University of Windsor
Department of Economics Working Paper Series
401 Sunset Avenue
Windsor, Ontario, Canada N9B 3P4
Administrator of Working Paper Series: Christian Trudeau
Contact: trudeauc@uwindsor.ca

From spanning trees to arborescences: new and extended cost sharing solutions

Eric Bahel (Virginia Polytechnic Institute and State University)
Christian Trudeau (University of Windsor)

Working paper 16-01

From spanning trees to arborescences: new and extended cost sharing solutions

Eric Bahel* Christian Trudeau†

January 9, 2016

Abstract

The paper examines minimal cost arborescence problems, which generalize the well-known minimal cost spanning tree (mcst) problems. We propose a new family of cost sharing methods that are easy to compute, as they closely relate to the network-building algorithm. These methods, called minimal incoming cost rules for arborescences (MICRAs), include as a particular case the extension of the folk solution introduced by Dutta and Mishra (2012). A simpler computational procedure thus obtains for this method. We also provide new axiomatizations of (a) the set of stable and symmetric MICRAs and (b) the folk solution. Finally, we closely examine two remarkable MICRAs. The first one relates to the cycle-complete rule for mcst problems introduced in Trudeau (2012). The second one contrasts with the folk rule by fully rewarding agents who help others connect to the source.

Keywords: arborescence problems; stable allocations, minimal incoming cost rules, leftover cost matrix.

JEL Classifications: C71, D63

1 Introduction

The well-studied minimal cost spanning tree (mcst) problem models any situation where (i) a number of agents need to be connected to a source (in order to obtain a good or service) and (ii) the cost of building or using any given edge is independent of its number of users. The problem was first studied from a game-theoretic point of view by Bird (1976).

*Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061-0316, USA. Email: erbahel@vt.edu.

†Department of Economics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, Canada. Email: trudeauc@uwindsor.ca

A natural extension of the mcst problem obtains when one relaxes the assumption that the direction of the flow does not affect the cost: indeed, in a network, connecting an agent i to another agent j typically does not cost the same as connecting j to i. This asymmetry occurs for instance if the cost on an edge is affected by the difference in altitude between the two nodes, or if a node is located upstream/downstream to the other on a river. In a communication network, it may also make sense to assume that the respective costs incurred to receive and send a message are different.

This natural extension is called the minimal cost arborescence (mca) problem. Finding the optimal network configuration in an mca problem can be done in polynomial time by the algorithm of Chu and Liu (1965) and Edmonds (1967). Dutta and Mishra (2012), who introduced mca problems to the economic literature, showed that the core of an mca problem is always nonempty; and they proposed a solution that is always in the core. Their method, which we call the Dutta-Mishra solution, is an extension of the folk solution (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007) for mcst problems. The Dutta-Mishra solution requires the computation, through the mca-building algorithm, of an irreducible cost matrix, such that we cannot reduce the cost of any edge without reducing the cost of the grand coalition. We then take the Shapley value of the characteristic function obtained from this irreducible matrix.

In this paper we extend their work by proposing a decomposition of any mca problem into a series of simple, partially symmetric games. This is achieved through an algorithm that combines nodes into supernodes in order to generate the optimal network configuration. At each stage of the algorithm, we extract the cost of the cheapest incoming edge for every supernode, and we divide this cost equally among the members of the supernode. We then obtain a "leftover" cost matrix, which takes into account the individual bargaining powers within the supernodes. We show that a sufficient condition for our allocation to be in the core of the mca problem is that the cost of the leftover matrix be shared in a stable way; and these cost shares should then be added to those obtained in the previous steps of the decomposition. We call any such rule a *Minimal incoming cost rule for arborescences*, or MICRA.

By construction the grand coalition always has a cost of zero to connect to the source under our leftover matrix; and hence a trivial core allocation obtains by assigning a cost share of zero to all agents, in effect ignoring the leftover matrix. We show that doing this allows one to recover the method proposed by Dutta and Mishra (2012). The advantage of our procedure lies in the fact that it is computationally much simpler, in particular because it does not require any use of the Shapley value. This new method also coincides with the folk solution when applied to an mcst problem, yielding a new interpretation of this well-known rule (albeit one that uses a decomposition of the mcst problem in a series of mca problems). Our approach is similar in spirit to that of the obligation rule (Feltkamp et al., 1994; Tijs et al., 2006; Bergantiños and Kar, 2010).

We also propose two rules that use the leftover matrix to adjust the shares of the agents, while still preserving core selection. A first approach is to apply to that leftover matrix the concept behind the cycle-complete solution for mcst problems (Trudeau, 2012). Suppose that the cost of directly connecting i to j is very high, and that there exist two distinct paths where the most expensive edge is cheaper than the direct connection cost. We modify the cost matrix by replacing the cost of the direct connection of i to j with that of the most expensive edge in this pair of paths. In mcst problems, this modification ensures the concavity of the associated cooperative cost game, which guarantees that the Shapley value (applied to this concave game) will always be in the core. We show that the result extends to mca problems and call the resulting rule the pair-of-paths (pop) rule.

The second method, called the *supplier rule*, is inspired by similar proposals for shortest path problems (Bahel and Trudeau, 2014) and veto games (Bahel, 2016). This new method fully compensates members who provide the best connection in each supernode; and, within the class of stable and symmetric MICRAs, it can therefore be viewed as opposite to the folk rule (which gives the best connection for free).

We also offer a characterization of the class of symmetric and stable rules, the MICRAs, which includes the three distinguished methods mentioned above (folk, pop and supplier rules). The characterization uses two well-known axioms, Core Selection and Symmetry, and two new ones (which are decribed next). The axiom Decomposition allows to allocate the cost of the Dutta-Mishra irreducible matrix and the cost of the leftover matrix independently. The axiom Minimal Group Connection states that if a group of agents are symmetric and can connect to each other at no cost, then they are jointly responsible for their incoming edges. In particular, the cost of their cheapest incoming edge can be shared first; and then one can focus on the remainder matrix representing the extra cost on other edges. The folk solution is uniquely characterized by requiring (in addition to the previous axioms) the non-negativity of the shares, or by discarding the leftover matrix. Our results generalize the work of Dutta and Mishra (2012) in a number of ways. First, their characterization of the folk solution applies to the restricted domain of simple cost matrices (those for which the algorithm stops at the first step), whereas ours applies to the full domain of cost matrices. Second, since Dutta and Mishra (2012) characterize a specific rule (the folk solution), their axioms are geared towards it. Our axioms are more general and lead to a large family of rules that includes the folk solution as a particular case.

The paper unfolds as follows. Section 2 introduces the model and the algorithm allowing to build the efficient network configuration. Section 3 defines and axiomatizes the family of symmetric and stable MICRAs; Sections 4 and 5 focus on some remarkable MICRAs. In section 4 we study and characterize the Dutta-Mishra solution, an extension of the folk solution for mcst problems. In section 5, we introduce two new rules: the pop rule and the supplier rule (described above). Section 6 provides our concluding comments. All elaborate proofs are available in the appendix (the simpler ones immediately follow their respective theorems).

2 The model

Let $N = \{1, \ldots, n\}$ be the set of agents who need to be connected to the source, which is denoted by 0, and define $N_0 \equiv N \cup \{0\}$. We will assume throughout that $|N| = n \geq 2$. Denote by (i, j) the directed edge from $i \in N_0$ to $j \in N$, and by $c_{ij} \geq 0$ its cost. Let $\mathbb{E} = \{(i, j) \in N_0 \times N |\ i \neq j\}$ be the set of edges. We call *cost matrix* any matrix of the form $c = (c_e)_{e \in \mathbb{E}}$, that is to say, $c \in \mathbb{R}_+^{\mathbb{E}}$. Let Γ be the set of cost matrices. A matrix $c \in \Gamma$ is *undirected* if $c_{ij} = c_{ji}$ for all $i, j \in N$. We use the symbol Γ^u to denote the set of undirected cost matrices (note that $\Gamma^u \subset \Gamma$).

A *cycle* p_{ll} is a set of $K \geq 3$ edges of the form (i_k, i_{k+1}), where $k \in [0, K-1]$, and such that $i_0 = i_K = l$ and $i_1, ..., i_{K-1}$ are distinct and different from l. A path p_{lm} between l and m is a set of $K \geq 2$ edges (i_k, i_{k+1}), with $k \in [0, K-1]$, such that $i_0 = l$, $i_K = m$ and $i_1, ..., i_{K-1}$ are distinct and different from l and m. Let $P_{lm}(N_0)$ be the set of all such paths between l and m. For any set of edges E, $[E]$ is the set of nodes that are adjacent to any edge in E.

An *arborescence* is a directed graph (without cycles) that contains a path from 0 to i, for any $i \in N$. As a graph,[1] an arborescence A is completely characterized by the set of edges it contains; and its associated cost is $c(A) \equiv \sum\limits_{e \in A} c_e$. Arborescences that achieve the minimal possible cost (to connect all agents in N) are called *minimal cost arborescences* (mca). One can find an mca in polynomial time using the algorithm of Chu and Liu (1965) and Edmonds (1967); and it need not be unique. We use the notation $C(N, c)$ to refer to the cost of any mca. Moreover, we write A^* to denote an arbitrary mca, and $\mathcal{A}^*(c)$ to describe the set containing all mca given the matrix c.

A *minimal cost arborescence problem* is a triple $(0, N, c)$, with $c \in \Gamma$. Furthermore, we call *minimal cost spanning tree problem* any triple $(0, N, c)$ such that $c \in \Gamma^u$. Unless otherwise specified, we will identify every mca problem with the corresponding cost matrix c (since 0 and N are fixed).

Definition 1 *A* ***cost sharing rule*** *for mca problems is a mapping* $y : \Gamma \to R^N$ *such that, for any* $c \in \Gamma$*, we have* $\sum\limits_{i \in N} y_i(c) = C(N, c)$.

Note from the above definition that we allow for negative cost shares. Indeed, it makes sense to reward agents who allow a cheaper access to the source for others.

For any coalition $S \subset N$, let $C(S, c)$ denote the minimal cost incurred (under the cost matrix c) to connect the members of S *without using any outside agent* $i \notin S$. We refer to $C(., c)$ as the *stand-alone cost function* associated with c.

Definition 2 *We call* ***core*** *of the mca problem associated with* $c \in \Gamma$ *the set*

$$Core(c) = \left\{ x \in \mathbb{R}^N |\sum_{i \in N} x_i = C(N, c) \text{ and } \sum_{i \in S} x_i \leq C(S, c),\ \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

[1] An example of arborescence is the graph $\{(0, i), i \in N\}$, where all agents are directly connected to the source.

In words, the core is the set of cost allocations satisfying the property that no coalition $S \subset N$ pays a joint share that is higher than its stand-alone cost.

2.1 Building the minimal cost arborescence

In mcst problems, one can build the network using a greedy algorithm where edges are added one by one. Unfortunately, this greedy algorithm typically does not work for general mca problems. For each $i \in N$, let $p(i) \in \arg\min_{j \in N_0} c_{ji}$. We say that a cost matrix $c \in \Gamma$ is *simple* if there exists an mca of the form $\{p(i), i)\}_{i \in N}$. In words, a cost matrix is simple if the greedy algorithm mentioned above (which consists in finding the cheapest incoming edge for each node) yields a well-defined arborescence —which is then an mca.

The network-building algorithm for mca problems, originally proposed by Chu and Liu (1965) and Edmonds (1967) and described in the economics literature by Dutta and Mishra (2012), consists in transforming the problem into a simpler one to which the greedy algorithm applies. For each agent we determine the minimum incoming cost, which we then subtract from the cost of every incoming edge. After this transformation, we look for (maximal) costless cycles, i.e., groups of agents who can connect with one another at zero cost; and these agents are then merged into what we call a *supernode*. We repeat this procedure as long as the transformed cost matrix is not simple.

Given a cost matrix c, we say that the set of agents $I = \{1, ..., K\}$ form a $c-$cycle if $c_{ii+1} = 0$ for all $i = 1, ..., K-1$ and $c_{K1} = 0$. We are now ready to describe the mca-building algorithm (we provide a full description because our cost sharing rules are defined using its outcome).

Algorithm 1 *(Chu and Liu (1965), Edmonds (1967))*
Stage 0: For each $j \in N$, *let* $\delta_j^0 = \min_{i \in N_0} c_{ij}$. *If* c *is a simple matrix, let* $T = 0$ *and terminate. Let* $N^0 \equiv \left\{N_1^0, ..., N_{n+1}^0\right\} \equiv \{0, 1, 2, ..., n\}$.
Stage 1. Define c^1 *such that* $c_{ij}^1 = c_{ij} - \delta_j^0$. *Construct a partition* $\left\{N_1^1, ..., N_{K^1}^1\right\}$ *of* N_0 *such that each* N_k^1 *is either a* c^1-*cycle of elements of* N_0 *or a singleton, with the constraint that no set of singletons form a* c^1*-cycle. By definition,* 0 *is always one of the singletons; and we write* $N_1^1 \equiv \{0\}$. *For* $k, l \in \left\{1, ..., K^1\right\}$ *and* $l \neq 1$, *let*

$$\tilde{c}_{kl}^1 = \min_{i \in N_k^1,\ j \in N_l^1} c_{ij}^1.$$

For each $k = 2, ..., K^1$, *let* $\delta_{N_k^1}^1 = \min_{j \in \{1,...,K^1\} \setminus k} \tilde{c}_{N_j^1 N_k^1}^1$.

If $\tilde{c}^1$ *is a simple matrix, let* $T = 1$ *and terminate.*

Stage t. Define c^t *such that* $c_{N_k^{t-1} N_l^{t-1}}^t = \tilde{c}_{N_k^{t-1} N_l^{t-1}}^{t-1} - \delta_{N_l^{t-1}}^{t-1}$ *for all* $k, l \in \left\{1, ..., K^{t-1}\right\}$. *Construct a partition* $\{N_1^t, ..., N_{K^t}^t\}$ *of* N^{t-1} *such that each* N_k^t *is either a* c^t-*cycle of elements of* N^t *or a singleton, with the constraint that no set of singletons form a* c^t*-cycle. For* $k, l \in \{1, ..., K^t\}$ *and* $l \neq 1$, *let*

$$\tilde{c}_{kl}^t = \min_{i \in N_k^t,\ j \in N_l^t} c_{ij}^t.$$

For each $k = 2, ..., K^t$, let $\delta^t_{N^t_k} = \min_{j \in \{1,...,K^t\} \backslash k} \tilde{c}^t_{N^t_j N^t_k}$. If $\tilde{c}^t$ is a simple matrix, let $T = t$ and terminate. Else, proceed to step $t + 1$.

Observe that the procedure ends in at most n steps, since at each step we either form a c^t-cycle or terminate. At the end, one has to "roll back" the algorithm to obtain the mca associated with the matrix c (as illustrated in the following example).

Example 1 *Figure 1 below describes an mca problem with 3 agents, with the nodes identified in the circles. The number beside each directed edge represents its cost.*

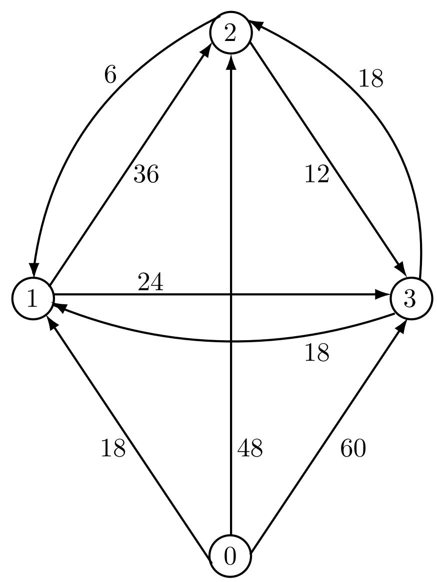

Figure 1: An example of mca problem

At stage 0, we have $\delta^0_1 = 6$, $\delta^0_2 = 18$ and $\delta^0_3 = 12$. Since the matrix is not simple, we proceed to stage 1.

Stage 1: Notice that we have $c^1_{23} = c^1_{32} = 0$, and thus $\{2, 3\}$ forms a c^1-cycle. It follows that $N^1 = \{\{0\}, \{1\}, \{2, 3\}\}$; and one obtains $\tilde{c}^1$ as depicted in Figure 2, and $\delta^1_1 = 0$, $\delta^1_{\{2,3\}} = 12$. The resulting matrix is still not simple; and we proceed to stage 2.

Stage 2: Observe that we have $c^2_{1\{23\}} = c^2_{\{23\}1} = 0$, and thus $\{1, \{2, 3\}\}$ form a c^2-cycle. One can then write $N^2 = \{\{0\}, \{1, 2, 3\}\}$; and $\tilde{c}^2$ is as depicted in Figure 2, with $\delta^2_{\{1,2,3\}} = 12$. Notice that c^2 is a simple matrix; and the procedure thus ends at $T = 2$.

We now proceed backwards to derive the mca. Under the matrix $\tilde{c}^2$, there is a cost of 12 to connect $\{1, 2, 3\}$ to the source, which is obtained by computing the smallest value between $\tilde{c}^1_{01} - \delta^1_1 = 12$ and $\tilde{c}^1_{0\{2,3\}} - \delta^1_{\{2,3\}} = 18$. Therefore, the edge $(0, 1)$ belongs to the mca.

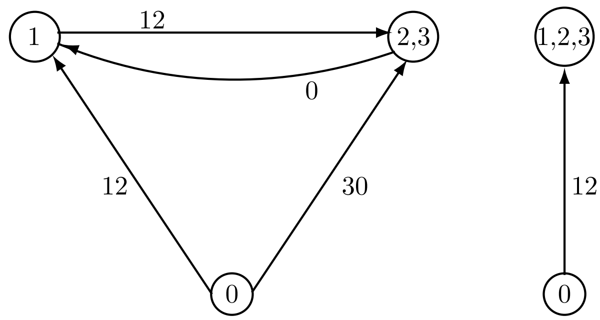

Figure 2: $\tilde{c}^1$ and $\tilde{c}^2$

At the beginning of stage 2, we merged $\{1\}$ *and* $\{2,3\}$ *into a supernode. Given that* $(0,1)$ *belongs to the mca, the edge* $(1,\{2,3\})$ *must belong to the mca of* $\tilde{c}^1$*. Recall that the cost of the edge* $(1,\{2,3\})$ *was obtained as the minimum between* $c_{12} - \delta_2^0 = 18$ *and* $c_{13} - \delta_3^0 = 12$*. Therefore, the edge* $(1,3)$ *belongs to the mca.*

Finally, at the beginning of stage 1, we merged $\{2\}$ *and* $\{3\}$ *into a supernode. Given that* $(0,1)$ *and* $(1,3)$ *already belong to the mca of* c*, it follows that the last edge of the mca of* c *is* $(3,2)$*.*

The lowest possible cost to connect all three agents is thus $c_{01}+c_{13}+c_{32} = 60$*.*

3 Cost sharing: minimal incoming cost rules

Using the algorithm defined in the previous section, we now define a new class of cost sharing rules for mca problems. Let $\bar{N}^t$ be the partition of N_0 that we obtain at Stage t. In other words, for all $i \in N$ and $t = 1, ..., T$, the set $\bar{N}^t(i)$ contains all agents that are in the same supernode as i at stage t. At any stage $t = 1, ..., T$, let the cost matrix $\bar{c}^t \in \Gamma$ be defined as follows: $\forall (i,j) \in E$,

$$\bar{c}^t_{ij} = \begin{cases} 0 & \text{if } i \in \bar{N}^t(j); \\ \delta^t_{\bar{N}^t(j)} & \text{otherwise.} \end{cases} \tag{1}$$

The study of mca problems by Dutta and Mishra (2012) focuses on the search for *irreducible cost matrices.* First defined for mcst problems by Bird (1976), an irreducible cost matrix is one for which it is impossible to reduce the cost of any edge without reducing the minimal cost of connecting the grand coalition.

For mca problems, we may have multiple irreducibles matrices associated with a given matrix. Dutta and Mishra (2012) constructed one irreducible matrix by using a procedure closely related to the algorithm described earlier.

The Dutta-Mishra irreducible matrix, which we call c^{DM}, is formally defined (using our notation) as follows: for all $(i,j) \in \mathbb{E}$, $c^{DM}_{ij} \equiv \sum_{t=0}^{T} \bar{c}^t_{ij}$. This matrix contains valuable information, which Dutta and Mishra (2012) use to propose a cost allocation. Our approach consists in decomposing the Dutta-Mishra irreducible matrix into a series of cost matrices $\bar{c}^t$, which allows us to take advantage of the very specific structure of these matrices: all members of a supernode are symmetric and can connect to each other at no cost. Thus, a natural way of sharing the cost generated by these matrices $\bar{c}^t$ is to split equally among members of each supernode the (common) cost to connect to outside members.

We construct a large family of rules that make use not only of the Dutta-Mishra irreducible matrix, but also of the additional information contained in the original cost matrix. To do so, let us define what we call the *leftover cost matrix* as follows: $\hat{c} \equiv c - c^{DM}$.

Remark 1 *Observe from the algorithm that the constructed mca for the initial matrix c is also an mca for the leftover matrix $\hat{c}$. Moreover, it is interesting to note that the cost of this mca under $\hat{c}$ is zero.*

Therefore a cost allocation for the cost matrix $\hat{c}$ is of the type: $a(\hat{c}) \in \mathbb{R}^N$, with $\sum\limits_{i \in N} a_i(\hat{c}) = 0$. Letting $\tilde{A}_0$ be the set containing all such zero-sum allocations[2] and $n^t(i) \equiv |\bar{N}^t(i)|$, we define the following family of cost sharing rules for mca problems.

Definition 3 *We call* minimal incoming cost rule for arborescences *(MICRA) any mapping $y : \Gamma \to \mathbb{R}^N$ such that, for any $c \in \Gamma$ and $i \in N$, we have:*

$$y_i(c) = \sum_{t=0}^{T} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} + a_i(\hat{c}), \tag{2}$$

for some $a(\hat{c}) \in \tilde{A}_0$.

Next, we introduce two natural axioms which are unanimously admitted when it comes to cooperative games and cost sharing.

Core Selection: A rule y satisfies Core Selection if $y(c) \in Core(c)$, for any cost matrix $c \in \Gamma$.
This classic requirement says that no coalition of players should jointly pay a share higher than their stand-alone cost. It is particularly relevant in the context of mca (or mcst) problems, for which it has been established that stable cost allocations always exist. A rule that satisfies Core Selection will be called *stable*.

Given a cost matrix $c \in \Gamma$, we say that two agents $i, j \in N$ are *c-symmetric* if, for any $(k,l) \in N_0 \times N$, we have $c_{ki} = c_{kj}$ and $c_{il} = c_{jl}$. The next axiom can then be defined as follows.

[2] Note that $\tilde{A}_0 \neq \emptyset$ since we obviously have $0_N \in \tilde{A}_0$.

Symmetry: A rule y satisfies Symmetry if $y_i(c) = y_j(c)$, for any $c \in \Gamma$ and any $i, j \in N$ that are c-symmetric.
The Symmetry property is a basic fairness requirement which guarantees "equal treatment of equals". Our characterization results would still hold if one replaced Symmetry with the (stronger) requirement *Anonymity*, i.e., the cost shares are independent of the agents' labels.

We are now ready to state our first result. It says that finding a MICRA that is a core selection amounts to finding stable cost allocations for minimal cost arborescences associated with a total cost of zero, which is a much simpler task given that choosing for instance $a(\hat{c}) = 0_N$ is always stable in such a case. Furthermore, a MICRA satisfies Symmetry **iff** it always assigns equal shares to agents that are symmetric under the leftover matrix $\hat{c}$. For a formal statement of the result, let Γ^0 be the set of cost matrices c such that $C(N, c) = 0$ (the minimal cost to connect all players is null). By definition we always have that $\hat{c} \in \Gamma^0$.

Theorem 1
(i) *A MICRA y satisfies Core Selection if and only if we have: $y(c) \in Core(c)$ for any matrix $c \in \Gamma^0$.*
(ii) *A MICRA y satisfies Symmetry if and only if we have: $y_i(c) = y_j(c)$ for any $c \in \Gamma^0$ and any i, j that are c-symmetric.*

The proof is provided in the appendix.

We call *S-MICRA* any minimal incoming cost rule that is both stable and symmetric. Obviously, a MICRA y is an S-MICRA if and only if it meets the conditions (i) and (ii) given in Theorem 1.

Let us now use our running example to illustrate the computation of the cost shares under a MICRA.

Example 2 *We revisit Example 1, with Figure 3 giving the decomposition of c into $\bar{c}^0$, $\bar{c}^1$, $\bar{c}^2$ and $\hat{c}$.*
As pointed out earlier, we have $\delta_1^0 = 6$, $\delta_2^0 = 18$ and $\delta_3^0 = 12$, $\delta_1^1 = 0$, $\delta_{\{2,3\}}^1 = 12$, as well as $\delta_{\{1,2,3\}}^2 = 12$.
Therefore, for any MICRA y, one gets: $\hat{y}_1 = 6 + 0 + \frac{12}{3} + a_1(\hat{c}) = 10 + a_1(\hat{c})$, $\hat{y}_2 = 18 + \frac{12}{2} + \frac{12}{3} + a_2(\hat{c}) = 28 + a_2(\hat{c})$ and $\hat{y}_3 = 2 + \frac{12}{2} + \frac{21}{3} + a_3(\hat{c}) = 22 + a_3(\hat{c})$, where $a_1(\hat{c}) + a_2(\hat{c}) + a_3(\hat{c}) = 0$. In addition, if y is an S-MICRA, we must have:[3] *$a(\hat{c}) \leq (0, 6, 24)$, $a_1(\hat{c}) + a_3(\hat{c}) \leq 0$, $a_2(\hat{c}) + a_3(\hat{c}) \leq 6$. It is straightforward to see that the allocation $\bar{a}(\hat{c}) = (0, 0, 0)$ and many others satisfy all of these requirements.*

Given the decomposition of c that we implicitly use to define our cost sharing methods, one essentially has two choices to make. The first one consists in allocating the connection cost corresponding to $\bar{c}^t$. This choice is a relatively

[3] Note that we do not have a pair of agents that are $\hat{c}$-symmetric; and therefore the property Symmetry implies no restriction on $a(\hat{c})$ here.

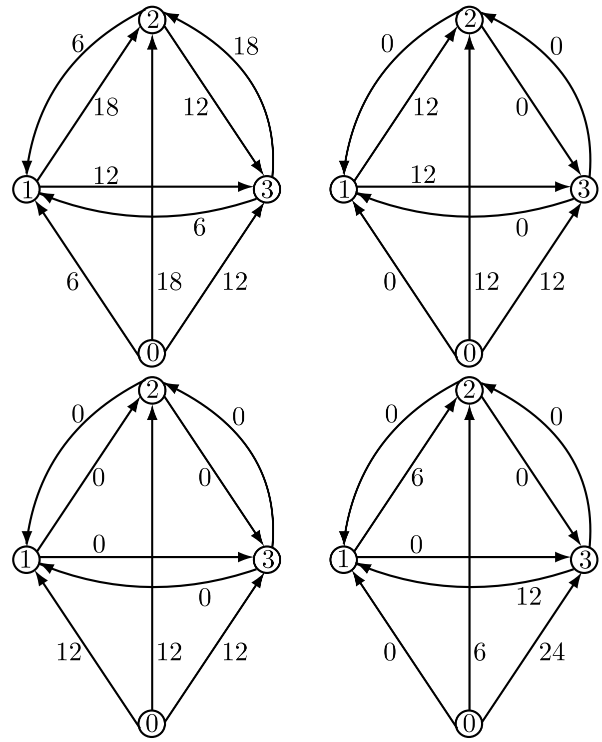

Figure 3: $\bar{c}^0$ (top left), $\bar{c}^1$ (top right), $\bar{c}^2$ (bottom left) and $\hat{c}$ (bottom right)

easy one, given that the core constraints impose that $\sum_{i\in \bar{N}_k^t} y_i(\bar{c}^t) = \delta^k_{\bar{N}_k^t}$. Then, since all agents in N_k^t are $\bar{c}^t$-symmetric, any rule satisfying Symmetry must split the total connection cost equally between them. The second choice is about sharing the cost of the leftover matrix $\hat{c}$. This question, to which there is an infinity of possible answers, is investigated in the next sections.

Before focusing on the leftover matrix, we provide an axiomatization of the family of S-MICRA. Let us first define the Reductionism axiom, which is well known in the mcst literature. Introduced formally by Bogomolnaia and Moulin (2010), this axiom says that cost shares should depend only on the values in the irreducible cost matrix. It is used directly or indirectly, through properties that imply it, in most of the axiomatizations of the (mcst version of) the folk solution (Trudeau, 2013). In the context of mca problems, although there are infinitely many irreducible matrices for any given $c \in \Gamma$, the Dutta-Mishra irreducible matrix is particularly appealing, as it is closely related to the mca-building algorithm.

Reductionism: For all $c \in \Gamma$, $y(N, c) = y(N, c^{DM})$

While acknowledging that the irreducible matrix contains some very important information, one may not be ready to discard the additional information contained in c (which happens under Reductionism). The next axiom therefore proposes to share the costs in two steps: the designer of the rule must first share the cost of the irreducible matrix; and then focus on the leftover matrix, which contains all the information thrown away while building the irreducible matrix.

Decomposition: For all $c \in \Gamma$, $y(N,c) = y(N,c^{DM}) + y(N,c-c^{DM})$.

For the third axiom, suppose that a group of agents have (i) no cost to connect to each other and (ii) symmetric costs to connect and be connected to agents outside the group. This implies that there exists an mca where they will all be connected to each other. Also, to connect them to others (and the source), they will have to pay at least the cheapest of their edges coming in from a node outside their group. Since there is a clear responsibility of the group for this incoming edge,[4] the next axiom states that in such a situation we can decompose the problem (of the agents in that group) by (i) subtracting the cost of the cheapest incoming edge from that of every incoming edge and (ii) sharing the cost of the remainder matrix separately.

Minimal Group Connection: Suppose that $S \subseteq N$ and $c \in \Gamma$ are such that any $i,j \in S$ are c-symmetric. Let c' be such that for all $i \in S$ and all $j \in N_0 \backslash S$, $c'_{ji} = \min_{l \in N_0 \backslash S} c_{li}$ and $c'_e = c_e$ else. Then, $y_i(N,c) = y_i(N,c') + y_i(N,c-c')$ for all $i \in S$.

Using these axioms, one can characterize the family of S-MICRAs as follows.

Theorem 2 *A cost sharing rule y satisfies Decomposition, Minimal Group Connection, Core Selection and Symmetry if and only if it is an S-MICRA.*

The proof is available in the appendix.

4 The folk solution: recovering the Dutta-Mishra extension

Given that we always have $C(N,\hat{c}) = 0$, a trivial way to obtain a symmetric and stable MICRA is to always pick $y(\hat{c}) = 0_N$. We call this particular MICRA the *(mca-extended) folk solution*, and we use the notation $\hat{y}^f$. The name comes from the fact that, for mcst problems in which all players can be connected at no cost, the folk solution always assigns a share of zero to every agent. Formally, for all $c \in \Gamma$ and all $i \in N$, let us define

$$\hat{y}_i^f(c) = \sum_{t=0}^{T} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)}. \tag{3}$$

[4] In the case of of mcst problems, it is not clear whether an edge (i,j) is needed to connect i to j or j to i. In mca problems, we have a clear distinction; and therefore it is very natural to link the cost shares to the cost of incoming edges, as it is those that are used to connect the agents.

Theorem 3 *The mca-extendend folk solution is an S-MICRA.*

Proof. By definition, the mca-extended folk solution is a MICRA. By Theorem 1, it is a S-MICRA if for any matrix $c \in \Gamma^0$ we have i) $y(c) \in Core(c)$ and ii) $y_i(c) = y_j(c)$ for all i, j that are c-symmetric. For all $c \in \Gamma^0$, we have $y_i^f(c) = 0$ for all $i \in N$. Since $C(S, c) \geq 0$ for all $S \subseteq N$ and $C(N, c) = 0$, $y^f(c) \in Core(c)$ and thus i) is satisfied. Trivially, condition ii) is satisfied. ■

In the mcst literature, the folk solution is obtained by taking the Shapley value of the coalitional game associated with the (unique) irreducible matrix (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007).

While there typically exist (for mca problems) multiple irreducible matrices associated with a given $c \in \Gamma$, Dutta and Mishra (2012) propose an extension of the folk solution by using the irreducible matrix c^{DM} defined in the previous section. Precisely, their method is defined as $y^{DM}(c) = Sh(c^{DM})$.

We show that, although the respective computational procedures differ, our (mca-extended) folk solution coincides with y^{DM}.

Theorem 4 *For any $c \in \Gamma$, we have $\hat{y}^f(c) = y^{DM}(c)$.*

Proof. Given the definitions of $\hat{y}^f(c)$ and $y^{DM}(c)$, it suffices to show that $Sh_i(\overline{c}^t) = \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)}$. We use the well-known definition of the Shapley value (as the average of all incremental rules) to prove this equality.

In the coalitional game $C(\cdot, \overline{c}^t)$, considering a permutation of agents where i is the first among all agents in $\bar{N}^t(i)$, one can compute his marginal contribution as $\delta_{\bar{N}^t(i)}$. If agent i is not the first among $\bar{N}^t(i)$ under the considered permutation then his marginal contribution is zero. Notice that the positions of agents in $N \backslash \bar{N}^t(i)$ do not affect the marginal contribution of i. Since the probability that agent i will be chosen first in $\bar{N}^t(i)$ is $\frac{1}{n^t(i)}$, we obtain $Sh_i(\overline{c}^t) = \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)}$. ■

It thus turns out that our approach provides (as a particular case) a much easier computational procedure for the Dutta-Mishra solution. Indeed, under our procedure, one does not need to make use of the Shapley value and, in addition, the shares can easily be assigned during the building of the optimal network.

Dutta and Mishra (2012) have shown that their method corresponds to the folk solution when the mca problem has an undirected matrix (i.e. the mca problem is also an mcst problem). The following corollary is straightforward to prove.

Corollary 1 *If $c \in \Gamma^u$, then $\hat{y}^f(c) = y^f(c)$.*

Hence, our mca-version of the folk solution does coincide with the mcst-version when applied to undirected mca problems. Our procedure thus provides a new way of defining the folk solution for mcst problems (interestingly, one that does not require to use the Shapley value). However, under our approach, the mcst problem may be transformed into a series of (directed) mca problems that are easy to solve.

A similar approach exists in the mcst literature, if one views the folk solution as an obligation rule (Feltkamp et al., 1994; Tijs et al., 2006; Bergantiños and Kar, 2010): using Prim's algorithm to build the optimal network, the cost of each edge added to the network is paid by the agents belonging to the two connected components linked by that edge, with all agents in a given connected component paying equal shares.

We provide two axiomatizations of the (mca-extended) folk solution. We need the two following axioms.

Group Independence: For all $c \in \Gamma$, if there exists $S \subseteq N$ such that $c_{ij} \geq c_{0j}$ and $c_{ji} \geq c_{0i}$ for all $i \in S$ and $j \in N \setminus S$, then $y_i(N, c) = y_i(S, c^S)$ for all $i \in S$.

The Group Independence axiom is implied by Core Selection. It states, for a coalition whose members do not offer a cheaper access to the source to any outsider (and vice-versa), that the cost shares of its members can be computed on the problem restricted to the group. It is worth noting that this is a weaker property than Separability, used by Dutta and Mishra (2012) for mca problems and Bergantiños and Vidal-Puga (2007), among others, for mcst problems. While Group Independence requires that no subcoalitions of the group have any gains to cooperate with non-members of the group, Separability only requires for the group itself to have no such gains. See Trudeau (2013) for a discussion of the links between Group Independence and other axioms in the context of mcst problems.

Non-negativity: For all $c \in \Gamma$, and all $i \in N$ $y_i(N, c) \geq 0$.

Although fairness requires that agents with advantageous locations be rewarded, it may be difficult in some contexts to enforce to idea that some demanders should receive money.

Since the folk solution is an S-MICRA, we build on Theorem 2. For this characterization of the folk solution, it turns out that we need only Group Independence instead of the stronger Core Selection.

Theorem 5 *A solution y satisfies Decomposition, Minimal Group Connection, Group Independence, Symmetry and Non-Negativity if and only if $y = \hat{y}^f$.*

Proof. Suppose that y satisfy the five axioms. By Decomposition $y(c) = y(c^{DM}) + y(c - c^{DM})$. By construction, for all $c \in \Gamma$, $C(N, c - c^{DM}) = 0$. Thus, by non-negativity, we must have that $y(c - c^{DM}) = 0_N$. To find $y(c^{DM})$ we proceed as in the proof of Theorem 2. Notice that whenever we used Core Selection, Group Independence is sufficient. We thus recover $\hat{y}^f$, as given by equation (3). ■

An alternative axiomatization of the folk solution is obtained by replacing Decomposition with the stronger Reductionism (which allows us to drop the Non-Negativity requirement).

Theorem 6 *A cost sharing rule y satisfies Reductionism, Minimal Group Connection, Group Independence and Symmetry if and only if $y = \hat{y}^f$.*

Proof. By Reductionism, we have $y(c) = y(c^{DM})$; and $y(c^{DM})$ can be computed exactly as in the proof of Theorem 2. ■

Note that our axiomatization of the folk solution applies to the full domain Γ of cost matrices. Dutta and Mishra (2012) offer axiomatizations of the folk solution on the restricted domain of simple cost matrices (those for which the algorithm stops at step $T = 0$). The folk solution thus makes each agent pay the cost of his cheapest incoming edge. They offer two characterizations on this restricted domain.

The first one uses Non-Negativity together with the property *Invariance*, which states that if the costs of all of an agent's incoming edges increase by a, then the share of that agent should increase by a. We can see a similarity with Minimal Group Connection (which is also built around the notion of responsibility for incoming edges).

Their second axiomatization uses two new properties. *Independence of Other Costs* states that the share of an agent should only depend on the cost of her incoming edges. *Strong Symmetry* states that two agents should have the same allocation as soon as they have symmetric costs on incoming edges (regardless of the costs on outgoing edges). Both properties are reminiscent of Reductionism (some costs do not matter) and Minimal Group Connection (reliance on incoming edges). As suggested by the names, our notion of Symmetry is weaker than Strong Symmetry.

5 Using the information of the leftover matrix

The approach used to compute the (mca-extended) folk solution is not the only one available. At each stage t, under the matrix $\bar{c}^t$, it costs nothing to link any two members of $\bar{N}_k^t$. In addition, every agent in $\bar{N}_k^t$ is implicitly given access to the cheapest edges between any member of $\bar{N}_k^t$ and an outside member. However, the individual bargaining powers within $\bar{N}_k^t$ depend on the matrix c and, more precisely, on the leftover matrix $\hat{c}$. Allocating non-zero shares for the leftover matrix $\hat{c}$ can thus allow to take care of these differences in bargaining power. Under this approach, it makes sense to reward agents who own these cheapest connections (across supernodes) to the detriment of those who have more expensive connections to other supernodes. Ignoring $\hat{c}$ will sometimes give cost shares that seem unreasonable, as illustrated by the following example.

Example 3 *We consider a problem with two agents, with the cost matrix represented on the left in Figure 4.*

We obtain $\delta_1^0 = 18$, $\delta_2^0 = 0$, $\delta_{(12)}^1 = 2$ *and the leftover matrix depicted on the right in Figure 4.*

Therefore, for any MICRA rule,

$$\begin{aligned}
\hat{y}_1(c) &= 18 + \frac{2}{2} + a(\hat{c}) \\
\hat{y}_2(c) &= 0 + \frac{2}{2} + a(\hat{c}).
\end{aligned}$$

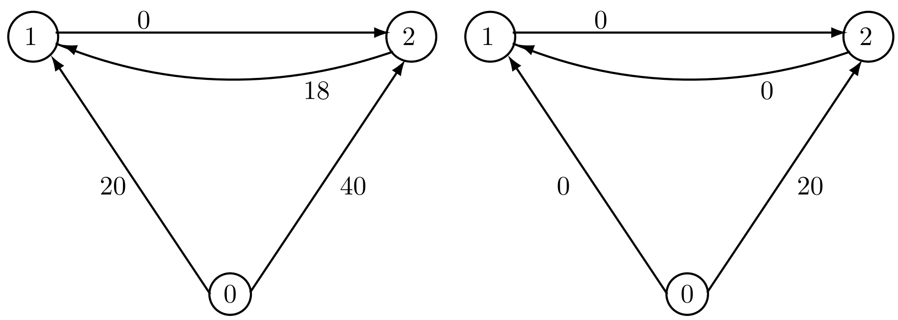

Figure 4: c and $\hat{c}$ in Example 3

In particular, we obtain $\hat{y}^f = (19, 1)$.

In the above example, agent 1, who has a stand-alone cost half of that of agent 2, ends up paying 19 times her share. Under this rule, agent 1 is punished for having a bad incoming edge from agent 2, while the latter is unharmed by her bad connection to the source.

5.1 The pair-of-paths rule for mca problem

A natural remedy would be to take the Shapley value of the cost game generated by $\hat{c}$. Doing this in the example above drastically changes the shares: agent 1 now pays 0 and agent 2 pays 20. While the obtained cost allocation is stable for this example, there is no guarantee in general that it will be in $Core(\hat{c})$. In the mcst literature, one alternative is to use the cycle-complete solution, which obtains by computing the Shapley value after making some changes to the matrix. These changes are less extreme than those required for the folk solution. For each pair of nodes, we look at pairs of paths that go from one to the other: we assign to the direct edge between these two nodes the cost of the most expensive edge on that pair of paths.

In the mcst literature, this procedure is done directly on the original matrix. This is unfeasible for mca problems, as it will often reduce the cost of connection for the grand coalition. We chose instead to apply the cycle-complete procedure on the leftover matrix. This proves enough to guarantee the concavity of the resulting cost game.

The following definition applies to general mca problems. Consider any distinct $i \in N_0$, $j \in N$; and denote by $\mathcal{P}_{ij}$ the set containing all pairs of paths $(p_{ij}, p'_{ij}) \in P^2_{ij}$ such that $[p_{ij}] \cap [p'_{ij}] = \{i, j, \}$. In addition, for any set of edges E, denote by $M(E) \equiv \max_{e \in E} c_e$ the cost of its most expensive edge.

In mcst problems, since the edges are not directed, a pair of paths from i

to j is the same as a pair of paths from j to i, which is the same as a cycle that includes both i and j, hence the name cycle-complete given to the rule. In mca problems, however, it is more appropriate to use the notion of pair of paths (pop).

Then for a any $c \in \Gamma$, we define the cost matrix c^{pop}, as follows:

$$c_{ij}^{pop} = \min_{(p_{ij}, p'_{ij}) \in \mathcal{P}_{ij}} M(p_{ij} \cup p_{ij}), \forall i \in N_0, \forall j \in N.$$

Definition 4 *The pair-of-paths (pop) rule for mca problems is the MICRA defined by*

$$\hat{y}_i^{pop}(c) = \sum_{t=0}^{T} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} + Sh_i(C(\cdot, \hat{c}^{pop})), \quad \forall i \in N, \forall c \in \Gamma.$$

Example 4 *In our running example, $\hat{c}^{pop}$ is as follows: all edges have the same cost as under $\hat{c}$, except for the edge $(0,3)$, which now has a cost of 6. To obtain this, consider the paths $p_{03} = \{(0,1),(1,3)\}$ and $p'_{03} = \{(0,2),(2,3)\}$; and notice that the cost of the most expensive edge for this pair of paths is 6.*

The resulting stand-alone game is such that $C(\{1\}) = C(N) = 0$ and $C(S) = 6$ otherwise. We thus obtain $\hat{y}^{pop}(\hat{c}^{pop}) = (-2,1,1)$, yielding $\hat{y}^{pop}(c) = (8,29,23)$.

Moreover, in Example 3, the pop rule coincides with the Shapley value, yielding much more compelling shares which reflect agent 2's disadvantageous position in the network: under both rules, agents 1 and 2 respectively pay 0 and 20. We now proceed to show that $\hat{y}^{pop}$ is a core selection (unlike the Shapley value).

Theorem 7 *The pair-of-paths (pop) rule is an S-MICRA.*

See the appendix for the proof.

It is worth noting that although the pop rule is built on the same concept as the cycle-complete solution for mcst problems, they do not always coincide on the set of undirected matrices. This is because of the way the matrix is decomposed into the sum of the Dutta-Mishra irreducible matrix and the leftover matrix, which might not have a common ranking of edges according to their cost. Therefore, the modifications made through the cycle-complete procedure may differ depending on whether they are done on the original matrix or the leftover matrix. We illustrate this possibility in the following example.

Example 5 *Consider the three-player mca problem associated with the* ***undirected*** *matrix c, which is illustrated by the leftmost network in Figure 5. One can use the algorithm described earlier to compute the matrices c^{DM} and $\hat{c}$ given in Figure 5.*

For undirected matrices, the difference between the cycle-complete solution (designed for mcst problems) and the pop rule (designed for mca problems) is

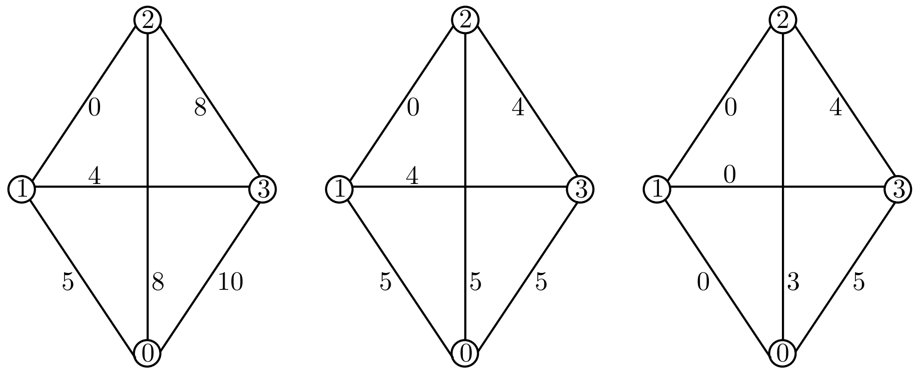

Figure 5: c (left), c^{DM} (center), and $\hat{c}$ (right)

that in the first case, we take the Shapley value of the game generated by c^{pop}, while in the second case, it is the Shapley value of the game generated by $\hat{c}^{pop}$. We thus examine how these two matrices are computed.

To obtain c^{pop} we only need to change edge $(0,3)$, that sees its cost drop to 8, as we have disjoint paths $(0,1),(1,3)$ and $(0,2),(2,3)$, with the cost of their most expensive edge being 8.

In order to derive $\hat{c}^{pop}$ we also only need to change edge $(0,3)$, using the same pair of paths. The most expensive edge on this pair of paths has a cost of 4, and therefore we change the cost to 4. Thus, one obtains $c_{03}^{DM} + \hat{c}_{03}^{pop} = 9 \neq c_{03}^{pop} = 8$; and hence the two matrices ($c^{DM} + \hat{c}^{pop}$ and c^{pop}) differ.

5.2 The supplier rule for mca problems

In this part, we define a new rule for mca problems. As will be argued, this method fully rewards the member of each supernode who provides access to others. As such, it is diametrically opposed to the folk rule.

Remark 2 *Notice that, at each step $t = 0, \ldots, T$ of the algorithm of Subsection 2.1, every non-singleton $\bar{N}_k^t$ is a $\hat{c}$-cycle, as well as a c^t-cycle (by construction). Hence, $\bar{N}_k^t$ is also a supernode under $\hat{c}$ (in the sense that all agents it contains can be connected to each other at no cost, without using outsiders).*

Recall that at the terminal stage T we have a simple matrix: one for which the mca is obtained by taking the minimal incoming cost of each supernode. As stated in Remark 1, the total cost of this mca under the leftover $\hat{c} \in \Gamma^0$ is 0 (for any matrix $c \in \Gamma$). For all $i \in N$, recall that: $\bar{N}^t(i)$ is the supernode containing i at stage $t = 1, \ldots, T$; and $n^t(i) \equiv |\bar{N}^t(i)|$. One can define $\alpha_i^t \equiv \min_{j \notin \bar{N}^t(i)} \hat{c}_{ji}$. In addition, let $i^t \in \bar{N}^t(i)$ be s.t. $\alpha_{i^t}^t = 0$.[5]

[5]Note that $i^t \in \bar{N}^t(i)$ s.t. $\alpha_{i^t}^t = 0$ always exists because $\hat{c} \in \Gamma^0$ (any mca for $\hat{c}$ is costless).

At the end of the algorithm, one can partition N using the mca $A^* \in \mathcal{A}(c)$ as follows.[6] Define $P \equiv \{k \in N | (0,k) \in A^*\}$ and, for all $k \in P$, let $N_k^{T+1} \equiv \{k\} \cup \{j \in N | \ \exists p_{kj} \in P_{kj}(N_0) \text{ s.t. } p_{kj} \subset A^*\}$ denote the set of (weak) *successors* of k in the mca A^*. It is easy to see that the collection $\{N_k^{T+1}\}_{k \in P}$ is a partition of N. We extend the notation defined above to this partition: if $i \in N_k^{T+1}$ then we will write $i^{T+1} = k$ and $\bar{N}^{T+1}(i) = N_k^{T+1}$, for any $i \in N$. Likewise, we denote by α_i^{T+1} the lowest cost to connect i to (the source or) some agent outside $\bar{N}^{T+1}(i)$. For convenience, we will also use $a \wedge b \equiv \min\{a,b\}$. Remark that $\bar{N}^t(i)$ weakly increases with t.

Using this notation, one can construct a cost allocation $a^s(\hat{c})$ in the following way. Initialize $i^{T+2} = 0$, $\beta_i^{T+2} = \min\limits_{i \in \bar{N}^{T+1}(i) \setminus i^{T+1}} \hat{c}_{0i}$ and, proceeding backward for any $t = T+1, \ldots, 1$, define the following for any $i \in N$:[7] $\beta_i^t = \frac{\beta_i^{t+1} \wedge \min_{i \in \bar{N}^t(i) \setminus i^t} \alpha_i^t}{n^t(i) - 1}$ if $i^{t+1} \notin \bar{N}^t(i)$, with $\beta_i^t = \beta^{t+1}$ if $i^{t+1} \in \bar{N}^t(i)$ and $\beta_i^t = 0$ if $\bar{N}^t(i) = \{i\}$; and

$$a_i^t(\hat{c}) = \begin{cases} \beta_i^t & \text{if } i^t \notin \{i, i^{t+1}\}; \\ -\beta_i^t(n^t(i) - 1) & \text{if } i = i^t \neq i^{t+1}; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Taking the sum of these $T+1$ allocations, write:

$$a_i^s(\hat{c}) = a_i^1(\hat{c}) + \ldots + a_i^{T+1}(\hat{c}). \tag{5}$$

We are now ready to introduce a new MICRA for mca problems, y^s. For any $c \in \Gamma$ and any $i \in N$,

$$y_i^s(c) = \sum_{t=0}^{T} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} + a_i^s(\hat{c}). \tag{6}$$

Remark that, for any $c \in \Gamma^0$, we have $y^s(c) = y^s(\hat{c}) = a^s(\hat{c})$.

Example 6 *In our running example, at stage $T+1 = 3$, we have $\bar{N}^3(1) = \{1,2,3\}$ and $a^3(\hat{c}) = (-6,3,3)$ since agent 1 has the best connection to the source and must be paid 1, the cost of the second-cheapest connection (equally divided between the agents 2 and 3). At stage $t = 2$, we have $a^2(\hat{c}) = (0,0,0)$, since $\bar{N}^3(1) = \bar{N}^2(1) = N$ (same supernode for $t = 2,3$). Note that in this example we have $\beta_i^4 = 6$ and $\beta_i^3 = \beta_i^2 = 3$ for any agent i. At stage $t = 1$, the supernode $\{1\}$ is trivial ($a_1^1(\hat{c}) = 0$ since $\beta_i^t = 0$ for any singleton supernode). For the supernode $\{2,3\}$, taking $i = 2$ gives $i^t = 3$: agent 3 has the lowest incoming cost and thus receives from agent 2 the amount $\beta_2^1 = \beta_2^2 \wedge \alpha_2^1 = 3$;*

[6] Recall that A^* is obtained as the outcome of the algorithm. Although there may be multiple mca in $\mathcal{A}(c)$, our allocation is the same regardless of the mca we pick.

[7] Remark from the definition of β_i^t that $(n^t(i) - 1)\beta_i^t = \min\limits_{i \in \bar{N}^t(i) \setminus i^t} \alpha_i^t$ is the second lowest incoming cost of the supernode $\bar{N}^t(i)$, *provided it is lower than* β_i^{t+1} —which is always satisfied at $t = T+1$ since $\beta_i^{T+2} = \min\limits_{i \in \bar{N}^{T+1}(i) \setminus i^{T+1}} \hat{c}_{0i} \geq \min\limits_{i \in \bar{N}^{T+1}(i) \setminus i^{T+1}} \alpha_i^{T+1}$.

and we have $y^1(\hat{c}) = (0, 3, -3)$. Taking the sum for all three periods thus gives $y^s(\hat{c}) = (-6, 6, 0)$. Going back to the initial matrix c, one finally obtains $y^s(c) = (10, 28, 22) + (-6, 6, 0) = (4, 34, 22)$.

Note from its definition that y^s is a MICRA. In order to study the other properties of y^s, we need to show the preliminary results below. Recall that $i^{T+2} \equiv 0$ and let $\gamma_i^t \equiv \P(i^t \neq i^{t+1})$, that is, $\gamma_i^t = 1$ if $i^t \neq i^{t+1}$ and $\gamma_i^t = 0$ else.

Lemma 1 *For any $c \in \Gamma$, $i \in N$ and $\tilde{t} = 1, \ldots, T+1$, we have:*

(i) $a_i^t(\hat{c}) \leq \beta_i^t \gamma_i^t \leq \frac{\beta_i^{t+1}}{n^t(i)-1}$ *for $t = 1, \ldots, T+1$ and $n^t(i) > 1$;*

(ii) $\sum_{t=1}^{\tilde{t}} a_i^t(\hat{c}) + \underbrace{\sum_{t=1}^{\tilde{t}} \sum_{j \in \bar{N}^t(i) \setminus \{i, i^{\tilde{t}}\}} a_j^t(\hat{c})}_{\geq 0} \leq \alpha_i^{\tilde{t}}, \quad$ *if $\gamma_i^{\tilde{t}} = 1$;*

(iii) $y_i^s(\hat{c}) + \sum_{t=1}^{T+1} \sum_{j \in \bar{N}^t(i) \setminus \{i, i^{T+1}\}} a_j^t(\hat{c}) \leq \hat{c}_{0i}$.

Interestingly, we are able to prove that the rule y^s is stable and symmetric, that is to say, it belongs to the family of S-MICRAs.

Theorem 8 *The supplier rule y^s is an S-MICRA.*

The respective proofs of the theorem and preceding lemma are available in the appendix.

As noted before, at each stage, the supplier rule assigns to the provider in each supernode the maximal subsidy (that preserves stability), which is equal to the second minimal incoming cost $\min_{i \in \bar{N}^{T+1}(i) \setminus i^{T+1}} \alpha_i^t$ (if it is lower than β^{t+1}). It therefore fully rewards intermediaries among S-MICRAs, with priority given to suppliers that are closer to the source in the mca. In this sense, the supplier rule can be viewed as diametrically opposed to the folk rule, which gives a subsidy of exactly zero to the best connection of each supernode (by discarding the leftover matrix).

6 Concluding remarks

The paper has introduced and studied a large family of cost sharing rules for mca problems, the MICRAs. These rules are obtained from the mca-building algorithm by decomposing the cost matrix in a series of partially symmetric matrices. The members of the family of MICRAs differ depending on their treatment of the leftover matrices, which represent the information that is overlooked during the constrution of the efficient network. A distinguished MICRA is the extension of the folk solution of Dutta and Mishra (2012), which ignores leftover matrices. In addition, we have introduced and examined two remarkable MICRAs that make use of leftover matrices. The first one, the pair-of-paths (pop) rule, is inspired by the cycle-complete rule for mcst problems. The second method, which we call supplier rule, fully rewards agents who allow others to connect at a low cost.

Moreover, we have proposed an axiomatization of the family of stable and symmetric MICRAs (S-MICRAs). The axioms used in our characterization are independent. Indeed, fixing a permutation of the agents, considering the cost function obtained from the pair-of-paths matrix and assigning to each agent his incremental cost using that permutation satisfies all properties but Symmetry. The equal division rule (sharing the total cost equally among all agents) satisfies all properties but Core Selection. Applying the pop rule if c is directed and the cycle-complete solution (for mcst problems) when c is undirected yields a rule that meets all properties except Minimal Group Connection. Finally, using the pop rule if c is in Γ^0 and the folk rule otherwise yields a method that satisfies all properties but Decomposition.

We have also proposed two characterizations for the (mca-extended) folk solution. The independence of these axioms is easily verified as well. And to see this, one can use cost sharing rules that are similar to those described above.

Some interesting questions for future research naturally arise from the present work. First, it would be interesting to find some compelling axioms allowing to characterize the pop and supplier rules. Note that an axiomatization of either rule is more difficult to obtain than that of the folk solution, since one needs to find natural properties allowing to treat all leftover matrices in a very specific way (whereas axioms like Reductionism simply demand to ignore them).

Second, the supplier rule has no known counterpart in the particular case of mcst problems. Therefore, finding and studying a variant that can be computed by using only the mcst-building algorithm (and of which the present version is an extension) would contribute to this literature.

References

Bahel, E. (2016). On the core and bargaining set of a veto game. *International Journal of Game Theory*, forthcoming.

Bahel, E. and Trudeau, C. (2014). Stable lexicographic rules for shortest path games. *Economics Letters*, 125(2):266–269.

Bergantiños, G. and Kar, A. (2010). On obligation rules for minimum cost spanning tree problems. *Games and Economic Behavior*, 69(2):224–237.

Bergantiños, G. and Vidal-Puga, J. (2007). A fair rule in minimum cost spanning tree problems. *Journal of Economic Theory*, 137(1):326–352.

Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, 6:335–350.

Bogomolnaia, A. and Moulin, H. (2010). Sharing the cost of a minimal cost spanning tree: Beyond the folk solution. *Games and Economic Behavior*, 69:238–248.

Chu, Y. and Liu, T. (1965). On the shortest arborescence of a directed graph. *Science Sinica*, 14:1396–1400.

Dutta, B. and Mishra, D. (2012). Minimum cost arborescences. *Games and Economic Behavior*, 74(1):120–143.

Edmonds, J. (1967). Optimum Branchings. *Journal of Research of the National Bureau of Standards*, 71:233–240.

Feltkamp, V., Tijs, S., and Muto, S. (1994). On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems. Technical Report 106, CentER DP 1994 nr.106, Tilburg University, The Netherlands.

Tijs, S., Branzei, R., Moretti, S., and Norde, H. (2006). Obligation rules for minimum cost spanning tree situations and their monotonicity properties. *European Journal of Operational Research*, 175(1):121–134.

Trudeau, C. (2012). A new stable and more responsible cost sharing solution for mcst problems. *Games and Economic Behavior*, 75(1):402–412.

Trudeau, C. (2013). Characterizations of the Kar and folk solutions for minimum cost spanning tree problems. *International Game Theory Review*, 15(2).

A Appendix - Proofs

A.1 Theorem 1

We first prove a preliminary result that will be helpful in showing Theorem 1. For all $t = 0, ..., T$ and $S \subseteq N$, let $R_S^t = \left\{ \bar{N}_k^t \in \bar{N}^t \left| S \cap \bar{N}_k^t \neq \emptyset \right. \right\}$.

Lemma 2 *For any $S \subseteq N$ and $t = 0, \ldots, T$ we have $C(S, \bar{c}^t) = \sum_{R \in R_S^t} \delta_R^t$.*

Proof. Let us consider $S \subseteq N$. For all $i \in S$, since $0 \notin \bar{N}^t(i)$ and there must be a path from 0 to i in any arborescence A, we can claim that there exists $(l^i, k^i) \in A$ s.t. $k^i \in \bar{N}^t(i)$, $l^i \notin \bar{N}^t(i)$. Next, fix $t \in \{0, \ldots, T\}$ and let $r = |R_S^t|$. Then there exists $i_1, \ldots, i_r \in S$ s.t. $R_S^t = \{\bar{N}^t(i_1), \ldots, \bar{N}^t(i_r)\}$. Hence, *for any arborescence* A that connects S, we have $\left\{(l^{i_1}, k^{i_1}), ..., (l^{i_r}, k^{i_r})\right\} \subseteq A$.

And, recalling the definition of $\bar{c}^t$ from (1), one may thus write

$$\bar{c}^t(A) \geq \underbrace{\bar{c}^t_{l^{i_1} k^{i_1}}}_{\delta^t_{\bar{N}^t(i_1)}} + ... + \underbrace{\bar{c}^t_{l^{i_r} k^{i_r}}}_{\delta^t_{\bar{N}^t(i_r)}} = \sum_{R \in R_S^t} \delta_R^t.$$

To conclude, we show that $A_S^t \equiv \bigsqcup_{m=1}^{r} \left[\{(0, i_m)\} \sqcup \left\{(i_m, j),\ j \in S \cap \bar{N}^t(i_m) \backslash i_m \right\}\right]$ attains the lower bound $\sum_{R \in R_S^t} \delta_R^t$ (and is hence an mca for S under $\bar{c}^t$). Indeed,[8] it is straightforward to see that A_S^t is an arborescence that connects S. And it comes from (1) that

$$\bar{c}^t(A_S^t) = \sum_{m=1}^{r} \underbrace{\bar{c}^t_{0 i_m}}_{\delta^t_{\bar{N}^t(i_m)}} = \sum_{R \in R_S^t} \delta_R^t,$$

since $\bar{c}^t_{i_m j} = 0$ for all $j \in \bar{N}^t(i_m) \backslash i_m$. ■

PROOF OF THEOREM 1

Let the cost sharing rule y be a MICRA and fix a cost matrix $c \in \Gamma$. Recall from Remark 1 and Definition 3 that all MICRAs satisfy the budget balance condition, that is, $\sum_{i \in N} y_i(c) = C(N, c) = C(N, c^{DM})$.

i) We first show that the condition $[y(c') \in Core(c')$ for any matrix $c' \in \Gamma^0]$ is sufficient for the MICRA y to satisfy Core Selection.

Fix $S \subseteq N$ and let A_S be an mca for S in c. We want to show coalitional rationality for S under y, that is, $c(A_S) \geq \sum_{i \in S} y_i(c) = \sum_{i \in S} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} + \sum_{i \in S} y_i(\hat{c})$. By

[8] Note that the arborescence A_S^t is formed by (i) connecting $i_1, \ldots, i_r$ directly to the source and then (ii) attaching to each one of them all other members of their supernode (at stage t) that belong to S.

definition of $\hat{c}$, we have $c = c^{DM} + \hat{c} = \sum\limits_{t=0}^{T} \overline{c}^t + \hat{c}$. Letting A_S^t and $\hat{A}_S$ be mcas connecting S under the respective cost matrices $\overline{c}^t$ and $\hat{c}$, one may then write

$$C(S,c) = c(A_S) = \sum_{t=0}^{T} \overbrace{\overline{c}^t(A_S)}^{\geq \overline{c}^t(A_S^t)} + \overbrace{\hat{c}(A_S)}^{\geq \hat{c}(\hat{A}_S)} \geq \sum_{t=0}^{T} C(S,\overline{c}^t) + C(S,\hat{c}). \tag{7}$$

Thus, we need to show that $\sum\limits_{i\in S} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} + \sum\limits_{i\in S} y_i(\hat{c}) \leq \sum\limits_{t=0}^{T} C(S,\overline{c}^t) + C(S,\hat{c})$. And given that $\sum\limits_{i\in S} y_i(\hat{c}) \leq C(S,\hat{c})$ by the assumption that $y(\hat{c}) \in Core(\hat{c})$ (since $\hat{c} \in \Gamma^0$), it suffices to show that, for any $t = 0,\ldots,T$, we have $\sum\limits_{i\in S} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} \leq C(S,\overline{c}^t)$. And this inequality is easily satisfied since

$$\sum_{i\in S} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)} = \sum_{R\in R_S^t} \delta_R^t \underbrace{\frac{|S\cap R|}{|R|}}_{\leq 1} \leq \sum_{R\in R_S^t} \delta_R^t = C(S,\overline{c}^t),$$

with the last equality coming from Lemma 2.

On the other hand, the condition $\left[y(c') \in Core(c'), \forall c' \in \Gamma^0\right]$ is obviously necessary for $y(\cdot)$ to be stable, since the axiom Core Selection requires the allocation $y(c')$ to be in $Core(c')$, for any cost matrix $c' \in \Gamma$ (including those associated with a total connection cost of 0).

ii) It is easy to see that if i,j are c-symmetric, then they are also $\overline{c}^t$-symmetric for $t = 0,1,..,T$ and $\hat{c}$-symmetric. Indeed, by the definition of a MICRA (see Definition 3) and the fact that i,j are $\overline{c}^t$-symmetric for $t = 0,1,..,T$, we have:

$$\sum_{t=0}^{T} \frac{\delta_{\bar{N}^t(i)}}{n^t(i)} = \sum_{t=0}^{T} \frac{\delta_{\bar{N}^t(j)}}{n^t(j)}.$$

Given that $c = \hat{c}$ **iff** $c \in \Gamma^0$, it follows that y satisfies Symmetry if and only if $y_i(c) = y_j(c)$ whenever $c \in \Gamma^0$ and $i,j \in N$ are c-symmetric. ■

A.2 Theorem 2

It is easy to verify that S-MICRAs satisfy the four properties. Conversely, we show below that if y satisfies the properties then it has to be an S-MICRA.

Fix $c \in \Gamma$ and let y be a cost sharing rule satisfying the four properties. By Decomposition, we have $y(c) = y(c^{DM}) + y(c - c^{DM})$. We first compute $y(c^{DM})$. To do so, one has to follow the mca-building algorithm; and the following notation is useful: $\dot{c}^t \equiv c^{DM} - \sum_{s=0}^{t-1} \overline{c}^s = \sum_{s=t}^{T} \overline{c}^s$.

Consider stage 1. For each $i \in N$, define $c^{\delta_i^0}$ as follows: for all $j \in N_0$, $c^{\delta_i^0}_{ji} = c^{DM}_{ij} - \delta_i^0$ and $c^{\delta_i^0}_e = c^{DM}_e$ otherwise. By Minimal Group Connection, $y_i(N, c^{DM}) = y_i(N, c^{\delta_i^0}) + y_i(c^{DM} - c^{\delta_i^0})$. By Core Selection, $y_i(N, c^{\delta_i^0}) =$

$y_i(\{i\},(c^{\delta_i^0})^{\{i\}}) = \delta_i^0$. Notice that by construction, $(\{i\},(c^{\delta_i^0})^{\{i\}}) = (\{i\},(\bar{c}^0)^{\{i\}})$ and that by Core Selection, $y_i(N,\bar{c}^0) = y_i(\{i\},(\bar{c}^0)^{\{i\}})$. Thus, $y_i(N,c^{DM}) = y_i(N,\bar{c}^0) + y_i(c^{DM} - \bar{c}^0) = y_i(N,\bar{c}^0) + y_i(c - \dot{c}^1)$.

At stage t, by construction, $\dot{c}^t$ is such that for all $i,j \in S \in \bar{N}^t$, i,j are $\bar{c}^t$-symmetric. By multiple applications of Minimal Group Connection and Core Selection as above, $y_i(N,c^{t-1}) = y_i(N,\dot{c}^t) + y_i(N,\bar{c}^{t-1})$ for all $i \in S$ and all $S \in \bar{N}^t$. By Core Selection, $y_i(N,\bar{c}^{t-1}) = y_i(S,(\bar{c}^{t-1})^S)$. By Symmetry and Core Selection, $y_i(S,(\bar{c}^{t-1})^S) = \frac{\delta_S^t}{|S|}$ for all $i \in S$ and $S \in \bar{N}^t$.

Combining, we obtain that

$$y_i(c^{DM}) = \sum_{t=0}^{T} \frac{\delta^t_{\bar{N}^t(i)}}{n^t(i)}$$

and $y_i(c) = y_i(c^{DM}) + y_i(c - c^{DM})$ for all $i \in N$. Therefore, y is a MICRA rule.

We conclude by finding the constraints on $y(c - c^{DM})$. By definition, $(c - c^{DM})^{DM}$ is the null matrix and, by Core Selection, this implies that we have $y\left((N, c - c^{DM})^{DM}\right) = 0_N$ (i.e., Decomposition imposes no further constraints). Suppose that S satisfies the conditions of Minimal Group Connection. By construction $C(N, c - c^{DM}) = 0$, $\min_{l \in N_0 \setminus S} c_{li} = 0$ and Minimal Group Connection imposes no further constraints. It is straightforward to see that, together, Core Selection and Symmetry require that $y(c - c^{DM})$ be symmetric and stable, which allows us to conclude that y is an S-MICRA.

A.3 Theorem 7

By definition, the pair-of-paths rule is a MICRA. By Theorem 1, a MICRA is a a S-MICRA if for any matrix $c \in \Gamma^0$ we have i) $y(c) \in Core(c)$ and ii) $y_i(c) = y_j(c)$ for all i,j that are c-symmetric.

i) Our proof strategy consists in writing $c \in \Gamma^0$ as a finite sum of cost matrices whose associated coalitional games are all concave.

Fix $c \in \Gamma^0$ and pick an ordering of all the edges in $\mathbb{E}$ $(\pi_1, ..., \pi_L)$ such that $c_{\pi_1} \leq c_{\pi_2} \leq ... \leq c_{\pi_L}$. Then let $c^{\pi_l} \in \Gamma^0$ be the cost matrix defined by $c^{\pi_l}_{\pi_m} = 1$ if $m \leq l$ and 0 otherwise. One can check that we have:[9]

$$c = \sum_{l=1}^{L} (c_{\pi_l} - c_{\pi_{l-1}}) c^{\pi_l};$$

$$C(N,c) = \sum_{l=1}^{L} (c_{\pi_l} - c_{\pi_{l-1}}) C(N, c^{\pi_l}) = 0 \text{ and}$$

$$C(S,c) \geq \sum_{l=1}^{L} (c_{\pi_l} - c_{\pi_{l-1}}) C(S, c^{\pi_l}) \text{ for all } S \subset N.$$

[9]By convention, we have $c_{\pi_0} = 0$.

It is easy to verify that for all $c \in \Gamma^0$, we have $c^{pop} = \sum_{l=1}^{L}(c^{\pi_l})^{pop}$. We show that the cooperative game induced by $(c^{\pi_l})^{pop}$ is concave (for any $l = 1, \ldots, L$).

Suppose by contradiction that $C(\cdot, c^{\pi_l})$ is not concave, then there exists $S \subset T \subset N$ and $k \in N \backslash T$ such that $C(S \cup \{k\}) - C(S) < C(T \cup \{k\}) - C(T)$.

(1) If $C(S \cup \{k\}) - C(S) = 1$ then we must have $C(T \cup \{k\}) - C(T) \leq 1$, because all edges under c^{π_l} have a cost of either 0 or 1. This contradicts the (non-concavity) assumption above.

(2) If $C(S \cup \{k\}) - C(S) = 0$ then there exists $i \in S$ such that $c_{ik} = 0$. It follows that $C(T \cup \{k\}) - C(T) \leq 0$, as $i \in S \subset T$, which contradicts the (non-concavity) assumption above.

(3) Otherwise, we must have $C(S \cup \{k\}) - C(S) < 0$. This implies that agent k allows at least one pair of agents (or an agent and the source) to connect to each other through a free path, something that they could not do without k. Formally, this means that there exist $i \in S_0$, $j \in S$ and $p_{ij}^{S\cup\{k\}}$ such that: (a) $M(p_{ij}^{S\cup\{k\}}) = 0$ and $[p_{ij}^{S\cup\{k\}}] \subseteq S \cup \{i, j, k\}$; (b) there is no path p_{ij}^{S} such that $M(p_{ij}^{S}) = 0$ and $[p_{ij}^{S}] \subseteq S \cup \{i, j\}$.

Since $C(S \cup \{k\}) - C(S) < C(T \cup \{k\}) - C(T)$, there also must exist p_{ij}^{T} such that $M(p_{ij}^{T}) = 0$ and $[p_{ij}^{T}] \subseteq T \cup \{i, j\}$. Next, consider the network $Z = p_{ij}^{S\cup\{k\}} \cup p_{ij}^{T}$. We will now show that there exist two nodes $l, m \in T_0$ for which Z contains $(p_{lm}, p'_{lm}) \in \mathcal{P}_{lm}$ such that $M(p_{lm} \cup p'_{lm}) = 0$ and $c_{lm} = 1$.

If $\left\{[p_{ij}^{S\cup\{k\}}] \cap [p_{ij}^{T}]\right\} = \{i, j\}$, then $Z \in \mathcal{P}_{ij}$, with $M(Z) = 0$ and $c_{ij} = 1$ by definition. Otherwise, since $k \in [p_{ij}^{S\cup\{k\}}]$ but $k \notin [p_{ij}^{T}]$, we have that there exist $l, m \in S$ (with $l = i$ and $m = j$ a possibility) such that we can write the path $p_{ij}^{S\cup\{k\}} = p_{il} \cup p_{lk} \cup p_{km} \cup p_{mj}$ and $p_{ij}^{S\cup\{k\}} = p_{il} \cup p_{lt} \cup p_{tm} \cup p_{mj}$, with $t \in T \backslash S$. Thus, $(p_{lk} \cup p_{km}, p_{lt} \cup p_{tm}) \in \mathcal{P}_{lm}$. By construction, $M(p_{lk} \cup p_{km} \cup p_{lt} \cup p_{tm}) = 0$. It remains to show that $c_{lm}^{\pi_l} = 1$. Suppose otherwise. Then $p'' \equiv p_{il} \cup p_{lm} \cup p_{mj}$ is such that $M(p'') = 0$ and $[p''] \subseteq S$, a contradiction.

Since we have obtained a contradiction in all 3 possible cases, we can conclude that $(c^{\pi_l})^{pop}$ is concave (for any $l = 1, \ldots, L$). Recalling that the sum operator preserves the concavity of games, one can then see that the cooperative game associated with c is also concave. It then follows that $Sh(C(\cdot, (c^{\pi_l})^{pop}) \in Core((c^{\pi_l})^{pop})$ and, hence, $\hat{y}^{pop}(c) \in Core(c)$.

ii) It is easy to verify that if i, j are c-symmetric, then they are also c^{pop}-symmetric. By the properties of the Shapley value, $y_i^{pop}(c^{pop}) = y_j^{pop}(c^{pop})$ and thus $y_i^{pop}(c) = y_j^{pop}(c)$.

A.4 Lemma 1

Fix $c \in \Gamma$ and $i \in N$. Statement *(i)* trivially follows from (4) and the definition of β_i^t. We prove the other statements below.

(ii) Let $\tilde{t} \in \{1, \ldots, T+1\}$ s.t. $\gamma_i^{\tilde{t}} = 1$ and observe first that $\sum_{t=1}^{\tilde{t}} \sum_{j \in \bar{N}^t(i)\backslash\{i, i^{\tilde{t}}\}} a_j^t(\hat{c}) \geq 0$. Indeed, noting from (4) that $a_j^t(\hat{c})$ may be negative [and equal to $-(n^t(i) -$

$1)\beta_i^t]$ for **at most** one $t \le \tilde{t}$, one may write:

$$\sum_{t=1}^{\tilde{t}} \sum_{j\in \bar{N}^t(i)\setminus\{i,i^{\tilde{t}}\}} a_j^t(\hat{c}) = \sum_{j\in \bar{N}^{\tilde{t}}(i)\setminus\{i,i^{\tilde{t}}\}} \left(\overbrace{a_j^{\tilde{t}}(\hat{c})}^{=\beta_i^{\tilde{t}}} + \overbrace{\sum_{\substack{t=1,\ldots,\tilde{t}-1 \\ \text{s.t. } j\in \bar{N}^t(i)\setminus\{i,i^{\tilde{t}}\}}} a_j^t(\hat{c})}^{\ge -(n^t(i)-1)\beta_i^t \ge -\beta_i^{\tilde{t}} \text{ by } (i)} \right) \ge 0.$$

Next, define

$$S_i^{\tilde{t}}(\hat{c}) \equiv \sum_{t=1}^{\tilde{t}} a_i^t(\hat{c}) + \sum_{t=1}^{\tilde{t}} \sum_{j\in \bar{N}^t(i)\setminus\{i,i^{\tilde{t}}\}} a_j^t(\hat{c}) = \sum_{j\in \bar{N}^{\tilde{t}}(i)\setminus i^{\tilde{t}}} a_j^{\tilde{t}}(\hat{c}) + \sum_{t=1}^{\tilde{t}-1} \sum_{j\in \bar{N}^t(i)\setminus i^{\tilde{t}}} a_j^t(\hat{c}).$$

If $i^{\tilde{t}} \notin \bar{N}^t(i)$ then we have from (4) that $\sum\limits_{j\in \bar{N}^t(i)\setminus i^{\tilde{t}}} a_j^t(\hat{c}) = \sum\limits_{j\in \bar{N}^t(i)} a_j^t(\hat{c}) = 0$, for any $t < \tilde{t}$. Moreover, if we instead have $i^{\tilde{t}} \in \bar{N}^t(i)$, it follows that $\gamma_i^t = 0$ and hence $\sum\limits_{j\in \bar{N}^t(i)\setminus i^{\tilde{t}}} a_j^t(\hat{c}) = 0$. Thus, in any case, one may simply write:

$$S_i^{\tilde{t}}(\hat{c}) = \sum_{j\in \bar{N}^{\tilde{t}}(i)\setminus i^{\tilde{t}}} a_j^{\tilde{t}}(\hat{c}). \tag{8}$$

It is now easy to examine the following two cases (recall that $\gamma_i^{\tilde{t}} = 1$).
Case 1: $i = i^{\tilde{t}}$.
Then (8) gives $S_i^{\tilde{t}}(\hat{c}) = \underbrace{-(n^{\tilde{t}}(i)-1)\beta_i^{\tilde{t}}}_{a_i^{\tilde{t}}(\hat{c})} + \sum\limits_{j\in \bar{N}^t(i)\setminus i} \beta_i^{\tilde{t}} = 0 = \alpha_i^{\tilde{t}}$.

Case 2: $i \ne i^{\tilde{t}}$.
In this case, (8) becomes $S_i^{\tilde{t}}(\hat{c}) = \sum\limits_{j\in \bar{N}^{\tilde{t}}(i)\setminus i^{\tilde{t}}} \underbrace{a_j^{\tilde{t}}(\hat{c})}_{=\beta_i^{\tilde{t}}} = \underbrace{(n^{\tilde{t}}(i)-1)\beta_i^{\tilde{t}}}_{\le \min\limits_{\substack{j\in \bar{N}^{\tilde{t}}(i) \\ j\ne i^{\tilde{t}}}} \alpha_i^{\tilde{t}}} \le \alpha_i^{\tilde{t}}$.

(iii) The last statement is a direct corollary of *(ii)* when $\tilde{t} = T+1$. Note that $\gamma_i^{T+1} = 1$ (since $i^{T+2} = 0 \ne i^{T+1}$) and $\hat{c}_{0i} \ge \alpha_i^{T+1} \equiv \min\limits_{j\notin \bar{N}^{T+1}(i)} \hat{c}_{ji}$.

A.5 Theorem 8

It is not difficult to see from the construction of $y^s(\hat{c}) = a^s(\hat{c})$ that we have $y_i(\hat{c}) = y_j(\hat{c})$ whenever two agents i and j are $\hat{c}$-symmetric. Using Theorem 1-(ii), we can hence claim that y^s satisfies Symmetry. Let us now show that y^s is stable as well. Using Theorem 1-(i), it suffices to show that $y^s(\hat{c}) \in Core(\hat{c})$, for any $c \in \Gamma$.

Fix $c \in \Gamma$ and recall that $\hat{c} \in \Gamma^0$. It is straightforward to see from the definition of y^s that $\sum\limits_{i\in N} y_i^s(\hat{c}) = 0$. Next, we sketch the proof of the following fact:[10] $\forall U \in 2^N \setminus \{\emptyset, N\}$, $\exists I_U^t \subset U$ (for any $t = 1, \ldots, T+1$) s.t. $U \subset \bigsqcup_{i \in I_U^t} \bar{N}^t(i)$ and $i' \notin \bar{N}^{T+1}(i)$ for any distinct $i, i' \in I_U^t$

$$\sum_{i\in U} y_i^s(\hat{c}) + \underbrace{\sum_{t=1}^{T+1} \sum_{i \in I_U^t} \sum_{j \in \bar{N}^t(i) \setminus (U \bigsqcup i^{T+1})} a_j^t(\hat{c})}_{\geq 0} \leq C(U, \hat{c}). \tag{9}$$

Note that, by Lemma 1-*(iii)*, the condition (9) holds for any $U = \{i\} \subset N$ (with $I_U^t = \{i\}$). Using induction, consider then $V \in 2^N \setminus \{\emptyset, N\}$ s.t. $|V| \geq 2$; and suppose that (9) is satisfied for any nonempty $U \subsetneq V$.

Let A^V be an mca for the problem $(0, V, \hat{c}^V)$, where $\hat{c}^V$ is the restriction of $\hat{c}$ to edges $(i,j) \in V_0 \times V$. In other words, we have $C(V, \hat{c}) = \hat{c}(A^V)$. Since V is finite, we may pick $l \in V$ that *has no successor* in A^V, that is, there exists **no** $m \in V$ s.t. $(l, m) \in A^V$. Let $k \in V_0$ be l's *predecessor* in A^V, that is to say, $(k, l) \in A^V$. Then, considering $U = V \setminus l$ and letting A^U be an mca for the problem $(0, U, \hat{c}^U)$, it is easy to see that $\hat{c}(A^V) = \hat{c}(A^U) + \hat{c}_{kl}$.

Given that $U \subsetneq V$, we know that (9) is satisfied for some sequence (I_U^t) such that $I_U^t \subset U$, $U \subset \bigsqcup_{i \in I_U} \bar{N}^{T+1}(i)$ and $i' \notin \bar{N}^{T+1}(i)$ for any distinct $i, i' \in I_U$. Let us show in what follows that (9) also holds for V and some sets (I_V^t).

• First, suppose that $k = 0$, i.e., $j \in P \equiv \{p \in N | (0,p) \in A^*\}$. Then we have:[11] $\hat{c}_{kl} = \hat{c}_{0l} = 0$ and $y_l^s(\hat{c}) + \sum_{t=1}^{T+1} \sum_{j \in \bar{N}^t(l) \setminus \{l, l^{T+1}\}} a_j^t(\hat{c}) \leq 0$ by the induction hypothesis. It is then easy to check that summing up this inequality and (9) for $U = V \setminus l$ gives the desired result for V, with $I_V^t = I_U^t \bigsqcup l$.

• Suppose now that $0 \neq k \in U$. We will assume, without loss of generality,[12] that $k \in I_U^t$ for any t. The following possibilities arise.

Case a: $k \notin \bar{N}^{T+1}(l)$ **or** $c_{0l} = 0$**.**

Then we have $\alpha_l^{T+1} \equiv \min_{k \notin \bar{N}^{T+1}(l)} \hat{c}_{il} \leq \hat{c}_{kl}$; and hence applying Lemma 1-*(iii)*,

$$y_l^s(\hat{c}) + \sum_{t=1}^{T+1} \sum_{j \in \bar{N}^t(l) \setminus \{l, l^{T+1}\}} a_j^t(\hat{c}) \leq \hat{c}_{kl}. \tag{10}$$

Taking the sum of (9) and (10), we get:

$$\left(\sum_{i \in U} y_i^s(\hat{c}) + y_l^s(\hat{c}) \right) + \sum_{t=1}^{T+1} \sum_{i \in I_U^t} \sum_{\substack{j \in \bar{N}^t(i) \\ j \notin U \bigsqcup i^{T+1}}} a_j^t(\hat{c}) + \sum_{t=1}^{T+1} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l, l^{T+1}}} a_j^t(\hat{c}) \leq \underbrace{\hat{c}(A^U) + \hat{c}_{kl}}_{=C(V, \hat{c})}.$$

[10] Observe that (9) is in fact stronger than coalitional rationality. We prove the important steps and leave some of the details (which are nonetheless explicitly mentioned) to the attention of the reader.

[11] Recall that $\hat{c}_e = 0$ for any $e \in A^*$, since we have $\hat{c} \in \Gamma^0$ and therefore $\hat{c}(A^*) = 0$.

[12] This assumption simplifies the notation; and it is not restrictive because Lemma 1-*(iii)* (the starting point of our induction) holds for any $i \in N$.

One can then check that (9) is satisfied for $V = U \bigsqcup l$ with $I_V^t = (I_U^t \setminus i^t) \bigsqcup l$ if $\exists i^t \in I_U^t \cap \bar{N}^t(l)$ and $I_V^t = I_U^t \bigsqcup l$ otherwise.

Case b: $k \in \bar{N}^{T+1}(l)$ **and** $c_{0i} > 0$**.**
Let us distinguish 2 final subcases (only the most complex one will be fully discussed).

Subcase b.1: $k \notin \bar{N}^1(l)$**.** Given that $k \in \bar{N}^{T+1}(l)$ and $\bar{N}^t(l)$ is weakly increasing in t, $\exists t^0 \in \{2, \dots, T\}$ s.t. $k \in \bar{N}^{t^0+1}(l)$ but $k \notin \bar{N}^t(l)$ if $t \leq t^0$. Assuming (**for now**) that $\exists t = 1, \dots, t^0$ s.t. $\gamma_l^t = 1$ and letting $\tilde{t}$ be the highest such t, we have $\alpha_l^{\tilde{t}} \equiv \min_{i \notin \bar{N}^{\tilde{t}}(l)} \hat{c}_{il} \leq \hat{c}_{kl}$. Hence, observing that $a_j^t(\hat{c}) = 0$ for any $j \in \bar{N}^t(l)$ and $t = \tilde{t}+1, \dots, t^0$ (if $t^0 > \tilde{t}$), one may use Lemma 1-*(ii)* to write

$$\sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l^{T+1}}} a_j^t(\hat{c}) = 0 + \sum_{t=1}^{\tilde{t}} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l^{T+1}}} a_j^t(\hat{c}) \leq \alpha_l^{\tilde{t}} \leq \hat{c}_{kl} \tag{11}$$

In addition, since $\bar{N}^t(k) = \bar{N}^t(l)$ **iff** $t \geq t^0 + 1$, we have:

$$\sum_{t=1}^{T+1} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin U \bigsqcup k^{T+1}}} a_j^t(\hat{c}) = \sum_{t=t^0+1}^{T+1} a_l^t(\hat{c}) + \sum_{t=t^0+1}^{T+1} \sum_{\substack{j \in \bar{N}^t(l) \\ j \notin V \bigsqcup l^{T+1}}} a_j^t(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin V \bigsqcup k^{T+1}}} a_j^t(\hat{c}).$$

Plugging (11) and $y_l^s(\hat{c}) = \sum_{t=1}^{t^0} a_l^t(\hat{c}) + \sum_{t=t^0+1}^{T+1} a_l^t(\hat{c})$ in the above, we get:

$$\begin{aligned}
\sum_{t=1}^{T+1} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin U \bigsqcup k^{T+1}}} a_j^t(\hat{c}) + \hat{c}_{kl} \geq{} & \overbrace{\sum_{t=t^0+1}^{T+1} a_l^t(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l^{T+1}}} a_j^t(\hat{c})}^{y_l^s(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l,\, l^{T+1}}} a_j^t(\hat{c})} + \\
& \sum_{t=t^0+1}^{T+1} \sum_{\substack{j \in \bar{N}^t(l) \\ j \notin V \bigsqcup l^{T+1}}} a_j^t(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin V \bigsqcup k^{T+1}}} a_j^t(\hat{c}) \\
\geq{} & y_l^s(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l,\, l^{T+1}}} a_j^t(\hat{c}) + \\
& \sum_{t=t^0+1}^{T+1} \sum_{\substack{j \in \bar{N}^t(l) \\ j \notin V \bigsqcup l^{T+1}}} a_j^t(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin V \bigsqcup k^{T+1}}} a_j^t(\hat{c}).
\end{aligned}$$

It thus follows that

$$\sum_{t=1}^{T+1} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin U \bigsqcup k^{T+1}}} a_j^t(\hat{c}) \geq y_l^s(\hat{c}) - \hat{c}_{kl} + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(l) \\ j \neq l,\, l^{T+1}}} a_j^t(\hat{c}) +$$

$$\sum_{t=t^0+1}^{T+1} \sum_{\substack{j \in \bar{N}^t(l) \\ j \notin V \bigsqcup l^{T+1}}} a_j^t(\hat{c}) + \sum_{t=1}^{t^0} \sum_{\substack{j \in \bar{N}^t(k) \\ j \notin V \bigsqcup k^{T+1}}} a_j^t(\hat{c}) \qquad (12)$$

Finally, recalling that $k \in I_U^t$ (for all t) and substituting (12) in (9) *applied to* $U = V \setminus l$, one can write (after reshuffling)

$$\sum_{i \in V} y_i^s(\hat{c}) + \sum_{t=1}^{T+1} \sum_{i \in I_V^t} \sum_{j \in \bar{N}^t(i) \setminus (V \bigsqcup i^{T+1})} a_j^t(\hat{c}) \leq \hat{c}(A^U) + \hat{c}_{kl},$$

where $I_V^t = (I_U^t \bigsqcup l) \setminus k$ for $t \geq t^0$, $I_V^t = (I_U^t \setminus i^t) \bigsqcup l$ if $t \leq t^0$ and $\exists i^t \in I_U^t$ s.t $l \in \bar{N}^t(i^t)$, and $I_V^t = I_U^t \bigsqcup l$ otherwise.

Note that the (omitted) case where $\gamma_l^t = 0$ for any $t \leq t^0$ is included below.

Subcase b.2: $k \in \bar{N}^1(l)$ or $\gamma_l^t = 0$ for any $t \leq t^0$. In this subcase, one has to use the same method, with some simplifications. Namely, if $k \in \bar{N}^1(l)$, then there is no need to use $\tilde{t}$; and also $a^t(\hat{c}) = 0$ if $\gamma_l^t = 0$ for any $t \leq t^0$. Either way, one obtains a stronger result than under Subcase b.1:

$$\sum_{i \in V} y_i^s(\hat{c}) + \sum_{t=1}^{T+1} \sum_{i \in I_V^t} \sum_{j \in \bar{N}^t(i) \setminus (V \bigsqcup i^{T+1})} a_j^t(\hat{c}) \leq \hat{c}(A^U) \leq \hat{c}(A^U) + \hat{c}_{kl} = \hat{c}(A^V).$$

Our induction argument is now complete. To conclude the proof, it remains to show that $\sum_{t=1}^{T+1} \sum_{i \in I_U^t} \sum_{j \in \bar{N}^t(i) \setminus (U \bigsqcup i^{T+1})} a_j^t(\hat{c}) \geq 0$ for all $U \subsetneq N$. This result is obtained by using the same procedure as in the proof of Lemma 1-*(ii)*.