we work with transferable cost games $(N, C)$, where $C$ is a real-valued function defined on all subsets $S \subseteq N$ satisfying $C(\emptyset) = 0$. We then define $v^C$ as follows: For all $S \subseteq N$, $v^C(S) = \sum_{i \in S} C(\{i\}) - C(S)$. An allocation $x$ for the cost game $C$ is equivalent to an allocation $x^C$ for the value game $v^C$ if $x_i^C = C(\{i\}) - x_i$ for all $i \in N$. We then say that $x \in Nu(C)$ iff $x^C \in Nu\left(v^C\right)$. We say that $x \in Core(C)$ iff $x^C \in Core(v^C)$. Finally, we say that $C$ is concave iff $-C$ is convex. It is straightforward to check that $C$ is concave iff $v^C$ is convex.

# 3 Clique games

Let $\mathcal{Q} = \left\{Q^1, \ldots, Q^K\right\}$ be a cover of $N$. For each $Q^k \in \mathcal{Q}$, the interior of $Q^k$, $Int\left(Q^k\right)$, is the set of agents that only belong to $Q^k$, i.e.

$$Int\left(Q^k\right) = \left\{i \in Q^k : i \notin Q^l \forall l \neq k\right\}.$$

We say that there exists a *path* between $Q^k$ and $Q^l$ if there exists $P^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ such that $Q^{k_1} = Q^k$, $Q^{k_M} = Q^l$ and $\left|Q^{k_m} \cap Q^{k_{m+1}}\right| = 1$ for all $m = 1, \ldots, M-1$. Analogously, we say that there exists a path between $Q^k$ and $Q^l$ through agent $i$ if there exists $P_i^{kl} = \left\{Q^{k_1}, \ldots, Q^{k_M}\right\}$ such that $Q^{k_1} \cap Q^{k_2} = \{i\}$. We say then that $P_i^{kl}$ is a path between $Q^k$ and $Q^l$ through agent $i$. The set of agents connected to $Q^k$ via a path through agent $i \in Q^k$

is denoted as

$$N_{k,i}^{P}=\left\{j\in N:\exists l,P_{i}^{kl}\text{ such that }j\in Q^{l}\right\}.$$

**Example 1** *Let $\mathcal{Q}=\{Q^1,Q^2,Q^3\}$ with $Q^1=\{1,2\}$, $Q^2=\{2,3,4\}$ and $Q^3=\{4,5,6\}$ (see Figure 1).*

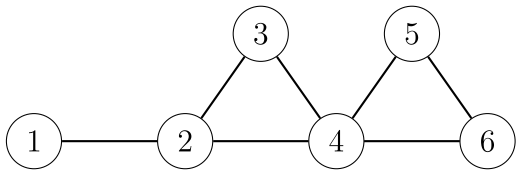


Figure 1: Example of a cover represented as cliques.

*In this case, $P_2^{13}=\{Q^1,Q^2,Q^3\}$ is a path between $Q^1$ and $Q^3$ through agent 2. The other paths are $P_2^{12}=\{Q^1,Q^2\}$, $P_2^{21}=\{Q^2,Q^1\}$, $P_4^{23}=\{Q^2,Q^3\}$, $P_4^{32}=\{Q^3,Q^2\}$, and $P_4^{31}=\{Q^3,Q^2,Q^1\}$. Moreover, $N_{1,1}^P=\emptyset$, $N_{2,2}^P=\{1,2\}$, $N_{1,2}^P=\{2,3,4,5,6\}$, $N_{2,4}^P=\{4,5,6\}$, and so on.*

We say that a game $\left(N,v^{\mathcal{Q}}\right)$ is a *clique game* if there exist $\mathcal{Q}=\left\{Q^1,\ldots,Q^K\right\}$ cover of $N$, $\{v_i\}_{i\in N}\subset\mathbb{R}_+$ and $\{v_Q\}_{Q\in\mathcal{Q}}\subset\mathbb{R}_+$ such that:

i) for all $k\in\{1,...,K\}$ and all $i,j\in Q^k$, $N_{k,i}^P\cap N_{k,j}^P=\emptyset$ (in words: there is at most one path between any two elements of $\mathcal{Q}$),

ii) for all $S\subseteq N$,

$$v^{\mathcal{Q}}(S)=\sum_{i\in S}v_i+\sum_{Q\in\mathcal{Q}(S)}\left(|Q\cap S|-1\right)v_Q \tag{1}$$

with $\mathcal{Q}(S)=\{Q\in\mathcal{Q}:S\cap Q\neq\emptyset\}$.

We write $\mathcal{Q}(i)$ for $\mathcal{Q}\left(\{i\}\right)$.

Let $\mathcal{C}$ be the set of all clique games.

We conclude this section by proposing two examples of clique games.

**Example 2** *(Trading goods) Suppose that* $N = \{1,2,3,4,5\}$*, with* 1 *and* 2 *being producers and* $3, 4$ *and* 5 *being buyers. Producer* 1 *has a capacity to produce two units at constant marginal cost* $c_1$ *while producer* 2 *can produce a single unit at cost* $c_2$*. Each buyer* $i$ *is interested by a single unit that she values at* $R_i$*. We suppose that the reserve prices of the buyers are larger than the marginal cost of the producers.*

*We further suppose that producers* 1 *and* 2 *have exclusive territories (because of vertical restraints or collusion) and that buyers* 3 *and* 4 *are on the territory of producer* 1 *and buyer* 5 *on the territory of producer* 2*. We also suppose that the producers' unused capacity can be sold to external buyers at price* $q$ *and that buyers have the option of buying from an external supplier at price* $p$*, with* $R_i > p > q > c_j$*.*

*When a coalition forms, trades occur between buyers and sellers in the same territory, with unsatisfied demands and unsold supply resolved on the outside market. For example, coalition* $\{1,2,3,5\}$ *will organize trades between* 1 *and* 3 *and* 2 *and* 5*, generating a surplus of* $R_3 + R_5 - c_1 - c_2$*. In addition, producer* 1 *sells its extra unit on the outside market, generating an additional surplus of* $q - c_1$*.*

*The game can thus be represented (see Figure 2) by a clique game, with cover* $\mathcal{Q} = \{\{1,2\},\{1,3\},\{1,4\},\{2,5\}\}$ *and* $v_1 = 2q - 2c_1$*,* $v_2 = q - c_2$*,* $v_i = R_i - p$ *for* $i = 3,4,5$*,* $v_{\{1,2\}} = 0$ *and* $v_Q = r \equiv p - q$ *otherwise.*

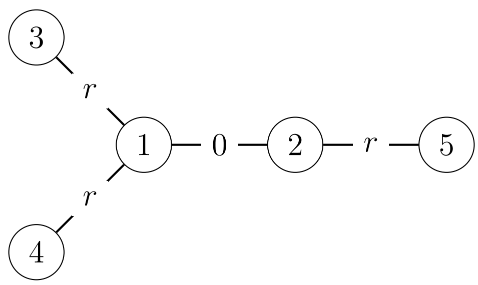


Figure 2: Clique cover of a trading goods game.

**Example 3** *(Job scheduling problem (Bahel and Trudeau, 2017)) Suppose that* $N = \{1,2,3,4\}$ *with each agent having jobs to schedule on a machine. Each job has fixed starting and finishing times, and a machine can only* *process one job at time. Each agent $i$ has utility $u_i$ per job completed, and machines can only be rented for the full time interval at cost $c$. Since $u_i > c$, a coalition will generate the most surplus by hiring the minimal number of machines needed to schedule all jobs of its members.*

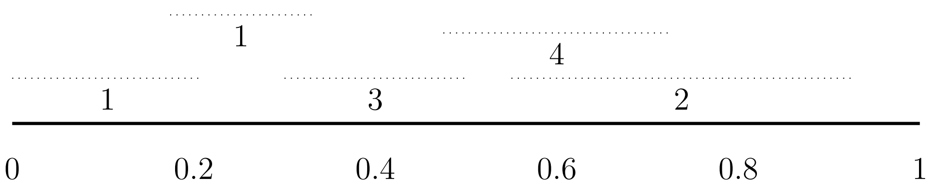


Figure 3: Example of a job scheduling problem.

Figure 3 provides an illustration of a job scheduling problem. In this example, agent 1 has two jobs to schedule, with others having a single job. Let $\mathcal{Q} = \{\{1, 2, 3\}, \{1, 4\}\}$, as those are the coalitions that can generate savings by scheduling (some of their) jobs on the same machine. The game is a clique game (Figure 4) with $v_1 = 2(u_1 - c)$, $v_i = u_i - c$ for $i = 2, 3, 4$ and $v_Q = c$ for all $Q \in \mathcal{Q}$.

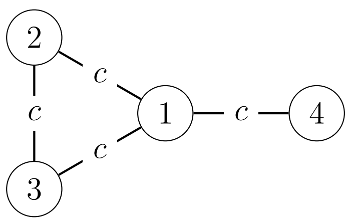


Figure 4: Clique cover of a job scheduling problem.

Not all job scheduling problems can be represented as a clique game however. If the two jobs of agent 1 are coming from different agents, we lose the representability by a clique.

## 4 Correspondence between the Shapley value and the nucleolus

In this section we show that for clique games, the Shapley value and the nucleolus coincide, and we provide a closed-form expression for their value. To get to this result, we first describe the marginal contributions in clique games.

**Lemma 1** *Given a clique game $(N, v^{\mathcal{Q}})$, the marginal contribution of player $i \in N$ to $S \subseteq N \setminus \{i\}$ is*

$$\Delta_i^{v^{\mathcal{Q}}}(S) = v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q.$$

**Proof.** By definition of a marginal contribution,

$$\begin{aligned}
\Delta_i^{v^{\mathcal{Q}}}(S) &= v^{\mathcal{Q}}(S \cup \{i\}) - v^{\mathcal{Q}}(S) \\
&\overset{(1)}{=} v_i + \sum_{Q \in \mathcal{Q}(S \cup \{i\})} \left(|Q \cap (S \cup \{i\})| - 1\right) v_Q - \sum_{Q \in \mathcal{Q}(S)} \left(|Q \cap S| - 1\right) v_Q \\
&= v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} \left[\left(|Q \cap (S \cup \{i\})| - 1\right) - \left(|Q \cap S| - 1\right)\right] v_Q \\
&+ \sum_{Q \notin \mathcal{Q}(S), i \in Q} \left(|Q \cap (S \cup \{i\})| - 1\right) v_Q \\
&= v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} \left[|Q \cap S)| - \left(|Q \cap S| - 1\right)\right] v_Q \\
&+ \sum_{Q \notin \mathcal{Q}(S), i \in Q} \left(|\{i\}| - 1\right) v_Q \\
&= v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q.
\end{aligned}$$

■

We are now ready for the main result of this section.

**Theorem 1** *For all $v^{\mathcal{Q}} \in \mathcal{C}$ and all $i \in N$,*

$$Sh_i\left(v^{\mathcal{Q}}\right) = \bar{y}_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

**Proof.** It is obvious from Lemma 1 that $v^Q$ is a convex game. Thus, the Shapley value is the average of extreme points of the core (Shapley, 1971; Ichiishi, 1981) and $Sh\left(v^{\mathcal{Q}}\right) = \bar{y}\left(v^{\mathcal{Q}}\right)$. We show that for all $i \in N$,

$$Sh_i\left(v^{\mathcal{Q}}\right) = Nu_i\left(v^{\mathcal{Q}}\right) = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q.$$

We suppose that for all $k \in \{1, ..., K\}$, $\bigcup_{i \in Q^k} N^P_{k,i} = N \setminus Int(Q^k)$, that is, there is a (unique) path between any two elements of $\mathcal{Q}$. Without that assumption, we can partition our agents into groups unconnected by paths, and we can compute the Shapley value and the nucleolus independently on each element of the partition.

We start with $Sh\left(v^{\mathcal{Q}}\right)$. Given $\pi \in \Pi$, under Lemma 1, the marginal contribution of agent $i$ to $P_i(\pi)$ is $v_i + \sum_{Q \in \mathcal{Q}(P_i(\pi)) \cap \mathcal{Q}(i)} v_Q$. For each $Q \in \mathcal{Q}(i)$, the probability that $Q \in \mathcal{Q}\left(P_i(\pi)\right) \cap \mathcal{Q}(i)$ is $\frac{|Q|-1}{|Q|}$. Summing up, we obtain the desired result.

We now focus on $Nu\left(v^{\mathcal{Q}}\right)$. Let $x \in \mathbb{R}^N$ defined as $x_i = v_i + \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q$ for all $i \in N$. We have that

$$\begin{aligned}
e\left(S, x, v^{\mathcal{Q}}\right) &= \sum_{i \in S} v_i + \sum_{i \in S} \sum_{Q \in \mathcal{Q}(i)} \frac{|Q|-1}{|Q|} v_Q - \sum_{i \in S} v_i - \sum_{Q \in \mathcal{Q}(S)} \left(|Q \cap S| - 1\right) v_Q \\
&= \sum_{Q \in \mathcal{Q}(S)} \left( \frac{|Q \cap S| \left(|Q| - 1\right)}{|Q|} - \left(|Q \cap S| - 1\right) \right) v_Q \\
&= \sum_{Q \in \mathcal{Q}(S)} \frac{\left(|Q \cap S| \left(|Q| - 1\right) - |Q \cap S| - 1\right) |Q|}{|Q|} v_Q \\
&= \sum_{Q \in \mathcal{Q}(S)} \frac{|Q| - |Q \cap S|}{|Q|} v_Q
\end{aligned}$$

for all $S \subset N$, $S \neq \emptyset$. Assume without loss of generality $\frac{v_{Q^1}}{|Q^1|} \leq \frac{v_{Q^2}}{|Q^2|} \leq \cdots \leq \frac{v_{Q^K}}{|Q^K|}$.

For each $i \in Q^1$, let $S_i^1 = N \setminus \left(N_{1,i}^P \cup \{i\}\right)$. Notice that for all $Q \in \mathcal{Q} \setminus \{Q^1\}$, we either have that $S_i^1 \cap Q = \emptyset$ or $S_i^1 \cap Q = Q$. In addition, $S_i^1 \cap Q^1 = Q^1 \setminus \{i\}$. Thus, $e\left(S_i^1, x, v^{\mathcal{Q}}\right) = \frac{v_{Q^1}}{|Q^1|}$. By construction, this is the lowest excess value. To see why, notice that any $S \subset N$ must have at least one $Q \in \mathcal{Q}(S)$ such that $|Q \cap S| < |Q|$. That generates an excess of $\frac{|Q|-|Q\cap S|}{|Q|} v_Q \geq \frac{v_Q}{|Q|} \geq \frac{v_{Q^1}}{|Q^1|}$.

For each $i \in Q^1$, let $T_i^1 = N_{1,i}^P \cup \{i\} = N \setminus S_i^1$. Take $\{T_i^1\}_{i \in Q^1}$. This is a partition of $N$. To see why, notice that each $T_i^1$ is nonemtpy (because $i \in T_i^1$ for all $i \in Q^i$), their union is $N$ (because all cliques are connected through a path), and they are pairwise disjoint (because of assumption i)). Thus, we have $|Q^1|$ coalitions whose complements have the minimal excess, with each agent belonging to exactly one of of these coalitions. Therefore, to increase the excess of one of these coalitions we would need to decrease the excess of another coalition, and the corresponding allocation could not be the nucleolus.

We repeat the process for all $Q^k$ to obtain that

$$\sum_{j \in S_i^k} Nu_j\left(v^{\mathcal{Q}}\right) = \sum_{j \in S_i^k} x_j \tag{2}$$

for all $Q^k \in \mathcal{Q}$ and all $i \in Q^k$. In case $i \in Int(Q^k)$ for some $Q^k \in \mathcal{Q}$, we have $S_i^k = N \setminus \{i\}$, from where (2) and efficiency of $x$ imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$.

In case $\mathcal{Q} = \{Q^1\}$, we have $N = Int\left(Q^1\right)$ and hence $Nu\left(v^{\mathcal{Q}}\right) = x$. So, we assume $|\mathcal{Q}| > 1$. From condition i) in the definition of clique games, there exist some $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Q = Int(Q) \cup \{i\}$ for all $Q \in \mathcal{Q}(i) \setminus \{Q^k\}$. This implies that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$. Under (2) and the efficiency of $x$, we deduce $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$. Repeating the same reasoning, we can always find a new $i \in N$ and $Q^k \in \mathcal{Q}(i)$ such that $Nu_j\left(v^{\mathcal{Q}}\right) = x_j$ for all $j \in Q^k \in \mathcal{Q}(i) \setminus \{Q^k\}$, so that (2) and the efficiency of $x$ imply $Nu_i\left(v^{\mathcal{Q}}\right) = x_i$, and so on until we get $Nu\left(v^{\mathcal{Q}}\right) = x$. ∎

We next establish the connection between clique games and the PS-games of Kar et al. (2009). We say that a game $(N, v)$ is a PS-game if there exists $a \in \mathbb{R}^N$ such that $\Delta_i^v(S) + \Delta_i^v(N \setminus (S \cup \{i\})) = a_i$ for all $i \in N$ and $S \subseteq N \setminus \{i\}$.

We show the condition needed for a clique game to also be a PS-game, which illustrates that not all clique games are PS-games.

**Proposition 1** *A clique game $v^{\mathcal{Q}}$ is a PS-game if and only if for all $Q \in \mathcal{Q}$ it holds either $|Q| \leq 2$ or $v_Q = 0$.*

**Proof.** Under Lemma 1, for any clique game $v^{\mathcal{Q}}$, we have that

$$\Delta_i^{v^{\mathcal{Q}}}(S) = v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q$$

and thus that $\Delta_i^{v^{\mathcal{Q}}}(S) + \Delta_i^{v^{\mathcal{Q}}}(N \setminus (S \cup \{i\}))$

$$\begin{aligned} &= v_i + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(i)} v_Q + v_i + \sum_{Q \in \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q \\ &= 2v_i + \sum_{Q \in \mathcal{Q}(N \setminus \{i\}) \cap \mathcal{Q}(i)} v_Q + \sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)} v_Q . \end{aligned}$$

Hence, $v^{\mathcal{Q}}$ is a PS-game if and only there exists $b \in \mathbb{R}^N$ such that

$$\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)} v_Q = b_i$$

for all $S \subseteq N \setminus \{i\}$. In this case, $a_i = 2v_i + \sum_{Q \in \mathcal{Q}(N \setminus \{i\}) \cap \mathcal{Q}(i)} v_Q + b_i$ for all $i \in N$.

Fix $i \in N$. Let $S \subseteq N \setminus \{i\}$ and $Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i)$. If $|Q| \leq 2$ then $|Q \cap S| \leq 1$ (because $i \in Q$ and $i \notin S$). Since $Q \in \mathcal{Q}(S)$, we deduce $Q \cap S = \{j\}$ for some $j \neq i$. Thus, $Q = \{i, j\} \subseteq S \cup \{i\}$, which contradicts that $Q \in \mathcal{Q}(N \setminus (S \cup \{i\}))$. Hence, $\mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i) \subseteq \{Q \in \mathcal{Q} : |Q| > 2\}$.

From this, we deduce that if $v_Q = 0$ for all $Q \in \mathcal{Q}$ such that $|Q| > 2$, then $b_i = 0$ for all $i \in N$.

Suppose now that there exists $Q \in \mathcal{Q}$ such that $|Q| > 2$ and $v_Q > 0$. Fix $i \in Q$. With $S = \emptyset$ we obtain $\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q = 0$ (as $\mathcal{Q}(S) = \emptyset$). With $S = \{j\}$, $j \in Q \setminus \{i\}$, we obtain $\sum_{Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup i)) \cap \mathcal{Q}(i)} v_Q \geq v_Q > 0$ (as $Q \in \mathcal{Q}(S) \cap \mathcal{Q}(N \setminus (S \cup \{i\})) \cap \mathcal{Q}(i))$ . ■

Moreover, not all PS games are clique games, as the next example shows:

**Example 4 (Example 3.12 in Kar et al. (2009))** *We consider the TU game $(N, v)$ with $N = \{1, 2, 3, 4\}$ and such that $v(S) = 0$ if $|S| = 1$, $1$ if $|S| = 2$, $\frac{3}{2}$ if $|S| = 3$, and $3$ if $S = N$. This is a PS game with $\Delta_i^v(S) + \Delta_i^v(N \setminus (S \cup \{i\})) = \frac{3}{2}$ for all $i$ and $S$. However, it is not a clique game. To see this, notice that $v(S) = 0$ if $|S| = 1$ imposes that $v_i = 0$ for all $i \in N$. Then, $v(S) = 1$ if $|S| = 2$ imposes that any pair $i, j$ belong to some clique $Q$ with $v_Q = 1$. The no-cycle condition of clique games (condition i)) leaves us with a single candidate for the set of cliques: $\mathcal{Q} = \{N\}$. But then $v(S) = |S| - 1$ for all $S$, which is different from the PS-game for $|S| = 3$.*

# 5 Minimum cost spanning tree problems

In this section we describe the minimum cost spanning tree problem, showing that an important subset of such games are also clique games. In turn, this allows us to link the nucleolus to some well-known cost sharing solutions for *mcst* problems.

## 5.1 The problem

We assume that the agents in $N$ need to be connected to a source, denoted by 0. Let $N_0 = N \cup \{0\}$. For any set $Z$, define $Z^p$ as the set of all non-ordered pairs $(i, j)$ of elements of $Z$. In our context, any element $(i, j)$ of $Z^p$ represents the edge between nodes $i$ and $j$. Let $c = (c_e)_{e \in N_0^p}$ be a vector in $\mathbb{R}_+^{N_0^p}$ with $N_0^p = (N_0)^p$ and $c_e \in \mathbb{R}_+$ representing the cost of edge $e$. Let $\Gamma$ be the set of all cost vectors. Since $c$ assigns cost to all edges $e$, we often abuse language and call $c$ a cost matrix. A *minimum cost spanning tree* (*mcst*) problem is a triple $(0, N, c)$. Since 0 and $N$ do not change, we omit them in the following and simply identify a *mcst* problem $(0, N, c)$ by its cost matrix $c$.

A cycle $p_{ll}$ is a set of $K \geq 3$ edges $(i_{k-1}, i_k)$, with $k \in \{1, \ldots, K\}$ and such that $i_0 = i_K = l$ and $i_1, \ldots, i_{K-1}$ distinct and different than $l$. A path $p_{lm}$ between $l$ and $m$ is a set of $K$ edges $(i_{k-1}, i_k)$, with $k \in \{1, \ldots, K\}$, containing no cycle and such that $i_0 = l$ and $i_K = m$. Let $P_{lm}(N_0)$ be the set of all such paths between nodes $l$ and $m$.

A spanning tree is a non-orientated graph without cycles that connects all elements of $N_0$. A spanning tree $t$ is identified by the set of its edges.

We call *mcst* a spanning tree that has a minimal cost. Note that the *mcst* might not be unique. Let $C(N, c)$ be the minimal cost of a *mcst*. Let $c^S$ be the restriction of the cost matrix $c$ to $S_0 \subseteq N_0$. Let $C(S, c)$ be the cost of the *mcst* of the problem $(S, c^S)$. Given these definitions, we say that $C$ is the stand-alone cost function associated with $c$.

For any cost matrix $c$, the associated cost game is given by $(N, C)$ with $C(S) = C(S, c)$ for all $S \subseteq N$. We then write, with some abuse of notation, $(N, c)$ instead of $(N, C)$ and say that it is a *mcst* game.

A variant of the *mcst* problem, called the public *mcst* problem, allows any coalition to use all nodes, including those belonging to agents outside of the coalition, to connect to the source. The *public cost function associated with* $c$ is defined as

$$C^{Pub}(S, c) = \min_{T \subseteq N \setminus S} C(S \cup T, c)$$

for all $S \subseteq N$. By contrast, we sometimes call $(N, c)$ the *private cost function associated with* $c$ and the *mcst* problem the *private mcst problem*.

Abusing language slightly, we use the term *mcst game* to designate the cost game generated by a *mcst* problem.

## 5.2 The irreducible and cycle-complete cost matrices

Given that a *mcst* game is typically not a concave game, its Shapley value is not always in the core. The following two modifications to the problem allow to transform the game into a concave one.

From any cost matrix $c$, we define the irreducible cost matrix $c^*$ as follows:

$$c^*_{ij} = \min_{p_{ij} \in P_{ij}(N_0)} \max_{e \in p_{ij}} c_e$$

for all $i, j \in N_0$.

From any cost matrix $c$, we define the cycle-complete cost matrix $c^{**}$ as follows:

$$c^{**}_{ij} = \max_{k \in N \setminus \{i,j\}} \left(c^{N \setminus \{k\}}\right)^*_{ij}$$

for all $i, j \in N_0$, and

$$c^{**}_{0i} = \max_{k \in N \setminus \{i\}} \left(c^{N \setminus \{k\}}\right)^*_{0i}$$

for all $i \in N$, where $\left(c^{N \setminus \{k\}}\right)^*$ indicates the matrix that we first restrict to agents in $N \setminus \{k\}$ before transforming into an irreducible matrix.

The cycle complete matrix can also be defined using cycles (Trudeau, 2012): for edge $(i, j)$, we look at cycles that go through agents $i$ and $j$. If there is one such cycle such that its most expensive edge is cheaper than a direct connection on edge $(i, j)$, we assign this cost to edge $(i, j)$.

Let $C^*$ be the characteristic cost function associated with the mcst problem $(N, c^*)$. Let $C^{**}$ be the characteristic cost function associated with the *mcst* problem $(N, c^{**})$. The Shapley values of $C^*$ and $C^{**}$ are respectively called the folk $(y^f(c))$ and cycle-complete $(y^{cc}(c))$ solutions.

## 5.3 Minimum cost spanning tree games and clique games

We are now ready to describe the set of *mcst* games that are also clique games.

**Lemma 2** *A mcst game $(N, c)$ is associated to a clique game if and only if there exist $\mathcal{Q}$ satisfying condition i) of clique games and $\{c_Q\}_{Q \in \mathcal{Q}} \subset \mathbb{R}$ that satisfy the following conditions:*

*a) $c_{ij} = c_Q$ for all $Q \in \mathcal{Q}$ and all $i, j \in Q$;*

*b)* $c_{ij} \geq \max\{c_{0i}, c_{0j}\}$ *for all* $i, j \in N$ *such that there exists no* $Q \in \mathcal{Q}$ *with* $i, j \in Q$*;*

*c) for all* $Q \in \mathcal{Q}$*, if* $c_{0,Q}^{min}, c_Q < c_{0,Q}^{max}$ *then* $|\arg\max_{j \in Q} c_{0j}| = |Q| - 1$*, where* $c_{0,Q}^{max} = \max_{j \in Q} c_{0j}$ *and* $c_{0,Q}^{min} = \min_{j \in Q} c_{0j}$*.*

**Proof.** Note first that condition b) can be replaced by:

b') $c_{ij} = \max\{c_{0i}, c_{0j}\}$ for all $i, j \in N$ such that there exists no $Q \in \mathcal{Q}$ with $i, j \in Q$.

To see why, notice that an edge $(i, j)$ with $c_{ij} > \max\{c_{0i}, c_{0j}\}$ is irrelevant in the sense that it does not affect the cost function $C$. Hence, the associated game $(N, v^C)$ does not change if we reduce $c_{ij}$ until equality holds. We then assume that $c$ has no irrelevant edges. This also forces to have $c_Q \leq \max\{c_{0i}, c_{0j}\}$ for all $Q \in \mathcal{Q}$ and all $i, j \in Q$.

We first show that the conditions generate a clique game. Suppose that we want to connect members of $S$ to the source. Conditions a) and b') make it never optimal to directly connect members of different cliques. Combination of the three conditions make it always better to connect members of the same clique to each other. Let $\{S_1, S_2, \ldots, S_K\}$ be a partition of $S$ such that if $i, j \in S_k$, then there exists a path between $i$ and $j$ for which the most expensive edge is $c_Q$, for some $Q \in \mathcal{Q}$. Then, the cost of coalition $S$ is

$$C(S, c) = \sum_{k=1}^{K} \min_{i \in S_k} c_{0i} + \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\, c_Q.$$

By condition c), if members of a clique have different costs to connect to the source, then all but one have the same high cost $c_{0,Q}^{max}$. We can thus simplify the cost of coalition $S$ to

$$C(S, c) = \sum_{i \in S} c_{0i} + \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\,(c_Q - c_{0,Q}^{max}).$$

We then have that $v^C$ is such that

$$v^C(S,c) = \sum_{Q \in \mathcal{Q}(S)} (|Q \cap S| - 1)\,(c_{0,Q}^{max} - c_Q)$$

which corresponds to a clique game with $v_i = 0$ for all $i \in N$ and $v_Q = c_{0,Q}^{max} - c_Q$.

We next show that these conditions are necessary. Without condition a), there exist $i, j, k \in Q$ such that $c_{ij} \neq c_Q$ but $c_{ik} = c_Q$. Then, $C(\{i,j\}, c) = \min\{c_{0i}, c_{0j}\} + c_{ij}$, $C(\{i,k\}, c) = \min\{c_{0i}, c_{0k}\} + c_Q$ and $C$ is no longer a clique game.

Without condition b), there exist $i, j$ belonging to different cliques such that $c_{ij} < \max\{c_{0i}, c_{0j}\}$. Then $C(\{i,j\}, c) = \min\{c_{0i}, c_{0j}\} + c_{ij}$ and $C$ is no longer a clique game.

Without condition c), there exists a clique $Q$ containing $m \geq 3$ agents and such that $|\arg\max_{j \in Q} c_{0j}| < m - 1$. There are thus at least two agents, say $i$ and $j$, with $c_{0,Q}^{min} \equiv c_{0i} \leq c_{0j} < c_{0,Q}^{max}$. Then, $C(\{i,j\}, c) = c_0 i + c_Q < c_{0i} + c_{0j} + (c_Q - c_{0,Q}^{max})$ and $C$ is no longer a clique game. ■

Let $\Gamma^c$ be the set of matrices generating clique *mcst* problems.

Consider the subset of *mcst* problems known as elementary *mcst* (*emcst*) problems: for any $i, j \in N_0$, $c_{ij} \in \{0, 1\}$. Let $\Gamma^e$ be the set of elementary cost problems.

It turns out that the intersection of clique and elementary *mcst* problems is the set of elementary cycle-complete problems,

**Lemma 3** $\Gamma^c \cap \Gamma^e = \Gamma^{ecc}$, *the set of elementary cycle-complete problems.*

**Proof.** "$\supseteq$" We need to show that elementary and cycle-complete *mcst* games are clique games. By definition, there exists a cover $\mathcal{Q}$ of $N$ that satisfies condition i) of clique games and such that $c_{ij} = 0$ if $i, j \in Q$ and $c_{ij} = 1$ otherwise. Thus, $c_Q = 0$ for all $Q \in \mathcal{Q}$ and conditions a) and b) of Lemma 2 are satisfied. Elementary cycle-complete matrices are such that for each $Q \in \mathcal{Q}$, either all members of $Q$ have a cost of zero to connect to the source, all members of $Q$ have a cost of one to the source, or a single agent in $Q$ has a cost of zero, with others having a cost of one to connect to the source. Otherwise, if agents $i$ and $j$ have a cost of zero, but not $k$, there are multiple paths of cost zero between the source and $k$. From this, condition c) of Lemma 2 only applies when a single agent in $Q$ has a cost of zero, with others having a cost of one to connect to the source, so that $|\arg\max_{j\in Q} c_{0j}| = |\{j \in Q : c_{j0} = 1\}| = |Q| - 1$.

"$\subseteq$" Let $c \in \Gamma^c \cap \Gamma^e$. Assume $c$ is not cycle-complete. Then, for some $i, j \in N_0$, we have that $c_{ij} = 1$ but there exist two distinct free paths between them. If $i, j \in N$, we cannot build $\mathcal{Q}$ that satisfies condition i) of clique games and conditions a) and b) in Lemma 2. If $j = 0$, we can assume that each node $k$ in these paths but two (one in each path) satisfy $c_{k0} = 1$. Let $i^0$ and $i^1$ be the nodes with $c_{i^0 0} = c_{i^1 0} = 0$. We also assume that $c_{\alpha\beta} = 0$ for all $\alpha, \beta \in N$ in the path (otherwise, we would be in the previous case). We have the following possibilities:

1. Both paths are contained in the same clique $Q \in \mathcal{Q}$. Then, condition c) in Lemma 2 implies $|\arg\max_{j\in Q} c_{0j}| = |Q| - 1$ and hence all nodes in $Q$ but one should have cost 1 to the source. But there are two nodes ($i^0$ and $i^1$) with cost zero to the source, which is a contradiction.

2. There exist two consecutive nodes $\alpha, \beta \in N$ that belong to different cliques. Since $c_{\alpha\beta} = 0$ and $\max\{c_{\alpha 0}, c_{\beta 0}\} = 1$, condition b) in Lemma 2 does not hold, which is a contradiction.

3. There exists a path of at least two cliques between $i^0$ and $i^1$. Clearly, each of these cliques should have at least two consecutive nodes. Moreover, condition i) of clique games implies that $i^0$ and $i^1$ belong to different cliques. Thus, there exist $j^0 \in N$ consecutive node to $i^0$ and such that $i^0, j^0 \in Q^0$ and $i^1 \in Q^1$ with $Q^0, Q^1$ different cliques. Condition b) in Lemma 2 implies that $0 = c_{j^0 i^1} \geq \max\{c_{0j^0}, c_{0i^1}\} = 1$, which is a contradiction.

■

We then have that, in any *mcst* problem whose cost matrix is elementary and cycle complete, the (pre)nucleolus, the Shapley value, the permutation-weighted average of the extreme points of the core and the cycle-complete rule coincide. Formally:

**Theorem 2** *For all* $c \in \Gamma^{ecc}$, $Nu(C) = Sh(C) = \bar{y}(C) = y^{cc}(c)$.

**Proof.** The correspondence between the nucleolus, the Shapley value and the permutation-weighted average of the extreme points of the core is obtained as a corollary of Theorem 1 and Lemma 3. Correspondence with the cycle-complete solution is by definition. ■

In addition, as soon as the cost matrix is elementary, the (pre)nucleolus, the permutation-weighted average of the extreme points of the core, and the cycle-complete rule coincide. Formally:

**Theorem 3** *For all* $c \in \Gamma^{e}$, $Nu(C) = \bar{y}(C) = y^{cc}(c)$.

**Proof.** Correspondence between the cycle-complete solution and $\bar{y}$ is shown in Trudeau and Vidal-Puga (2017). We show the correspondence between the nucleolus and the cycle-complete solution. It is immediate that $C^{**} \leq C$. We show that if $C^{**}(S) < C(S)$, then the excess of coalition $S$ is ignored in the calculation of $Nu(C^{**})$.

As shown in Trudeau and Vidal-Puga (2017), there exists $T \subseteq N \setminus S$ such that $C^{**}(S) = C(S \cup T) + C(N \setminus T) - C(N) < C(S)$. This can we rewritten as

$$\begin{aligned} \sum_{i \in S \cup T} C(i) - C(S \cup T) + \sum_{i \in N \setminus T} C(i) - C(N \setminus T) - \sum_{i \in N} C(i) + C(N) &> \sum_{i \in S} C(i) - C(S) \\ v^C(S \cup T) + v^C(N \setminus T) - v^C(N) &> v^c(S) \\ x(S \cup T) - v^C(S \cup T) + x(N \setminus T) - v^C(N \setminus T) - x(N) + v^C(N) &< x(S) - v^C(S) \\ e(S \cup T, x, v^C) + e(N \setminus T, x, v^C) - e(N, x, v^C) &< e(S, x, v^C) \\ e(S \cup T, x, v^C) + e(N \setminus T, x, v^C) &< e(S, x, v^C). \end{aligned}$$

Therefore, the excess of $S$ is not taken into account when we find $Nu(C)$. We also have that

$$C^{**}(S \cup T) + C^{**}(N \setminus T) - C^{**}(N) \leq C(S \cup T) + C(N \setminus T) - C(N) = C^{**}(S)$$

leading to conclude, in the same manner as above, that

$$e(S \cup T, x, C^{**}) + e(N \setminus T, x, C^{**}) \leq e(S, x, C^{**})$$

and thus that the excess of $S$ is not taken into account when we find $Nu(C^{**})$. Therefore, $Nu(C)$ and $Nu(C^{**})$ depend on the same excesses, and we must have that $Nu(C) = Nu(C^{**})$. Since $Nu(C^{**}) = y^{cc}(c)$, we also have that $Nu(C) = y^{cc}(c)$. ■

If we look at public *mcst* games instead of private *mcst* games, we obtain similar correspondence results. First, we consider the subset of elementary irreducible games, for which $C^{Pub} = C$. We have correspondence between the (pre)nucleolus, the Shapley value, the permutation-weighted average of the extreme points of the core and the folk solution.[2]

**Corollary 1** *For all elementary and irreducible matrices c,* $Nu(C) = Sh(C) = \bar{y}(C) = y^f(c)$.

For elementary *mcst* games, for which $C^{Pub}$ is typically different from $C$, we obtain the following result:

**Theorem 4** *For all* $c \in \Gamma^e$, $Nu\left(C^{Pub}\right) = \bar{y}\left(C^{Pub}\right) = y^f(c)$.

The proof is similar to the proof of Theorem 3 and is omitted.

[2]A related result is provided by Subiza et al. (2016). They provide a closed-form solution for the folk solution in a class of *mcst* games that are a subset of clique games in which links between agents have a cost that is either high or low. Their result is a simplification of our closed-form expression for their family. They extend by considering games in which the set of agents can be partitioned in independent groups, such that they can all be connected separately to the source, applying their conditions on every group. One could do the same thing in our setting.

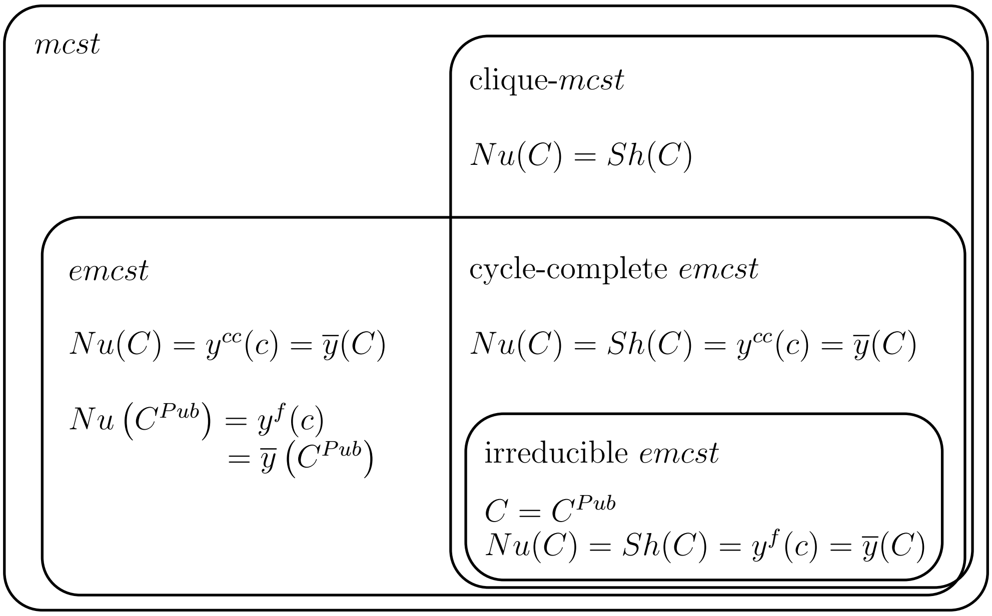


Figure 5: Summary of the results for *mcst* problems.

The results of this section are summarized in Figure 5. The set of clique-*mcst* games are those described in Subsection 5.2.

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