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Pollution Permit Sharing Games

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ABSTRACT: We consider a pollution permit sharing problem in which a finite number of countries, each identified by a unique technology that transforms the pollution permits into an output, share a given amount of permits. We define a Pollution Permit Sharing (PPS) game that assigns to each coalition of countries the maximal value of output they can generate collectively with their technologies and permits available to them. We show that the game is totally balanced. We also show that, for the well-known *Cap and Trade* (CAT) mechanism, namely the *competitive equilibrium allocation* which generates an efficient allocation of the permits, its corresponding (net) output distribution is in the *core* of the PPS game.

We consider two other coalitional games whose definitions depend on the availability of either the total permits or the technologies. The Aspiration Upper Bound with given Permits (AUBP) game assigns to each coalition the maximal value they can generate by using the technologies available from all countries, but only with the permits available to the coalition. And the Aspiration Upper Bound with given Technologies (AUBT) game assigns to each coalition the maximal value they can generate using only the technologies available to the coalition with the permits available from all countries. We show that the core of the AUBP game is nonempty. More importantly, we show that the competitive equilibrium allocation violates the above two aspiration upper bound restrictions. Finally, we suggest the Shapley value as one of the possible alternative solutions to the permit sharing problem.

JEL classification: C71, D60, Q20

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1 Introduction

The problem of climate change has emerged as one of the most urgent issues the world is facing. There have been international efforts to resolve the problem. Extending the United Nations Framework Convention on Climate Change (UNFCCC), many countries participating in the Kyoto Protocol voluntarily set the limits on the maximum amount of the green house gases (GHGs) they emit. The cap or the permits on the GHGs emission creates a unique allocation problem.

We describe the problem of climate change in a simple model of sharing pollution permits: Each country has a given level of permits and a technology that transforms the permits into an output.[1]

The technology is assumed to be represented by a concave function: the output is increasing with decreasing rate with respect to the input. Countries may pool their permits and their technologies to produce the joint outputs to be shared among the participants. We ask how these countries can take advantage of these possibilities of joint production and determine the allocation of the joint outputs. More importantly, we ask what interesting restrictions these possibilities of joint production impose on the solutions of the problem.

We define first the so-called *Pollution Permit Sharing* (PPS) game, where the value of each coalition is the maximal output the countries in the coalition can jointly achieve by using their own technologies and their own permits. We show that the PPS game is totally balanced (**Theorem 1**).

One well-known solution to the permit sharing problem has stood out in both theory and practice. The Cap and Trade (CAT) solution or the competitive equilibrium allocation has received wide support among economists and policy makers. The competitive equilibrium allocation is achieved through a decentralized process of economic decision making and provides an efficient outcome that improves the participants' welfare compared to the initial allocation.[2] We show that the competitive equilibrium allocation is in the

[1]Note that we treat the permit as an input for the production of output. And the level of permit is the only input.

[2]Moulin in his book "Cooperative Microeconomics" calls for three modes of the cooperation between selfish economic agents: *direct agreements*, *justice*, and *decentralized behavior*. Hence the competitive allocation meets two of the three modes required in Moulin (1995): *direct agreements* and *decentralized behavior*.

core of the PPS game.[3]

Although the competitive equilibrium allocation is efficient, and it meets two of the three modes required in Moulin (1995): *direct agreements* and *decentralized behavior*, its justice or fairness based on certain normative criteria hasn't been questioned formally in the context of pollution permit sharing as far as we are aware of.[4] We introduce upper bound restrictions that are naturally expected for a solution. For example, a country may be required to receive no more than the level of output it can produce by utilizing the total combined permits available to all participants with its own technology. The competitive equilibrium allocation violates this requirement.

Incorporating the idea of upper bound requirements, we may naturally introduce two new games to the problem. Consider the maximum level of output a country can produce with its own technology if it is allowed to utilize the total permits distributed to all participating countries. Call it the country's aspired level of output. Since the aggregate amount of pollution permits is fixed and shared among the participating countries, the presence of others should impose a negative externality on the share of output the country would receive. Then it is natural to require that everyone shares the cost imposed by the negative externality due to the entitlements possessed by others (Moulin, 1991). In other words, the aspiration level of output provides an upper bound a country should receive. Similarly we can extend the idea of the aspiration level for an individual country to coalitions of countries and accordingly define a game called the *Aspiration Upper Bounds with given Technologies* (AUBT) game.[5]

Alternatively, we can define the *Aspiration Upper Bounds with given Permits* (AUBP) game. Suppose that a country or a coalition of countries can use *all* the available technologies of all countries with the permits assigned

[3] Shapley and Shubik (1969) consider a market game of a pure exchange economy and show that the competitive equilibrium utility vector is in the core of the game. However, in their market game model, it is assumed that all agents' utilities are transferrable and can be added up. In our model, since all countries produce the same homogenous good and use a homogenous input, the transferability of the payoffs comes naturally, compared to the transferability of utility as in Shapley and Shubik's.

[4] There has been criticism of the competitive equilibrium mechanism with regard to its manipulability via technology in a production economy (Suh, 2001) as well as via endowments in an exchange economy (Postlewaite, 1979).

[5] Ambec and Sprumont (2002) consider a similar aspiration upper bound game in a river sharing problem.

to the country or the coalition. The maximal output the country or the coalition can generate this way provides an upper bound that the country or countries can obtain. Apparently, the AUBP game is quite different from the AUBT game.

We require that a permit sharing solution should meet the above two upper bound restrictions. This requirement rules out the competitive equilibrium allocation. This is shown in **Theorems 2** and **3**, respectively.

Our paper is related to a handful of papers in the literature, for example, from the early Shapley and Shubik (1969) to the recent Bahel and Trudeau (2014). Shapley and Shubik (1969)'s market game is defined only for an exchange economy where agents' utilities are transferable. They show that the competitive equilibrium allocation (payoffs) is in the core of the game. In the competitive equilibrium allocation, an agent with zero amount of endowment could only end up with zero utility but not positive. In contrast, in our model, a country with zero amount of permits but with a productive technology can obtain a positive reward from the cooperative production. Bahel and Trudeau (2014) consider a discrete cost sharing model with multiple technologies. They define a production allocation game (PAG) that associates each coalition with the minimal costs of producing the aggregate demand of the coalition. They show that the core of the PAG is nonempty. However, our PPS game is defined as the maximal total output a coalition of countries can jointly achieve using their technologies and their aggregate permits. We also assume that our technologies are represented by strictly concave, strictly increasing, and differentiable functions with divisible inputs.

The rest of the paper is organized as follows. In Section 2, we introduce the Pollution Permit Sharing (PPS) game and show that it is totally balanced in **Theorem 1**. In Section 3, we define the Aspiration Upper Bound with given Permits (AUBP) game and the Aspiration Upper Bound with given Technologies (AUBT) game, respectively. And our main results in Section 3 show that the the competitive equilibrium allocation is not in the core of either of the two games in **Theorems 2** and **3**. In Section 4, we introduce the Shapley value as a solution concept for the problem and provide an example that compares the Shapley value and the competitive equilibrium allocation. In Section 5, we conclude the paper.

2 The model

Let $N = \{1, 2, ..., n\}$ be the set of a finite number of countries. Each country, $i \in N$, is endowed with an initial amount of pollution permit $\overline{x}_i \in R_+$. The total amount is thus $\sum_{i \in N} \overline{x}_i$. Each country i has a production function that transforms permit (as input) into the output of a homogeneous good, $f_i : R_+ \to R_+$. Throughout the paper, we assume that for each $i = 1, ..., n$, f_i is strictly increasing, strictly concave, and differentiable, with $f_i(0) = 0$ and $f_i'(0) = +\infty$. We also speak interchangeably between countries and agents in the paper.

It is assumed that the permits do not have intrinsic value to a country. And the welfare or the utility of a country is determined only by the level of output share the country receives. Hence we do not introduce the utility function of a country in this model.

We define a coalitional form game based on the total level of output produced by a coalition of countries and call it the *Pollution Permit Sharing* (PPS) game.

Definition 1 *The Pollution Permit Sharing (PPS) game: For each $S \subseteq N$, define*

$$\begin{aligned} P(S) &= \max \sum_{i \in S} f_i(x_i) \qquad (1) \\ & s.t. \ \sum_{i \in S} x_i = \sum_{i \in S} \overline{x}_i. \end{aligned}$$

An *allocation* for a game P is a vector $y \in R_+^n$ such that

$$y_1 + \cdots + y_n = P(N).$$

Given a game P, an allocation, $y = (y_1, ..., y_n)$, is called a *core* allocation if

$$\sum_{i \in S} y_i \geq P(S), \forall S \subseteq N.$$

The set of all core allocations is called the core of the game.

A game P is called convex if, for any $i \in N$, and $S \subset T \subseteq N$ and, $i \notin T$,

$$P(S \cup i) - P(S) \leq P(T \cup i) - P(T).$$

In general, the PPS game defined in (1) is not convex. This is shown by the following example.

Example 1. Let $N = \{1, 2, 3\}$. Let

$$f_1(x_1) = \sqrt{x_1}, \quad \overline{x}_1 = 1,$$
$$f_2(x_1) = \sqrt{x_2}, \quad \overline{x}_2 = 1,$$
$$f_3(x_3) = \frac{1}{2}\sqrt{x_3}, \quad \overline{x}_3 = 0.$$

It is easy to check that

$$P(1) = 1, P(2) = 1, P(3) = 0,$$
$$P(1,2) = 2, P(1,3) = \frac{\sqrt{5}}{2}, P(2,3) = \frac{\sqrt{5}}{2},$$
$$P(1,2,3) = \frac{\sqrt{18}}{2}.$$

Since

$$P(1,3) - P(3) > P(1,2,3) - P(2,3),$$

this shows that the game is not convex.

In the special case that all agents have the same technologies and the same initial amounts of permits, we can show that the PPS game is convex. Formally, we have

Proposition 1 *If all agents have the same technologies, $f_i = f, i = 1, ..., n$ and the same initial amounts $\overline{x}_i = \overline{x}/n, i = 1, ..., n$, then the game is convex.*

Proof. Under the assumption,

$$P(S) = |S| f(\frac{\overline{x}}{n}).$$

It is obvious that P is a convex game. Q.E.D.

The following example shows that when agents have different initial amounts, the game may not be convex even they all have the same technologies .

Example 2. Assume $f_i(x) = f(x) = \sqrt{x}, i = 1, 2, 3$. Assume $\overline{x}_1 = 1, \overline{x}_i = 0, i = 2, 3$. It is easy to show that

$$P(1) = 1, P(2) = 0, P(3) = 0,$$

$$P(1,2) = \sqrt{2}, P(1,3) = \sqrt{2}, P(2,3) = 0,$$

$$P(1,2,3) = \sqrt{3}.$$

Since

$$P(1,2,3) - P(1,3) < P(1,2) - P(1),$$

the game is not convex.

Note that, although both games are not convex in the two previous examples, it can be shown that both games have a nonempty core. Now we ask whether or not any permit sharing game has a nonempty core. The answer is positive. Moreover, we have the following theorem.

Theorem 1 *The PPS game defined in (1) is totally balanced.*

Proof. Let p^* be the equilibrium price of the permits.[6] Thus, for each $i = 1, ..., n$, $x_i(p^*)$ solves

$$\max f_i(x_i) - p^*(x_i - \overline{x}_i),$$

and

$$\sum_{i=1}^{n} x_i(p^*) = \sum_{i=1}^{n} \overline{x}_i.$$

Now we show that $y = (y_1, ..., y_n)$, where

$$y_i = f_i(x_i(p^*)) - p^*(x_i(p^*) - \overline{x}_i),$$

is a core allocation.

First, consider the function

$$F(x_S) = \sum_{i \in S} f_i(x_i).$$

[6] We omit the straightforward proof for the existence of the equilibrium price p^*. Also note that we normalize the output price to be 1 in the paper.

It is easy to see that $F(x_S)$ is a concave function of x_S since each $f_i, i \in S$ is concave. Therefore,

$$F(x_S) \le F(x_S^*) + \nabla F(x_S^*)^T (x_S - x_S^*), \forall x_S, x_S^*.$$

Note that if x^* is the equilibrium allocation of the permits, we must have

$$\nabla F(x_S^*) = (f_i'(x_i(p^*)))_S = (p^*)_S.$$

Therefore,

$$\begin{aligned}
F(x_S^*) + \nabla F(x_S^*)^T (x_S - x_S^*) &= \sum_{i \in S} f_i(x_i(p^*)) + (p^*, ..., p^*)_S (x_S - x_S(p^*)) \\
&= \sum_{i \in S} f_i(x_i(p^*)) + p^*(\sum_{i \in S} x_i - \sum_{i \in S} x_i(p^*)) \\
&= \sum_{i \in S} f_i(x_i(p^*)) - p^*(\sum_{i \in S} x_i(p^*) - \sum_{i \in S} x_i) \\
&\ge F(x_S), \forall x_S.
\end{aligned}$$

However, for any x_S such that

$$\sum_{i \in S} x_i = \sum_{i \in S} \overline{x}_i,$$

we have

$$\begin{aligned}
\sum_{i \in S} f_i(x_i(p^*)) - p^*(\sum_{i \in S} x_i(p^*) - \sum_{i \in S} x_i) &= \sum_{i \in S} f_i(x_i(p^*)) - p^*(\sum_{i \in S} x_i(p^*) - \sum_{i \in S} \overline{x}_i) \\
&\ge F(x_S), \forall x_S \text{ s.t.} \sum_{i \in S} x_i = \sum_{i \in S} \overline{x}_i.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i \in S} y_i &= \sum_{i \in S} f_i(x_i(p^*)) - p^*(\sum_{i \in S} x_i(p^*) - \sum_{i \in S} \overline{x}_i) \\
&\ge P(S).
\end{aligned}$$

This proves that y is a core allocation. That the game is totally balanced follows naturally.

This completes the proof of the theorem. Q.E.D.

In a (pure) exchange economy, an agent with zero initial endowment could only obtain zero utility. In our model, a country equipped with a technology but with no permits can share its technology and may also be rewarded with a positive share of the joint outputs. Such an outcome is naturally expected in the cooperative game model of joint production compared to the market game with only consumption goods. To demonstrate this point, we introduce the *strict core* concept, where each coalition receives a strictly positive amount of output share.

Definition 2 *(Strict Core) An allocation, y, is called a strict core allocation if*

$$\sum_{i \in S} y_i > P(S), \forall S \subset N, S \neq N.$$

Proposition 2 *For the pollution permit game with identical technologies (i.e., $f_i = f, i = 1, ..., n$) and with the initial allocation of the permits, $\overline{x}_1 = \overline{x}, \overline{x}_i = 0, i = 2, ..., n$, the game has a strict core allocation.*

Proof. Let

$$y_1 = nf(\frac{\overline{x}}{n}) - \epsilon,$$

$$y_2 = \cdots = y_n = \frac{\epsilon}{n-1},$$

where

$$\epsilon = \frac{nf(\frac{\overline{x}}{n}) - (n-1)f(\frac{\overline{x}}{n-1})}{n}.$$

Now we show that y is a strict core allocation. For any $S \neq N$, $S \subset N$ and $1 \in S$, we have

$$\begin{aligned}
\sum_{i \in S} y_i &= nf(\frac{\overline{x}}{n}) - \epsilon + (|S| - 1)\frac{\epsilon}{n-1} \\
&= nf(\frac{\overline{x}}{n}) - [1 - \frac{(|S|-1)}{n-1}]\frac{nf(\frac{\overline{x}}{n}) - (n-1)f(\frac{\overline{x}}{n-1})}{n} \\
&= nf(\frac{\overline{x}}{n})[1 - \frac{(n-|S|)}{n(n-1)}] + [\frac{(n-|S|)}{n(n-1)}](n-1)f(\frac{\overline{x}}{n-1})
\end{aligned}$$

$$
\begin{aligned}
&> (n-1)f(\frac{\overline{x}}{n-1})[1-\frac{(n-|S|)}{n(n-1)}]+[\frac{(n-|S|)}{n(n-1)}](n-1)f(\frac{\overline{x}}{n-1}) \\
&= (n-1)f(\frac{\overline{x}}{n-1}) \\
&\geq |S|f(\frac{\overline{x}}{|S|}) \\
&= P(S),
\end{aligned}
$$

and for any $S \neq N$, $S \subset N$ and $1 \notin S$, we have

$$\sum_{i \in S} y_i = |S|\epsilon > 0.$$

This proves that y is a strict core allocation. This completes the proof of the proposition. Q.E.D.

3 The Aspiration Upper Bound Games

In a resource allocation problem of an economy, it is natural to expect that the welfare an agent or a group of agents could receive is limited by the resources available to the whole economy. We may require a solution to respect such a limit or an upper bound for the share of a coalition that naturally arises by what the whole economy can offer. This normative requirement could play a crucial role in characterizing a solution. In Ambec and Sprumont (2002), they define an aspiration upper bound game for a river sharing problem and use it together with another lower bound game to uniquely identify a welfare allocation method which corresponds (uniquely) to a resource allocation.

In this section, we introduce two aspiration upper bound games for our problem. The first is called the Aspiration Upper Bound with given Permits game (AUBP). The second is called the Aspiration Upper Bound with given Technologies game (AUBT). The AUBP game assigns to each coalition of countries the maximal value they can generate by using the technologies of *all* countries but only with the given permits owned by the countries in the coalition. The AUBT game, on the other hand, assigns to each coalition of countries the maximal value they can generate by using the permits of *all* countries but only with the given technologies owned by the countries in the coalition. We will examine the implications of these two games along with the PPS game for the permit sharing problem.

We begin with the Aspiration Upper Bound with given Permits game.

Definition 3 *The Aspiration Upper Bound with given Permits (AUBP) game: For each $S \subseteq N$, define*

$$\begin{aligned} \overline{P}(S) &= \max \sum_{i=1}^{n} f_i(x_i) \\ s.t. & \sum_{i=1}^{n} x_i = \sum_{i \in S} \overline{x}_i. \end{aligned} \tag{2}$$

A game is concave if, for any $i \in N$, and $S \subset T \subseteq N$ and, $i \notin T$,

$$\overline{P}(S \cup i) - \overline{P}(S) \geq \overline{P}(T \cup i) - \overline{P}(T).$$

The proposition below shows that the AUBP game is concave.

Proposition 3 *The Aspiration Upper Bounds with given Permits game defined in (2) is concave.*

Proof. Consider the function

$$F(x_1, ..., x_n) = \sum_{i=1}^{n} f_i(x_i).$$

Since $f_i, i = 1, ...n$ are concave, thus F is concave.

For any $t \geq 0$, define

$$\begin{aligned} \overline{P}(t) &= \max F(x_1, ..., x_n) \\ & \sum_{i=1}^{n} x_i = t. \end{aligned}$$

It suffices to prove that $\overline{P}$ is a concave function of t.[7]

[7]Under the assumption on the production functions of the agents, it can be shown that $\overline{P}(t)$, as a function of t, is strictly increasing and continuously differentiable. That function $\overline{P}(t)$ is concave, to be proved here, implies that the game $\overline{P}$ is concave.

Let $t_1 \geq 0, t_2 \geq 0$, and $\lambda \in [0,1]$. Then

$$\overline{P}(\lambda t_1 + (1-\lambda)t_2) = \max F(x_1, ..., x_n) \quad \sum_{i=1}^{n} x_i = \lambda t_1 + (1-\lambda)t_2. \tag{3}$$

Consider the following two problems

$$\overline{P}(t_1) = \max F(x_1, ..., x_n) \quad \sum_{i=1}^{n} x_i = t_1,$$

and

$$\overline{P}(t_2) = \max F(x_1, ..., x_n) \quad \sum_{i=1}^{n} x_i = t_2.$$

Denote $x^1 = (x_1^1, ..., x_n^1)$ and $x^2 = (x_1^2, ..., x_n^2)$ the two solutions of the above two problems, respectively. Then,

$$\lambda x^1 + (1-\lambda)x^2$$

is a feasible solution for the problem (3). Therefore,

$$\begin{aligned} \overline{P}(\lambda t_1 + (1-\lambda)t_2) &\geq F(\lambda x^1 + (1-\lambda)x^2) \\ &\geq \lambda F(x^1) + (1-\lambda)F(x^2) \\ &= \lambda\overline{P}(t_1) + (1-\lambda)\overline{P}(t_2). \end{aligned}$$

This proves the concavity of $\overline{P}$. It follows that the game $\overline{P}$ is concave.

This completes the proof. Q.E.D.

The core of the AUBP game is now defined as the set of allocations $y = (y_1, ..., y_n)$ such that

$$\sum_{i \in S} y_i \leq \overline{P}(S), \ \forall S \subset N, \text{ and } \sum_{i \in N} y_i = \overline{P}(N). \tag{4}$$

Since any concave game has a nonempty core, thus the core of the AUBP game is nonempty. Moreover, it is totally balanced.

However, the competitive equilibrium allocation, which is in the core of the game P, is in general not in the core of the AUBP game $\overline{P}$.

Theorem 2 *In general, the competitive equilibrium allocation is not in the core of the AUBP game*

Proof. We show the theorem by the following counterexample. Suppose that $\overline{x}_1 = 0$ and all other countries have positive amounts of initial permits. Then, since

$$\begin{aligned} \overline{P}(\{1\}) &= \max \sum_{i=1}^{n} f_i(x_i) \\ & \sum_{i=1}^{n} x_i = 0, \end{aligned}$$

we have $\overline{P}(\{1\}) = 0$. But apparently, in the competitive equilibrium allocation, agent 1 obtains a positive value

$$y_1 = f_1(x_1^*) - p^*(x_1^* - 0) > f_1(0) = 0 = \overline{P}(\{1\}).$$

This shows that the competitive equilibrium allocation violates the Aspiration Upper Bounds. This proves the theorem. Q.E.D.

The theorem says that, in general the competitive equilibrium allocation cannot be in both the core of the PPS game P and the core of the AUBP game $\overline{P}$.

Now we ask if there are other allocations in the core of P that are also in the core of $\overline{P}$. The answer, in general, is negative. This is shown by the following example.

Example 3. Let $N = \{1, 2, 3\}$. Let

$$f_1(x_1) = \sqrt{x_1}, \overline{x}_1 = 1,$$

$$f_2(x_2) = 2\sqrt{x_2}, \overline{x}_2 = 0,$$

$$f_3(x_3) = \sqrt{x_3}, \overline{x}_3 = 1.$$

Then, it is easy to calculate that

$$P(1) = 1, P(2) = 0, P(3) = 1,$$

$$P(1,2) = \sqrt{5}, P(1,3) = 2, P(2,3) = \sqrt{5},$$
$$P(1,2,3) = 2\sqrt{3}.$$

For any core allocation $y = (y_1, y_2, y_3)$, since $y_1 + y_2 \geq \sqrt{5}$ and $y_2 + y_3 \geq \sqrt{5}$ we must have

$$y_3 \leq 2\sqrt{3} - \sqrt{5}, y_1 \leq 2\sqrt{3} - \sqrt{5}.$$

Therefore,

$$y_2 \geq 2\sqrt{3} - 2(2\sqrt{3} - \sqrt{5}) = 2\sqrt{5} - 2\sqrt{3} > 0.$$

On the other hand, if y is also in the core of the AUBP game $\overline{P}$, we must have

$$y_2 \leq \overline{P}(2) = 0 = \max f_1(x_1) + f_2(x_2) + f_3(x_3) \text{ s.t. } x_1 + x_2 + x_3 = 0,$$

a contradiction.

However, if all countries have the same technologies and same initial amounts of permits, then the PPS game P would be convex. Since the AUBP game $\overline{P}$ is concave, hence we can apply the so-called *sandwich theorem* (see Martínez-de-Albéniz and Rafels, 2004) to show that the cores of the above two games have a nonempty intersection.

Proposition 4 *If all countries have the same technologies and the same amounts of initial permits, then there exist allocations $y \in R^n_+$ that are in both the core of the Pollution Permit Sharing game P and the core of the Aspiration Upper Bound with given Permits game $\overline{P}$, i.e., there are $y = (y_1, ..., y_n)$ such that*

$$\overline{P}(S) \geq \sum_{i \in S} y_i \geq P(S), \ \forall S \subset N,$$

$$\overline{P}(N) = \sum_{i \in N} y_i = P(N).$$

Another sufficient condition for the nonempty intersection of the cores of the two games is obtained when all countries have the identical technologies and one country owns all the permits. This is shown in the next proposition.

Proposition 5 *For the pollution permit game with identical technologies (i.e., $f_i = f, i = 1, ..., n$), and with the initial allocation of the permits, $\overline{x}_1 = \overline{x}, \overline{x}_i = 0, i = 2, ..., n$, the core of the PPS game and the core of the AUBP game have a nonempty intersection.*

Proof. Let

$$y_1 = nf(\frac{\overline{x}}{n}), y_i = 0, i = 2, ..., n.$$

Now we check that $y = (y_1, 0, ..., 0)$ is in both the core of the PPS game P and the core of the AUBP game $\overline{P}$.

For any $S \subset N$, and if $1 \in S$, we have

$$P(S) \leq \sum_{i \in S} y_i \leq \overline{P}(S) = nf(\frac{\overline{x}}{n}),$$

and if $1 \notin S$, we have

$$0 = P(S) = \sum_{i \in S} y_i \leq \overline{P}(S) = 0.$$

This proves the proposition. Q.E.D.

Now, we turn to the Aspiration Upper Bounds with given Technologies (AUBT) game.

Definition 4 *The Aspiration Upper Bound with given Technologies (AUBT) game: For each $S \subseteq N$, define*

$$\begin{aligned} \widehat{P}(S) &= \max \sum_{i \in S} f_i(x_i) \\ s.t. & \sum_{i \in S} x_i = \sum_{i=1}^{n} \overline{x}_i. \end{aligned} \tag{5}$$

In this game, each coalition is allowed to use all the permits available in the economy but is restricted to using only their own technologies. The core of the AUBT game is similarly defined as the definition of the core of the AUBP game.

It is easy to see that the AUBT game is in general not convex.[8] This is shown by the following example.

Example 4. Let $N = \{1, 2, 3\}$. Let

$$f_1(x_1) = \sqrt{x_1}, \quad \overline{x}_1 = 1,$$

$$f_2(x_1) = \sqrt{x_2}, \quad \overline{x}_2 = 1,$$

$$f_3(x_3) = \frac{1}{2}\sqrt{x_3}, \quad \overline{x}_3 = 0.$$

It is easy to show that

$$\widehat{P}(1) = \sqrt{2}, \widehat{P}(2) = \sqrt{2}, \widehat{P}(3) = \frac{1}{2}\sqrt{2},$$

$$\widehat{P}(1,2) = 2, \widehat{P}(1,3) = \sqrt{\frac{5}{2}}, \widehat{P}(2,3) = \sqrt{\frac{5}{2}},$$

$$\widehat{P}(1,2,3) = \frac{\sqrt{18}}{2}.$$

Since

$$\widehat{P}(1,3) - \widehat{P}(3) > \widehat{P}(1,2,3) - \widehat{P}(2,3),$$

this shows that the game is not convex.

Now we show that the competitive equilibrium allocation also violates the upper bound restrictions in the AUBT game.

Theorem 3 *In general, the competitive equilibrium allocation is not in the core of the Aspiration Upper Bound with given Technologies game defined in (5) when the latter is nonempty.*

Proof. The theorem is proved by the following example.

Consider $N = \{1, 2\}$ and

$$f_1(x_1) = \frac{1}{10}\sqrt{x_1}, \quad \overline{x}_1 = 2,$$

[8]It is an open question whether or not the AUBT game is concave. In fact, it is unclear whether or not the game is balanced.

$$f_2(x_1) = \sqrt{x_2}, \quad \overline{x}_2 = 1.$$

It is easy to calculate that

$$x_1^* = \frac{3}{101}, x_2^* = \frac{300}{101}, p^* = \frac{1}{2\sqrt{\frac{300}{101}}},$$

and thus

$$v_1 = f_1(x_1^*) - p^*(x_1^* - \overline{x}_1) = 0.5888 > \widehat{P}(1) = \frac{1}{10}\sqrt{3},$$

violating the upper bound for agent 1.

It is easy to check that the game $\widehat{P}$ has a nonempty core. This completes the proof. Q.E.D.

4 The Shapley Value: an Example

We have considered the competitive equilibrium allocation in the permit sharing problem. And we have introduced certain properties expected in the model and evaluated the allocation according to the properties. It would be interesting to consider other game-theoretic solutions to the problem. One example of such solutions concept we think interesting in the permit sharing problem is the Shapley value. We provide the definitions of the Shapley value that correspond to the three different games introduced above. And we provide an example of a comparison of the three Shapley values together with the competitive equilibrium allocation.

Definition 5 *For the pollution permit sharing problem, the Shapley values of the PPS game, the $AUBP$ game, and the $AUBT$ game, are defined below, respectively,*

$$y_i^{PPS} = \sum_{S \subseteq N-i} \frac{|S|!(|N|-|S|-1)!}{|N|!}[P(S \cup \{i\}) - P(S)], i = 1, ..., n, \tag{6}$$

where $P(S)$ is defined in (1),

$$y_i^{AUBP} = \sum_{S\subseteq N-i} \frac{|S|!(|N|-|S|-1)!}{|N|!}[\overline{P}(S\cup\{i\}) - \overline{P}(S)], i = 1, ..., n, \quad (7)$$

where $\overline{P}(S)$ is defined in(2), and

$$y_i^{AUBT} = \sum_{S\subseteq N-i} \frac{|S|!(|N|-|S|-1)!}{|N|!}[\widehat{P}(S\cup\{i\}) - \widehat{P}(S)], i = 1, ..., n, \quad (8)$$

where $\widehat{P}(S)$ is defined in (5).

To compare the above three solutions along with the competitive equilibrium allocation, consider the following example. Let $N = \{1, 2\}$. Let

$$f_1(x_1) = \sqrt{x_1}, \qquad \overline{x}_1 = 1,$$
$$f_2(x_2) = \frac{1}{2}\sqrt{x_2}, \qquad \overline{x}_2 = 2.$$

It is easy to show that

$$P(1) = 1; P(2) = \frac{\sqrt{2}}{2}; P(1,2) = \sqrt{\frac{15}{4}}.$$

$$\overline{P}(1) = \frac{\sqrt{5}}{2}; \overline{P}(2) = \sqrt{\frac{5}{2}}; \overline{P}(1,2) = \sqrt{\frac{15}{4}}.$$

$$\widehat{P}(1) = \sqrt{3}; \widehat{P}(2) = \frac{\sqrt{3}}{2}; \widehat{P}(1,2) = \sqrt{\frac{15}{4}}.$$

The Shapley value of the PPS game is the following

$$y_1^{PRS} = \frac{1}{2} + \frac{1}{2}(\sqrt{\frac{15}{4}} - \frac{\sqrt{2}}{2}) \approx 1.1147,$$

$$y_2^{PRS} = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}(\sqrt{\frac{15}{4}} - 1) \approx 0.8218.$$

The Shapley value of the AUBP game is the following

$$y_1^{AUBP} = \frac{1}{2}\frac{\sqrt{5}}{2} + \frac{1}{2}(\sqrt{\frac{15}{4}} - \sqrt{\frac{5}{2}}) \approx 0.7367,$$

$$y_2^{AUBP} = \frac{1}{2}\sqrt{\frac{5}{2}} + \frac{1}{2}(\sqrt{\frac{15}{4}} - \frac{\sqrt{5}}{2}) \approx 1.1998.$$

The Shapley value of the AUBT game is the following

$$y_1^{AUBT} = \frac{1}{2}\sqrt{3} + \frac{1}{2}(\sqrt{\frac{15}{4}} - \frac{\sqrt{3}}{2}) \approx 1.4013,$$

$$y_2^{AUBT} = \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}(\sqrt{\frac{15}{4}} - \sqrt{3}) \approx 0.5352.$$

On the other hand, the competitive equilibrium allocation gives the following

$$v_1 = f_1(x_1) - p^*(x_1 - \overline{x}_1) = \sqrt{\frac{12}{5}} - \frac{1}{4\sqrt{\frac{3}{5}}}(\frac{12}{5} - 1) = 1.0974,$$

$$v_2 = f_2(x_2) - p^*(x_2 - \overline{x}_2) = \frac{1}{2}\sqrt{\frac{3}{5}} - \frac{1}{4\sqrt{\frac{3}{5}}}(\frac{3}{5} - 2) = 0.8391.$$

The reader can see that agents are treated similarly in between the Shapley value of the PPS game and the competitive equilibrium allocation. However, the respective Shapley values between the AUBP game and the AUBT game are diagonally different and the former favors agent 2 while the latter agent 1. But both are very different from the competitive equilibrium allocation.

5 Conclusion

The Cap and Trade allocation or the price solution through the market has been suggested as a main allocation mechanism in reducing the GHGs. To apply the price solution, first the privatization of the common resource is needed or the rights to pollute must be clearly assigned, possibly by voluntary participation or by a certain international negotiation process.

The international negotiations toward reaching such an agreement have been going on for over the last two decades and its progress has been very slow. Although some partial agreements have been reached and the markets for pollution permits have started to operate, the negotiation process of sharing the burden of emission reduction is still ongoing.

Acknowledging the importance and the difficulty of reaching such an agreement over the assignment of pollution permits, we have applied the model of cooperative game theory which does not necessarily or directly require the assignment of the permits among the countries (e.g., in the AUBT game model) and evaluated the price solution in the model.

We have introduced three kinds of games, each of which captures certain aspects of coalition formation possibilities and their implications on the allocations: the PPS game, the AUBP game and the AUBT game. In the PPS game, the production of a coalition is restricted by the technologies and the permits available to the coalition. We have shown that the well-known competitive equilibrium allocation is in the core of the PPS game. And in the AUBP game and the AUBT game, the production of each coalition is confined by either the technologies or the permits available to the coalition. The competitive equilibrium allocation violates two naturally defined aspiration upper bound conditions. Formally speaking, we have shown that the competitive equilibrium allocation is neither in the core of the AUBP game nor in the core of the AUBT game.

It would be worthwhile to find other interesting solutions that satisfy some of the reasonable properties or axioms expected in the permit sharing problem. We propose the Shapley value as an example of alternative solutions to the permit sharing problem. It would be interesting to find out reasonable properties or axioms that are satisfied by the Shapley value solution for each respective game defined in the paper. It would be also interesting to investigate whether the Shapely value solution may be characterized by some of the properties.

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