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Marcelo Arbex (University of Windsor)
Christian Trudeau (University of Windsor)

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Optimal tax policy under heterogeneous environmental preferences

Marcelo Arbex* Christian Trudeau[†]

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Abstract

We model a federation of two heterogeneous jurisdictions where agents value consumption vs. nature differently. Consumption obtained through pollution-inducing production also generates a negative externality on neighbors. We show that even with a decentralized policy we can obtain first-best efficiency by choosing a combination of pollution taxes in both regions and lump-sum transfers. Moreover, we show that optimal pollution taxes are determined only by the externality parameters, independent of agents' preferences for consumption and nature.

Keywords: Externalities, environmental preferences, optimal taxation.

JEL Classification: D62; H23; H87; Q58.

^{*,†} Department of Economics, University of Windsor, 401 Sunset Avenue, Windsor, ON, N9B 3P4, Canada. Email: arbex@uwindsor.ca. † Corresponding Author; Email: trudeauc@uwindsor.ca. We thank Maria Galego, Marcelin Joanis, Marcel Oestreich, Emilson Silva, Tracy Snoddon and participants at the 2015 CEA Meetings for helpful comments and discussions. We retain responsibility for any errors.

1 Introduction

We study a two-region federation model where jurisdictions face trade-offs between consumption, obtained through pollution-generating economic activities, and the quality of the environment. Pollution not only damages the local environment, but also creates negative externalities on neighbors. We view environmental externalities as generators of public "bads", along the lines of Meade (1952) and his concept of "atmospheric externalities" (Sandmo (2011)). We show that when regions are heterogeneous in three dimensions (nature endowment, damage spillovers and valuation of consumption vs. nature), we can achieve first-best efficiency by using pollution tax rates and a lump-sum transfer together.¹

Optimal pollution tax rates are determined only by externality parameters (Pigouvian taxation). It aims to charge a jurisdiction with the social cost of their consumption, and its tax rate is thus increasing in the damage it causes on its neighbor. The optimal transfer plays a redistributive role and is affected by each region's endowment of nature and the degree of environmental damage spillovers. The lump-sum transfer is from one region to the other, irrespective of the economic decisions taken by the jurisdictions, while the pollution tax has built-in liability, with the polluter compensating its neighbor.

The inefficiency of decentralized policymaking has long been established as the norm in theoretical public and environmental economics literature on production efficiency in the face of
externalities (Pigou (1920); Samuelson (1954)). In a model with heterogeneous jurisdictions and
interjurisdictional environmental damage spillovers, Ogawa and Wildasin (2009) find that decentralized policymaking leads to efficient resource allocation, even in the complete absence of
corrective interventions by governments or coordination of policy. Decentralized policymaking
can still result in globally efficient allocations, even when preferences and production technologies
differ among jurisdictions and governments have information and care only about local environmental impacts. Fell and Kaffine (2014) argue that Ogawa and Wildasin (2009)'s result hinges on
the fact that in their model, there is a fixed sum of environmental damages across jurisdictions,
and their central result breaks down if the model is modified in ways that make the environmental
damage endogenous. Even though in our model the total environmental damage is affected by the
policy choices, the decentralized outcome is still efficient.

2 The Economy

We consider an economy where two jurisdictions i = 1, 2 are inhabited by a large number of agents with identical preferences. We define all variables in per capita terms and consider the case of a constant population. Each region is endowed with an initial environment of quality

¹There is an extensive literature dedicated to alternative policies: carbon taxation, cap-and-trade, tradable permits, and regulations related to pollution control (see, for instance, Montgomery (1972); Baumol and Oates (1988), and Muller and Mendelsohn (2009)).

 N_i , which is then reduced by environmental damages (i.e., pollution) e_i linked to production. Each unit of output produced in jurisdiction i, labelled as Y_i , results in one unit of environmental damage there. Production in jurisdiction i has a negative atmospheric externality and causes environmental damage in the other jurisdiction. The degree of environmental damage spillovers from the other jurisdiction is captured by a region specific parameter $\beta_j \in [0, 1]$, so that the environmental damage experienced by region i is given by

$$e_i = Y_i + \beta_i Y_i. \tag{1}$$

In our economy, if β_i is positive, local economic activity causes damage not only to the local environment but in other jurisdictions as well. Oates and Schwab (1988) assume no interjurisdictional environmental spillovers and environmental quality in any jurisdiction depends only on local economic activity, i.e., $\beta_j = 0$ in equation (1). The upper limit of $\beta_i = 1$ corresponds to complete spillovers, where a unit of output produced in jurisdiction i does just as much damage elsewhere as it does locally. The analyses of Ogawa and Wildasin (2009) and Fell and Kaffine (2014) are restricted to the case $\beta_1 = \beta_2 = \beta$.

The cumulative level of environmental damage is

$$\sum_{i=1}^{2} e_i = \sum_{i=1}^{2} (1 + \beta_i) Y_i. \tag{2}$$

We do not assume that the sum of the environmental damage is equal to an exogenous constant. In such a case, the planner can only shift environmental damages across jurisdictions, but not reduce aggregate damages. We depart from Ogawa and Wildasin (2009) by allowing the planner to choose (indirectly) the optimal level of environmental damage in each region. Hence, the choice of consumption-nature quality allocations is affected by the heterogeneity of the regions with respect to their preferences for environmental quality and consumption.

The utility function of the representative agent residing in jurisdiction i is denoted as $u_i(c_i, n_i)$, where c_i is the agent's consumption of a private good in jurisdiction i, $n_i = N_i - e_i$ denotes nature quality enjoyed locally. We make the usual assumptions on utility functions (differentiable, increasing and strictly quasi-concave functions). We allow for agents in different jurisdictions to have different preferences for nature versus consumption.

3 The centralized and decentralized problems

3.1 The centralized problem

Consider a federation where the central government cares equally about agents in both regions and can directly choose production and consumption in both jurisdictions, with the constraint that $c_1 + c_2 \leq Y_1 + Y_2$. Using the fact that $n_i = N_i - Y_i - \beta_j Y_j$, we reexpress the problem in terms

of consumption and nature levels, c_i and n_i . Hence, the planner chooses a first-best allocation by solving the following problem:

$$\max_{\{c_1, c_2, n_1, n_2\}} \sum_{i=1}^{2} u_i (c_i, n_i)$$

under the constraint that

$$c_1 + c_2 \le \frac{(N_1 - n_1)(1 - \beta_1) + (N_2 - n_2)(1 - \beta_2)}{1 - \beta_1 \beta_2}.$$
 (3)

From the first order conditions of this problem we obtain the following conditions for first-best efficiency:

$$\frac{\partial u_1(c_1, n_1)}{\partial c_1} = \frac{\partial u_2(c_2, n_2)}{\partial c_2} \tag{4}$$

$$\frac{\partial u_{1}(c_{1}, n_{1})}{\partial c_{1}} = \frac{\partial u_{2}(c_{2}, n_{2})}{\partial c_{2}} \tag{4}$$

$$\frac{\partial u_{1}(c_{1}, n_{1})}{\partial n_{1}} (1 - \beta_{2}) = \frac{\partial u_{2}(c_{2}, n_{2})}{\partial n_{2}} (1 - \beta_{1})$$

$$\frac{\partial u_{1}(c_{1}, n_{1})}{\partial c_{1}} = \frac{\partial u_{1}(c_{1}, n_{1})}{\partial n_{1}} + \beta_{1} \frac{\partial u_{2}(c_{2}, n_{2})}{\partial n_{2}}$$
(6)

$$\frac{\partial u_1(c_1, n_1)}{\partial c_1} = \frac{\partial u_1(c_1, n_1)}{\partial n_1} + \beta_1 \frac{\partial u_2(c_2, n_2)}{\partial n_2}$$
 (6)

Condition (4) and (5) impose that the marginal utilities for consumption and nature, respectively, are equalized across regions. According to condition (6), the marginal utility for consumption should be equal to the marginal disutility created by the additional pollution it generates. The feasibility constraint, equation (3), must hold with equality. From this system of equations, we can find the first-best optimal allocations c_1^* , c_2^* , Y_1^* , Y_2^* , n_1^* and n_2^* .

3.2 The decentralized problem

Jurisdictions play a game between themselves and with the government. First, the government announces the value of its policy instruments. Second, with this information households in both jurisdictions determine their productions, which cause externalities for both regions. Finally, given the decisions of the government and private agents, consumption, nature level and welfare are determined. All parameters $(N_i, \beta_i)_{i=1,2}$ are common knowledge. We solve the game by backward induction.

For each region i, t_i is the lump-sum transfer households make to (or receive from) the central government and p_i is the pollution tax they pay on each unit of output region i produces. We assume that region 1's transfer always equals the negative of region 2's transfer, i.e., $\sum_{i=1}^{2} t_i = 0$. We also assume that revenue collected with the pollution tax in region i is transferred to households in region $j.^2$

When the central government can make lump-sum transfers and levy a pollution tax, house-

²For alternative federal, inter-jurisdictional settings see, for instance, Boadway et al. (2013) and Silva and Caplan (1997). The assumption that the central government returns all proceeds of its policies is particularly realistic if we think of supra-national agreements or cases where revenues originating from pollution policies are earmarked to compensate regions that have been polluted.

holds in jurisdiction i face the following budget constraint

$$c_i = Y_i(1 - p_i) - t_i + p_i Y_i. (7)$$

Hence, region i's household problem is:

$$\max_{Y_i} u_i \left(Y_i (1 - p_i) - t_i + p_j Y_j, N_i - Y_i - \beta_j Y_j \right). \tag{8}$$

The first-order condition of the region i's household is:

$$\frac{\partial u_i(c_i, n_i)}{\partial c_i} (1 - p_i) - \frac{\partial u_i(c_i, n_i)}{\partial n_i} = 0.$$
(9)

At the first-best optimal allocations, equation (9) implies:

$$\frac{\partial u_i(c_i^*, n_i^*)/\partial c_i}{\partial u_i(c_i^*, n_i^*)/\partial n_i} = \frac{1}{(1 - p_i)},\tag{10}$$

which combined with equation (6), simplifies to:

$$\frac{1}{(1-p_i)} = 1 + \beta_i \frac{\partial u_j(c_j^*, n_j^*)/\partial c_j}{\partial u_i(c_i^*, n_i^*)/\partial n_i}.$$
 (11)

The above equation together with equation (5), and after some manipulation, implies that the optimal pollution tax in jurisdiction i is not affected by the households preferences for consumption and nature:

$$p_i^* = \frac{\beta_i (1 - \beta_j)}{1 - \beta_i \beta_j}.\tag{12}$$

Interestingly, the optimal pollution tax is determined only by the (atmospheric) externality parameters (β_1, β_2) . The pollution tax p_i^* aims to charge jurisdiction i with the social cost of their consumption, and it is thus increasing in β_i and decreasing in β_i .

Using the region i's household budget constraint, expression (7), in terms of n_i and n_j , as well as the optimal pollution tax rate (12), we obtain

$$t^* \equiv t_1^* = -t_2^* = \frac{N_1 - n_1^*}{1 - \beta_1 \beta_2} (1 - \beta_1) - c_1^*. \tag{13}$$

We summarize our results in the following theorem:

Theorem 1. For any utility functions u_1 and u_2 and any parameters $(N_i, \beta_i)_{i=1,2}$, the planner can obtain the first-best allocation as a result of the decentralized problem by choosing $p_i^* = \frac{\beta_i(1-\beta_j)}{1-\beta_i\beta_j}$ for i=1,2 and $t^* = \frac{N_1-n_1^*}{1-\beta_1\beta_2}(1-\beta_1)-c_1^*$, with n_1^* and c_1^* obtained by solving the centralized problem.

4 Conclusions

We show that even with a decentralized policy we can obtain first-best efficiency by choosing a combination of pollution tax and lump-sum transfers when preferences for consumption and nature are heterogeneous. The pollution tax rates depend only on the externality parameters. While the optimal lump-sum transfer typically depends on preferences, for Cobb-Douglas functions, it can be easily shown that it is also independent of preferences.³ Then, in a version of the game in which preferences are private information, we would obtain strategy-proofness for free: the central government does not even need to know these preferences.

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³For more details, see the Supplemental Material.

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Supplemental Material

1 A special case: Cobb-Douglas preferences

For further analysis, we need more specific preferences to be able to solve for the optimal value of the lump-sum transfers as well as the equilibrium levels for consumption, nature and welfare. We use Cobb-Douglas preferences.

1.1 Optimal Environmental Taxation

For the purpose of this section, we assume that the utility function of the representative agent residing in jurisdiction i is denoted as

$$u_i(c_i, n_i) = \theta_i \ln c_i + (1 - \theta_i) \ln n_i \tag{1}$$

where $0 < \theta_i < 1$ represents the weight put on consumption by the resident in jurisdiction i. We allow for agents in different jurisdictions to have different preferences for nature versus consumption. To make sure that jurisdictions put at least some value, as minimal as it might be, on consumption and nature, θ_i cannot be equal to 0 or 1. Without loss of generality, we suppose that $\theta_1 \geq \theta_2$, meaning that agents in region 2 care at least as much about nature than agents in jurisdiction 1.

From the first order conditions of the general problem we obtain the following conditions for first-best efficiency:

$$\frac{\partial u_1(c_1, n_1)}{\partial c_1} = \frac{\partial u_2(c_2, n_2)}{\partial c_2} \tag{2}$$

$$\frac{\partial u_1(c_1, n_1)}{\partial c_1} = \frac{\partial u_2(c_2, n_2)}{\partial c_2}$$

$$\frac{\partial u_1(c_1, n_1)}{\partial n_1} (1 - \beta_2) = \frac{\partial u_2(c_2, n_2)}{\partial n_2} (1 - \beta_1)$$
(2)

$$\frac{\partial u_1(c_1, n_1)}{\partial c_1} = \frac{\partial u_1(c_1, n_1)}{\partial n_1} + \beta_1 \frac{\partial u_2(c_2, n_2)}{\partial n_2}$$

$$\tag{4}$$

the optimal pollution tax in jurisdiction i is not affected by the households preferences for consumption and nature:

$$p_i^* = \frac{\beta_i (1 - \beta_j)}{1 - \beta_i \beta_j}.\tag{5}$$

Interestingly, the optimal pollution tax is determined only by the (atmospheric) externality parameters (β_1, β_2) . The pollution tax p_i^* aims to charge jurisdiction i with the social cost of their consumption, and it is thus increasing in β_i and decreasing in β_i .

Using the region i's household budget constraint, in terms of n_i and n_j , as well as the optimal pollution tax rate (5), we obtain

$$t^* = \frac{N_1 - n_1^*}{1 - \beta_1 \beta_2} (1 - \beta_1) - c_1^*. \tag{6}$$

Using directly equations (2), (3) and (4) yields the optimal levels of consumptions and nature:

$$c_i^* = \frac{\theta_i \left(N_i (1 - \beta_i) + N_j (1 - \beta_j) \right)}{2(1 - \beta_i \beta_j)}, \tag{7}$$

$$n_i^* = \frac{(1 - \theta_i) \left(N_i (1 - \beta_i) + N_j (1 - \beta_j) \right)}{2(1 - \beta_i)}.$$
 (8)

These expressions with (6) allow us to obtain the optimal lump-sum transfer as follows:

$$t^* = \frac{(N_1 - N_2) - N_1 \beta_1 + N_2 \beta_2}{2(1 - \beta_1 \beta_2)} \tag{9}$$

Replacing for these optimal values, we obtain the following expression for the welfare in each

jurisdiction:

$$W_i^* = \begin{pmatrix} \theta_i \ln(\theta_i) + (1 - \theta_i) \ln(1 - \theta_i) + \ln(N_i(1 - \beta_i) + N_j(1 - \beta_j)) \\ -\ln(2) - \theta_i \ln(1 - \beta_i\beta_j) - (1 - \theta_i) \ln(1 - \beta_i) \end{pmatrix}.$$

The most striking result from expressions (5) and (9) is that the optimal pollution tax rates and the optimal lump-sum transfer are not affected by the households preferences for consumption and nature when preferences are Cobb-Douglas, as in equation (1). In this case, the substitution and income effects are of the same magnitude. This is not true for all utility functions.

When preferences are Cobb-Douglas, the jurisdictions' ex-ante heterogeneity with respect to environmental quality and consumption does not affect the design of optimal policies. In an imperfect information version of the game where the preferences of the jurisdictions would be private information, jurisdictions would have no incentives to misreport their preferences and, in fact, the federation does not even need to know them. We thus obtain strategy-proofness.

The optimal transfer t^* is affected by each region endowment of nature (N_1, N_2) and the degree of environmental damage spillovers (β_1, β_2) . Lump-sum transfers in our economy play a redistributive role and are larger the higher is the difference in the nature endowment of the regions. The planner compares adjusted stocks of nature $N_1(1-\beta_1)$ and $N_2(1-\beta_2)$ and the transfer goes from the region with the largest adjusted stock of nature to its neighbor. The magnitude of this transfer depends on both the difference in adjusted stocks of nature and on the externality parameters.

If regions are homogenous with respect to their production damage spillover, i.e., $\beta_1 = \beta_2 = \beta$, it would be optimal to transfer resources in a lump-sum manner from the high nature endowment region to the region with low nature endowment. The optimal transfer in this case is $(N_1 - N_2)/(2(1 + \beta))$. Notice that the region that receives the transfer is not necessarily the jurisdiction that prefers more consumption or nature, as preferences do not affect optimal policies. As long as the environmental damage experienced by both regions is affected in the same way by their neighbors' production, that is, same β , the optimal pollution tax rate is equal to $\beta/(1 + \beta)$.

1.2 Comparative Statics

We now discuss how the optimal policies depend on the initial stocks of nature and damage spillovers. The optimal lump-sum transfer, equation (9), is affected by a region's initial endowment of nature (N_i) and its externality parameter β_i in opposite directions. For instance, a higher stock of nature in region 1 increases the lump-sum transfer, meaning that jurisdiction 1 will either transfer more resources to or it will receive less from region 2 (the sign of the transfer being determined by the difference in the adjusted nature endowment of the regions). On the other hand, if the degree of environmental damage spillovers from jurisdiction 1 (β_1) increases, the weighted nature stock available for redistribution decreases, and consequently, the optimal transfer decreases. Regarding the optimal pollution tax, equation (5), the effect of the externalities parameters on this policy is straightforward since a pollution tax is intended to correct for the pollution (externality) that affects nature quality across jurisdictions. The pollution tax p_i^* levied on jurisdiction i's households is (i) increasing on β_i , the environmental damage its production inflicts on region j's households, and (ii) decreasing on the negative externality generated by its neighbor (β_i) . Finally, our results suggest that the planner tends to distribute the burden of policy interventions across jurisdictions. For instance, a high pollution tax levied on jurisdiction 1, due to an increase in this region's externalities (increase in β_1), is compensated by a lower lump-sum transfer - either meaning that jurisdiction 1's transfers to region 2 is smaller or that region 1 receives more in a lump-sum manner from region 2.

Optimal allocations and welfare are increasing in the initial endowment of nature in both regions. The more region i's households prefer consumption vis- \hat{a} -vis nature (θ_i) the more they consume (c_i) , while nature quality enjoyed in jurisdiction i is decreasing in θ_i . Preferences of region j's households (θ_j) do not affect optimal allocations - and thus welfare - of jurisdiction i's households. The effects of the degrees of environmental damage spillovers β_i and β_j on consumption c_i are unclear and depend on the signs of $(N_i - \beta_j N_j)$ and $(N_i \beta_i - N_j)$, respectively. The environmental quality households in jurisdiction i enjoy (n_i) is affected by β_i and β_j in the same way as the pollution tax p_i^* paid by them. That is, n_i is increasing on β_i , the environmental damage its production inflicts on region j's households, and it is decreasing on the negative externality

generated by its neighbor (β_j) . The intuition for this result is as follows. If β_i goes up it increases the pollution tax levied on jurisdiction i's households, discouraging production in that region. By producing less, region i creates less pollution and harm its own nature less. The net effect is an increase of environmental quality enjoyed by jurisdiction i's households (n_i) . Regarding the effect of β_j , the same intuition applies but in opposite direction.

These results can be verified by the following expressions:

$$\frac{\partial t^*}{\partial \beta_1} = -\frac{2(1-\beta_2)(N_1+N_2\beta_1)}{4(1-\beta_1\beta_2)^2} < 0 \qquad \frac{\partial c_i^*}{\partial N_i} = +\frac{\theta_i(1-\beta_i)}{2(1-\beta_i\beta_j)} > 0 \qquad \qquad \frac{\partial n_i^*}{\partial N_i} = +\frac{(1-\theta_i)(1-\beta_i)}{2(1-\beta_i)} > 0$$

$$\frac{\partial t^*}{\partial \beta_2} = +\frac{2(1-\beta_2)(N_1+N_2\beta_1)}{4(1-\beta_1\beta_2)^2} > 0 \qquad \frac{\partial c_i^*}{\partial N_j} = +\frac{\theta_i(1-\beta_j)}{2(1-\beta_i\beta_j)} > 0 \qquad \qquad \frac{\partial n_i^*}{\partial N_j} = +\frac{(1-\theta_i)(1-\beta_j)}{2(1-\beta_i)} > 0$$

$$\frac{\partial t^*}{\partial N_1} = +(1-\beta_1) > 0 \qquad \qquad \frac{\partial c_i^*}{\partial \theta_i} = +\frac{\left(N_i(1-\beta_i)+N_j(1-\beta_j)\right)}{2(1-\beta_i\beta_j)} > 0 \qquad \qquad \frac{\partial n_i^*}{\partial \theta_i} = -\frac{\left(N_i(1-\beta_i)+N_j(1-\beta_j)\right)}{2(1-\beta_i)} < 0$$

$$\frac{\partial t^*}{\partial N_2} = -(1-\beta_2) < 0 \qquad \qquad \frac{\partial c_i^*}{\partial \theta_j} = 0 \qquad \qquad \frac{\partial n_i^*}{\partial \theta_j} = 0$$

$$\frac{\partial n_i^*}{\partial \theta_j} = 0 \qquad \qquad \frac{\partial n_i^*}{\partial \theta_j} = 0$$

$$\frac{\partial n_i^*}{\partial \theta_j} = 0 \qquad \qquad \frac{\partial n_i^*}{\partial \theta_j} = 0$$

$$\frac{\partial n_i^*}{\partial \theta_j} = -\frac{(1-\theta_i)N_j(1-\beta_j)}{2(1-\beta_i)^2} > 0$$