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Help us to help you: how consumer data can alter quality races

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Abstract

Recent technological changes have made it easy for firms to collect data on their consumers, which in turns allows them to improve the efficiency of their R&D. We explore the strategic interaction that occurs when two firms compete in a vertically-differentiated market to acquire this data and invest in R&D to set the quality of their product. Among our results, we find that if the initial quality lead is not too large, there exists equilibria where the laggard is able to reverse the lead by being particularly aggressive in acquiring this consumer data. While total welfare is higher when the initial leader maintains its lead, consumers prefer leapfrogging.

1 Introduction

Recent technological changes have made it easy for firms to collect data on their consumers and elicit feedback from them. There are many potential applications for this data (Varian (2014)), chief among them is using the collected data to improve the efficiency of R&D, by providing improvements to the product that correspond to the needs of the consumers. Google, for example, is known for releasing products to the public before they are done in order to get such feedback.¹

Given the value of this data, we can expect that firms will alter their behavior to acquire this data. Surprisingly, the economic literature is very much silent on this. An exception is a working paper by Baye and Sapi (2014) that looks at the decision to invest in data-collecting devices. We focus on the impact that this consumer data has on price competition, as the firms compete to acquire the data, which in turn will be used to increase the efficiency of R&D in the following period.

In a two-period model where firms amass data and invest in R&D in the first period and compete with the updated quality in the second period, we show that multiple equilibria can occur, including some where the initial laggard in terms of quality can leapfrog its competitor. We get this result even though our model is completely deterministic and gives no advantage to the laggard.

The vertical differentiation competition at the heart of our framework is built on Tirole (1988). The high-quality firm earns more profit than its rival (a result that holds for more general frameworks, see Lehmann-Grube (1997)), giving incentives to firms to fight to become that second-period leader. On the other hand, saving the investment cost and differentiating the product generate the incentive for firms to stay behind. It thus becomes a coordination game, in which it might be better to act less aggressively and maximize differentiation of the goods in the second period if the rival firm acts aggressively to become the leader. We are able to solve analytically for all pure-strategy subgame perfect Nash equilibria.

The proposed model is part of the literature on vertical differentiation with endogenous quality initiated by Shaked and Sutton (1982). Our main distinction is that the firms can make their quality improvement technology more efficient by fighting for customers and the data they generate. Common conclusions of vertical differentiation models include different qualities observed in equilibrium and a high-quality advantage. There are various predictions, depending on the exact set-up, if laggards (Grossman and Helpman (1991), Segerstrom (1991)) or leaders (Segerstrom and Zolnerek (1999)) have more incentives to invest in R&D. As a contrast, we show that both are possible, unless the initial quality lead is so large, in which case the leader is the only one to invest.

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¹“Small-batch innovation: What the Lean Startup model can do for your R&D”, Melissa Campeau, Financial Post, October 23, 2014.

More generally, the model is part of the large literature on price competition in dynamic oligopolies. Maskin and Tirole (1988) provide a general framework. The many applications of this model include dynamic oligopolies with learning-by-doing (Cabral and Riordan (1994)), network and compatibility or switching costs effects (Katz and Shapiro (1994), Farrell and Klemperer (2007)). As in the above models, our model yields possible pricing below marginal cost.

The literature on lead reversals include Fudenberg et al. (1983) and Harris and Vickers (1987), who study possible leapfrogging in patent races, and Budd et al. (1993), who build a model to study increasing asymmetry in a dynamic oligopoly where competition depends on effort levels (R&D, advertising, etc.) of firms. Athey and Schmutzler (2001) build a general model of dynamic competition with investment which encompasses many of the previous models. The general conclusions in the above models are that it requires either a large advantage in terms of investment effectiveness for the follower or significant uncertainty to obtain leapfrogging. In general, the leading firm tends to increase its dominance.

The paper is structured as follows. We first describe the model set - up in Section 2. We then derive the multiple equilibria in Section 3. Section 4 demonstrates the conditions for each equilibria and Section 5 examines the welfare consequences. The final section concludes the paper.

2 The model

We study a two-period duopoly model. The two firms compete in price over two periods in this vertically differentiated market. In the first period, firms have the opportunity to invest to improve the quality of their good in the second period. Given that sales in the first period allow the producers to acquire either direct feedback (consumer evaluation and comments) or indirect feedback (data on usage and bugs, ability to do test designs, etc.), the effectiveness of investment is increasing with first-period sales.

More precisely, in period 1, the development phase, p_i , x_i and π_i denote, respectively, the price, quantity and profits of firm i . Let y_i be the investment to improve quality. We suppose that the marginal cost of production is 0.² Let v_i be the quality of good i in period 1. It can be interpreted as the reserve price of the agent with the highest valuation for quality. Let v_a and v_b be given. We suppose that $\Delta v \equiv v_a - v_b \geq 0$; that is, firm a has an initial lead in quality.

In period 2, the maturity phase, p'_i , x'_i , v'_i and π'_i denote the key variables. In particular, the quality of the product of firm i improves in the following manner: $v'_i = v_i + \sqrt{y_i} \sqrt{(1 + x_i)}$, for $i = a, b$. The marginal productivity of the investment is thus decreasing in the investment itself and increasing with the first-period sales. Let $\Delta v' = |v'_a - v'_b|$. Time is discounted by $\beta \in [0, 1]$. Let $\Pi_i = \pi_i + \beta \pi'_i$ be the present value of profits of firm i .

The price competition in each period is modeled as a classic vertically differentiated model: each consumer k is characterized by a marginal willingness to pay for this value γ_k , uniformly distributed over the support $[0, 1]$, with the population size normalized to 1. Let u_i^k be the utility of consumer k when he buys the product of firm i : $u_i^k = u + \gamma_k v_i - p_i$. A consumer buys one unit or buys nothing. If he buys nothing, his utility is 0. The good is non-durable and the valuation u is high enough so that each agent buys one unit in each period.

The timing of the game is as follows: at the beginning of period 1, firms observe v_a and v_b and then simultaneously choose p_i and y_i . At the beginning of period 2 they observe v'_a and v'_b and then simultaneously choose p'_i .

3 Maturity phase and potential equilibria

The equilibrium concept is subgame perfect Nash-equilibrium, so we proceed by backward induction and start by looking at the second period.

3.1 Maturity phase

We first solve the second period price competition. Suppose that $\Delta v'$ is known. We denote by L the leading firm (in terms of quality) in the maturity phase, with F being the follower. Note that even though we assume that firm

²Since marginal cost is equal to zero, negative prices allow for pricing under marginal cost. The assumption on marginal cost eliminates variables and allows for a clearer exposition.

a has the lead initially ($v_a \geq v_b$), this order need not hold in the second period. The profits for the leader and the follower in the second stage are $\pi'_i = p'_i x'_i$ for $i = L, F$, with $x'_L = 1 - \frac{p'_L - p'_F}{\Delta v'}$ and $x'_F = \frac{p'_L - p'_F}{\Delta v'}$.³

Lemma 1. *In the maturity phase (period 2), there exist a unique pure strategy Nash equilibrium $(p_L^*, p_F^*) = \left(\frac{2\Delta v'}{3}, \frac{\Delta v'}{3}\right)$. It is such that $\pi_L^* = \frac{4\Delta v'}{9}$ and $\pi_F^* = \frac{\Delta v'}{9}$.*

The proof is easily obtained, given that in the second period, since there are no further investments in quality, the competition is as in a classic vertically differentiated market. Each firm maximizes its own profit with respect to price and simultaneously solving the resulting best response functions yields the above result: the leading firm sells twice as much as the follower, making four times the profits of the follower.

3.2 Development phase: 4 potential types of equilibria

In the first stage, firms maximize the present value of profits $\Pi_i = \pi_i + \beta\pi'_i$. Notice that in the maturity phase the profits of both firms are increasing in the difference in quality, meaning that firms have incentives to vertically differentiate. However, since the leader makes four times as much profits as the follower, the follower might challenge the lead. We thus obtain different types of potential equilibria, depending whether or not firms try to conserve/take the lead. What is clear is that firms will behave differently if they want to be in the lead (high quality if in lead, low quality if not). Further, if a firm tries to be the leader but fails, it cannot be a Nash equilibrium, as it would prefer to change its strategy. The potential equilibria type also depends on corner solutions: since sales are always between 0 and 1, there is a limit to the advantage a firm can obtain with its first period sales.

We first consider equilibria in which firm a keeps the lead. In this case profits are:

$$\begin{aligned}\Pi_a^C &= p_a \left(\frac{\Delta v + p_b - p_a}{\Delta v} \right) + \frac{4\beta}{9} \left(\Delta v + \sqrt{y_a} \sqrt{\frac{2\Delta v + p_b - p_a}{\Delta v}} - \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}} \right) - y_a, \\ \Pi_b^C &= p_b \left(\frac{p_a - p_b}{\Delta v} \right) + \frac{\beta}{9} \left(\Delta v + \sqrt{y_a} \sqrt{\frac{2\Delta v + p_b - p_a}{\Delta v}} - \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}} \right) - y_b.\end{aligned}$$

Given that firm a keeps the lead, it is obvious that it is optimal for firm b to choose $y_b = 0$. We then obtain the following two equilibria types:⁴

- *Lead consolidation with market split (CS):* $0 < x_a < 1$, $0 < x_b < 1$, $y_a > 0$ and $y_b = 0$. Solving the FOCs with respect to p_a , p_b and y_a yields equilibrium values $p_a^{CS} = \frac{2\Delta v}{3} - \frac{7\beta^2}{243}$, $p_b^{CS} = \frac{\Delta v}{3} - \frac{2\beta^2}{243}$, $y_a^{CS} = \frac{20\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}$.
- *Lead consolidation with market capture (CC):* $x_a = 1$, $x_b = 0$, $y_a > 0$ and $y_b = 0$. Solving the FOC with respect to y_a yields equilibrium values $p_a^{CC} = p_b^{CC} = P^{CC}$, $y_a^{CC} = \frac{8\beta^2}{81}$.

We next consider (potential) equilibria where firm b leapfrogs firm a in terms of quality. For this case profits are:

$$\begin{aligned}\Pi_a^L &= p_a \left(\frac{\Delta v + p_b - p_a}{\Delta v} \right) + \frac{\beta}{9} \left(-\Delta v - \sqrt{y_a} \sqrt{\frac{2\Delta v + p_b - p_a}{\Delta v}} + \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}} \right) - y_a, \\ \Pi_b^L &= p_b \left(\frac{p_a - p_b}{\Delta v} \right) + \frac{4\beta}{9} \left(-\Delta v - \sqrt{y_a} \sqrt{\frac{2\Delta v + p_b - p_a}{\Delta v}} + \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}} \right) - y_b.\end{aligned}$$

Given that firm b takes the lead, it is obvious that it is optimal for firm a to choose $y_a = 0$. We then obtain the following two equilibria types:

- *Leapfrogging with market split (LS):* $0 < x_a < 1$, $0 < x_b < 1$, $y_a = 0$ and $y_b > 0$. Solving the FOCs with respect to p_a , p_b and y_a yields equilibrium values $p_a^{LS} = \frac{2\Delta v}{3} - \frac{2\beta^2}{243}$, $p_b^{LS} = \frac{\Delta v}{3} - \frac{7\beta^2}{243}$, $y_b^{LS} = \frac{16\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}$.
- *Leapfrogging with market capture (LC):* $x_a = 0$, $x_b = 1$, $y_a = 0$ and $y_b > 0$. Solving the FOC with respect to y_b yields equilibrium values $p_a^{LC} = P^{LC}$, $p_b^{LC} = P^{LC} - \Delta v$, $y_b^{LC} = \frac{8\beta^2}{81}$.

³If $\Delta v' = 0$, we assume that whoever has the lowest price captures the full market and sells one unit. If both prices are equal, the market is split equally.

⁴It is easy to verify that the firm that wants to be the leader in the next period will always choose to sell some units in the first period.

4 Equilibria

We can now derive the main result.

Theorem 1. *We have the following equilibria, given initial values of Δv and β .*

- i) *If $\Delta v \geq \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15} \right)$, the unique equilibrium is the lead consolidation equilibrium with market split.*
- ii) *If $\frac{5\beta^2}{81} \leq \Delta v < \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15} \right)$, the set of equilibria consists of the lead consolidation equilibrium with market split and the leapfrogging equilibrium with market split.*
- iii) *If $\frac{5\beta^2}{162} \leq \Delta v < \frac{5\beta^2}{81}$, the set of equilibria consists of lead consolidation equilibria with market capture, with $P^{CC} \in \left[-\frac{4\beta^2}{81} + \Delta v, \frac{\beta^2}{81} \right]$ and the leapfrogging equilibrium with market split.*
- iv) *If $0 \leq \Delta v < \frac{5\beta^2}{162}$, the set of equilibria consists of lead consolidation equilibria with market capture, with $P^{CC} \in \left[-\frac{4\beta^2}{81} + \Delta v, \frac{\beta^2}{81} \right]$ and the leapfrogging equilibrium with market capture, with $P^{LC} \in \left[-\frac{4\beta^2}{81} + 2\Delta v, \frac{\beta^2}{81} \right]$.*

Proof is in the appendix. While the proof is space-intensive, it follows a simple procedure. For each type of equilibria, we find conditions for equilibrium by making sure that the resulting sales and quality differences are compatible with the given type, that firms do not have incentives to sell more or less (without changing their investment) and that firms have no incentives to change their strategy with respect to the lead.

Figure 1 summarizes the result of Theorem 1. Thus, if the initial quality lead is large (and/or if firms are impatient), the only equilibria is one such that firm a consolidates its lead. If $\Delta v < \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15} \right)$, we always have some equilibria in which firm b manages to leapfrog its competitor. Given the relatively small lead, a particularly aggressive pricing strategy in the first period allows firms to gain valuable consumer data, which in turns improves the efficiency of its investment. Faced with an aggressive competitor, firm a will optimally decide to ease up on competition and not invest in improving its quality, in order to become the low-quality alternative in the next period, reaping the benefits of quality differentiation.

This is not the only equilibrium however, as for all values of the parameters, there is also an equilibrium in which firm a consolidates its lead. If Δv is relatively high (or again, if firms are impatient), firms share the market in the first period. If firm a consolidates its lead, it is not worthwhile to engage in cutthroat competition in the first period to attract all consumers, as the cost to acquire them would not be compensated for by the gains in investment efficiency, and subsequently, in the increased quality lead in the second period. In cases where firm b takes the lead, the reason for market sharing is not the same: since there is a (relatively) large advantage for firm a in the first period, acquiring new consumers is especially costly. Firm b needs to offer a discount of Δv to attract all consumers. Even though it would lead to a larger differentiation in the second period, it is not cost-effective to attract after more consumers in the first period. For smaller initial leads, these effects are reversed and firms do compete for all consumers in the first period.

Figure 2 describes the resulting quality difference in the second period as a function of the initial quality lead and whether or not firm a keeps the lead. In all cases the quality lead in the second period is larger than in the first. The curves are linear for small values of Δv , as it corresponds to the market capture cases. For larger values of Δv , the curves become non-linear as we move to the market splitting equilibria, where investments are less than maximal. It is not surprising to see that with an equilibrium with lead consolidation the second-period quality lead is increasing in Δv . For the leapfrogging case, it is decreasing in Δv , as not only the size of the leap increases, but the investment stays constant (for market capture equilibria) or declines (for market splitting equilibria). We reach a point (in our example around $\Delta v = 0.124$) where leapfrogging equilibria are no longer feasible.

It is easy to see that if data acquisition had no impact on R&D effectiveness, the equilibrium prices in period 1 would be $p_a^* = \frac{2\Delta v}{3}$ and $p_b^* = \frac{\Delta v}{3}$. If we consider equilibria with market splitting, it is obvious that the competition to acquire data from consumers pushes the prices down from these levels. However, in the equilibria with market capture, it is not as straightforward. Figure 3 shows the equilibrium prices when we are in the lead consolidation equilibria and with $\beta = 1$. When $\Delta v \leq 0.06$, we have a market capture equilibria, and the region in gray represents all possible equilibrium prices. Otherwise, we have a market splitting equilibria. The dashed line represents the price of firm a when data acquisition has no impact. A first observation is that pricing under marginal cost (here negative prices) is possible if the initial quality lead is small. This is not particularly surprising, as there is a strong gain to differentiate in the second period and a high-level of competition to become the leader. A second and

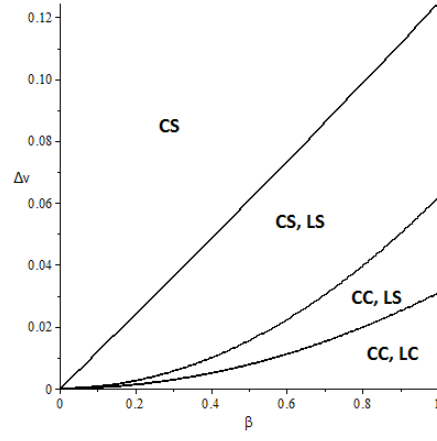


Figure 1: Types of equilibria as a function of Δv and β

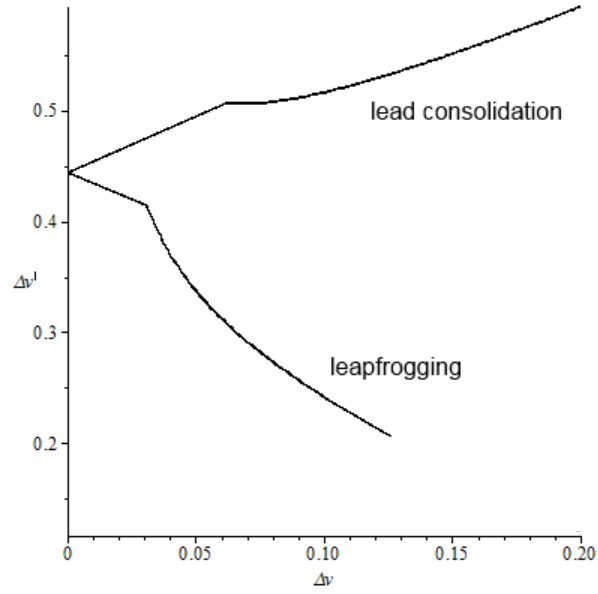


Figure 2: Quality lead in the second period ($\Delta v'$) as of a function of initial quality lead(Δv), with $\beta = 1$

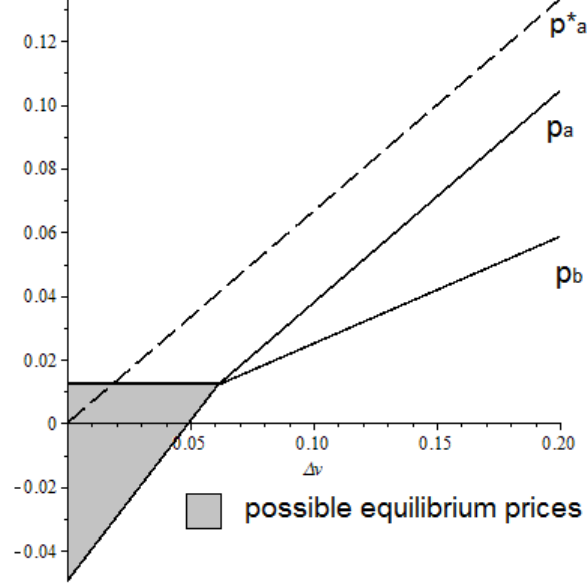


Figure 3: Equilibrium prices in lead consolidation equilibria as a function of initial quality lead(Δv), with $\beta = 1$

more surprising observation is that for very small values of Δv , the equilibrium price for firm a might be larger than without R&D-improving data acquisition. This can be explained by the fact that while firms have strong incentives to become the leader, their first goal is to differentiate themselves, which might lead one of the firms to be completely non-aggressive in terms of pricing in order to guarantee that the rival will acquire the maximum amount of data and will maximize differentiation in the next period. Similar observations can be made for the leapfrogging equilibria.

5 Welfare consequences

If Δv is large enough, the only equilibria is the lead consolidation equilibrium. However, for low values of Δv , we have the coexistence of consolidation and leapfrogging equilibria. Which equilibrium arises is an issue of coordination. If the government can intervene to guide coordination, which equilibrium should it facilitate?

We examine this question from the perspective of the total welfare criteria. The total welfare follows as the sum of consumer surplus and profits over two periods, where consumer surplus is the multiplication of the average utility and the total number of consumers. Given that a government might place different weights on consumers and firms, we also examine separately the impacts of the equilibrium type on consumer and producer surpluses.

Let I and I' represent, respectively, the agent indifferent between buying from the low and high-quality firms in the first and second period. Agents with values for quality in $[0, I]$ buy from the low-quality firm and those in $[I, 1]$ buy from the high-quality firm. In the lead consolidation case we have that $I = \frac{p_a - p_b}{v_a - v_b}$ and $I' = \frac{p'_a - p'_b}{v'_a - v'_b}$.

Let CS and CS' represent, respectively, the consumer surplus in periods 1 and 2. In the lead consolidation case we have that

$$\begin{aligned} CS &= (1 - I)(u + \left(\frac{1 + I}{2}\right)v_a - p_a) + I(u + \frac{I}{2}v_b - p_b) \\ CS' &= (1 - I')(u + \left(\frac{1 + I'}{2}\right)v'_a - p'_a) + I'(u + \frac{I'}{2}v'_b - p'_b). \end{aligned}$$

Let $ACS = CS + \beta CS'$ be the aggregate consumer surplus and $W = ACS + \Pi_a + \Pi_b$ be total welfare. Profit values for all equilibria types can be found in the proof of Theorem 1. In the lead consolidation with market split

(CS) equilibrium, we have

$$\begin{aligned} ACS^{CS} &= \frac{-13122v_a^2(1+\beta) + 85293v_av_b(1+\beta) - 2268(v_a - v_b)\beta^2 + 118098u(v_a - v_b)(1+\beta) - 35\beta^4 - 72171v_b^2(1+\beta)}{118098(v_a - v_b)} \\ W^{CS} &= \frac{52488v_a^2(1+\beta) - 45927v_av_b(1+\beta) + 10530(v_a - v_b)\beta^2 + 118098u(v_a - v_b)(1+\beta) + 95\beta^4 - 6561v_b^2(1+\beta)}{118098(v_a - v_b)}. \end{aligned}$$

In the lead consolidation with market capture (CC) equilibrium, we have

$$\begin{aligned} ACS^{CC} &= u(1+\beta) + \frac{v_a}{2} - P^{CC} + \frac{11\beta v_b}{18} - \frac{\beta v_a}{9} - \frac{4\beta^2}{81} \\ W^{CC} &= u(1+\beta) + \frac{v_a}{2} + \frac{\beta v_b}{18} + \frac{4\beta v_a}{9} + \frac{8\beta^2}{81}. \end{aligned}$$

In the leapfrogging equilibria, firm 2 is the high-quality firm in period 2, and thus $I' = \frac{p'_b - p'_a}{v'_b - v'_a}$. CS is as before but

$$CS' = (1 - I')(u + \left(\frac{1 + I'}{2}\right)v'_b - p'_b) + I'(u + \frac{I'}{2}v'_a - p'_a).$$

In the leapfrogging with market split (LS) equilibrium, we have that

$$\begin{aligned} ACS^{LS} &= \frac{-13122(v_a^2 - \beta v_b^2) + 85293v_av_b(1 - \beta) - 2106(v_a - v_b)\beta^2 + 118098u(v_a - v_b)(1 + \beta) - 35\beta^4 - 72171(v_b^2 - \beta v_a^2)}{118098(v_a - v_b)} \\ W^{LS} &= \frac{6966(v_a - v_b)\beta^2 - 45927v_av_b(1 - \beta) + 118098u(v_a - v_b)(1 + \beta) + 95\beta^4 - 6561(v_b^2 - \beta v_a^2) + 52488(v_a^2 - \beta v_b^2)}{118098(v_a - v_b)}. \end{aligned}$$

In the leapfrogging with market capture (LC) equilibria we have that

$$\begin{aligned} ACS^{LC} &= u(1+\beta) - \frac{1}{2}v_b - P^{LC} + v_a + \frac{11}{18}\beta v_a - \frac{1}{9}\beta v_b - \frac{4}{81}\beta^2 \\ W^{LC} &= u(1+\beta) + \frac{v_b}{2} + \frac{\beta v_a}{18} + \frac{4\beta v_b}{9} + \frac{8\beta^2}{81}. \end{aligned}$$

We are now ready to compare the various equilibria. Notice that in the lead consolidation with market capture equilibria and the leapfrogging with market capture equilibria consumer surplus depends on the equilibrium prices. For simplicity, we assume that in all cases firms coordinate on the highest compatible price, $P^{CC} = P^{LC} = \frac{\beta^2}{81}$.

Theorem 2. *For all values of the parameters where we have coexistence of the lead consolidation and leapfrogging equilibria, we have that:*

- i) *the total welfare is higher with the lead consolidation than with the leapfrogging equilibria, regardless of the values of P^{CC} and P^{LC} ;*
- ii) *the consumer surplus is higher with the leapfrogging than with the lead consolidation equilibria, if $P^{CC} = P^{LC} = \frac{\beta^2}{81}$.*
- iii) *the producer surplus is higher with the consolidation than with the leapfrogging equilibria, if $P^{CC} = P^{LC} = \frac{\beta^2}{81}$.*

Proof. From Theorem 1, we have three relevant intervals to consider. If $\frac{5\beta^2}{81} \leq \Delta v < \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15} \right)$ (interval

1) we need to compare CS and LS. If $\frac{5\beta^2}{162} \leq \Delta v < \frac{5\beta^2}{81}$ (interval 2) we need to compare CC and LS. Finally, if $\frac{5\beta^2}{162} \leq \Delta v < \frac{5\beta^2}{81}$ (interval 3) we need to compare CC and LC.

i) In interval 1, subtracting W^{LS} from W^{CS} yields: $W^{CS} - W^{LS} = \frac{\beta(567(v_a - v_b) + 44\beta)}{1458} > 0$. In interval 3, subtracting W^{LC} from W^{CC} yields: $W^{CC} - W^{LC} = \frac{(v_a - v_b)(9 + 7\beta)}{18} > 0$. In interval 2, subtracting W^{LS} from W^{CC} yields: $W^{CC} - W^{LS} = \frac{(6561 + 45927\beta)(v_a - v_b)^2 + 4698\beta^2(v_a - v_b) - 95\beta^4}{118098(v_a - v_b)} > 0$ for $\frac{5\beta^2}{162} < \Delta v < \frac{5\beta^2}{81}$. Thus, total welfare is always higher with the lead consolidation equilibrium.

ii) In interval 1, subtracting $ACSCS$ from $ACSL^S$ yields: $ACSL^S - ACSCS = \frac{\beta(1053(v_a - v_b) + 2\beta)}{1458} > 0$. In interval 3, subtracting $ACSCC$ from $ACSL^C$ yields: $ACSL^C - ACSCC = \frac{(v_a - v_b)(9 + 13\beta)}{18} > 0$. In interval 2, substituting $P^{CC} = \frac{\beta^2}{81}$ into $ACSL^S - ACSCC$ yields: $-\frac{(72171 - 85293\beta)(v_a - v_b)^2 - 5184\beta^2(v_a - v_b) + 35\beta^4}{118098(v_a - v_b)} > 0$ for $\frac{5\beta^2}{162} < \Delta v < \frac{5\beta^2}{81}$. Thus, consumers prefer the leapfrogging to the lead consolidation equilibria.

iii) Follows from i) and ii). $\square$

In the leapfrogging equilibria, the average price level is lower and the quality level is higher than with the lead consolidation. This generates the result that the leapfrogging improves the consumer surplus compared to the case with the lead consolidation. Yet, the resulting lower price and higher investment level hurt profits. The loss in profits dominates the increase in consumer surplus so that total welfare is lower with leapfrogging than with lead consolidation. If a government puts equal weights on consumer and producer surpluses, it will try to encourage the lead consolidation equilibrium. However, if it puts enough weight on the consumer surplus, it will instead push for leapfrogging.

6 Conclusion

We model a two – period duopoly in which two asymmetric firms have the control over price and R&D investments, with the effectiveness of R&D increasing with data acquired from consumers. While we always have equilibria where the initial leader consolidates its lead, we also have equilibria where there is leapfrogging. Our model is deterministic and the follower has no *a priori* advantage over the leader. In fact, in our model, the the follower starts with a disadvantage: sales, and thus data acquisition, are more difficult in the first period. The addition of R&D-improving data acquisition to quality races allows us to obtain results that differ considerably from the literature.

From a pure total welfare point of view, lead consolidation is more desirable than leapfrogging, but results in a transfer from consumers to firms. If there is a bias towards consumer surplus then coordination on the leapfrogging equilibria should be encouraged.

Results are obtained in a model that supposes full market coverage: consumers always buy from either firm a or b . If we remove this assumption we lose the ability to solve analytically. However, intuitively, results should not differ: the follower will now make less than four times the profits of the leader in the second period (as it now loses consumers to the outside option), giving firms more incentives to become the leader. While consumers will be more difficult to attract in the first period, aggressive pricing will be more efficient for firm b than firm a , as it is then able to gain consumers who have low-value for quality (who bought nothing) and relatively high-value for quality (who bought from firm a). By opposition a low price for firm a will only allow to gain consumer on one side (relatively low value for quality, who bought from firm b). This asymmetry will allow for leapfrogging.

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Appendix

A Proof of Theorem 1

For the four types of potential equilibria we derive the conditions on Δv and β (and P, if applicable) that make sure that we have no profitable deviations.

1) lead consolidation equilibrium with market split

We need to find conditions such that 1) we have $0 \leq x_a^{CS} \leq 1$, 2) firms a and b have no incentives to deviate by moving to a corner solution and 3) firm b has no incentives to take the lead.

Step 1.1: verifying that $0 \leq x_a^{CS} \leq 1$

We have that $x_a^{CS} = \frac{\Delta v + p_b - p_a}{\Delta v}$. Substituting the equilibrium values, we obtain $x_a^{CS} = \frac{2}{3} + \frac{5\beta^2}{243\Delta v}$. It is easy to see that $x_a^{CS} \geq 0$ is trivially satisfied. We have that $x_a^{CS} \leq 1$ if

$$\Delta v \geq \frac{5\beta^2}{81} \quad (1)$$

Step 1.2: verifying that firms have no incentives to move to a corner solution in the first period.

At the equilibrium values, we have that

$$\begin{aligned} \Pi_a^{CS} &= \frac{4\Delta v}{9}(1+\beta) + \frac{56\beta^2}{729} + \frac{25\beta^4}{59049\Delta v} \\ \Pi_b^{CS} &= \frac{\Delta v}{9}(1+\beta) + \frac{23\beta^2}{729} + \frac{40\beta^4}{59049\Delta v} \end{aligned}$$

Step 1.2a: Firm a has no sales

It then chooses $p'_a = p_b + \Delta v$ and $y'_a = \frac{4\beta^2}{81}$. Its profits become $\Pi_a^{1.2a} = \frac{4\beta}{9} \left(\Delta v + \frac{2\beta}{9} \right) - \frac{4\beta^2}{81}$. It can be verified that $\Pi_a^{CS} - \Pi_a^{1.2a} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 1.2a.

Step 1.2b: Firm a captures the market

It then chooses $p'_a = p_b$ and $y'_a = \frac{8\beta^2}{81}$. Its profits become $\Pi_a^{1.2b} = \frac{\Delta v}{3} - \frac{2\beta^2}{243} + \frac{\beta}{9} \left(\Delta v + \frac{4\beta}{9} \right) - \frac{16\beta^2}{81}$. It can be verified that $\Pi_a^{CS} - \Pi_a^{1.2b} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 1.2b.

Step 1.2c: Firm b captures the market

It then chooses $p'_b = p_a - \Delta v$. Its profits become $\Pi_b^{1.2c} = \left(-\frac{\Delta v}{3} - \frac{7\beta^2}{243} \right) + \frac{\beta}{9} \left(\Delta v + \sqrt{\frac{20\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}} \right)$. It can be verified that $\Pi_b^{CS} - \Pi_b^{1.2c} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 1.2c.

Step 1.2d: Firm b has no sales

It then chooses $p'_b = p_a$. Its profits become $\Pi_b^{1.2d} = \frac{\beta}{9} \left(\Delta v + \sqrt{2} \sqrt{\frac{20\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}} \right)$. It can be verified that $\Pi_b^{CS} - \Pi_b^{1.2d} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 1.2d.

Step 1.3: verifying that firm b has no incentives to take the lead

Step 1.3a: Firm b takes the lead and captures the market

It then chooses $p'_b = p_a - \Delta v$ and $y'_b = \frac{8\beta^2}{81}$. Its profits become $\Pi_b^{1.3a} = \left(-\frac{\Delta v}{3} - \frac{7\beta^2}{243} \right) + \frac{\beta}{9} \left(-\Delta v - \sqrt{\frac{20\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}} + \frac{4\beta}{9} \right) - \frac{8\beta^2}{81}$. It can be verified that $\Pi_b^{CS} - \Pi_b^{1.3a} \geq 0$ for all values of β and Δv . There are no beneficial deviations of type 1.3a.

Step 1.3b: Firm b takes the lead and shares the market

The profit becomes

$$\Pi_b^{1.3b} = p_b \left(\frac{p_a - p_b}{\Delta v} \right) + \frac{4\beta}{9} \left(-\Delta v - \sqrt{y_a} \sqrt{\frac{2v + p_b - p_a}{\Delta v}} + \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}} \right) - y_b$$

It then chooses p'_b and y'_b , higher order polynomials that can take complex values. Numerical analysis allows us to conclude (by finding the correct local maximum) that in the region there are no profitable deviations of type 1.3b.

Our only constraint to obtain a lead consolidation equilibrium with market split is thus to satisfy condition (1).

2) Lead consolidation equilibria with market capture

We need to show that: 1) firms a and b have no incentives to deviate by moving away from the corner solution and 2) firm b has no incentives to deviate by trying to take the lead

Step 2.1: verifying that firms have no incentives to move away from the corner solution in the first period.

At the equilibrium values, we have that

$$\begin{aligned}\Pi_a^{CC} &= P + \frac{4\beta \Delta v}{9} + \frac{8\beta^2}{81} \\ \Pi_b^{CC} &= \frac{\beta \Delta v}{9} + \frac{4\beta^2}{81}.\end{aligned}$$

Step 2.1a: Firm a shares the market

It then chooses $p'_a = \frac{P+\Delta v}{2} - \frac{2\beta^2}{81}$ and $y_b = \frac{2\beta^2}{6561\Delta v}(243\Delta v + 81P + 4\beta^2)$. Replacing, we have that $\Pi_a^{CC} - \Pi_a^{2.1a} = \frac{-(81P+4\beta^2-81\Delta v)^2}{26244\Delta v} < 0$. Firm a would thus always have incentives to deviate.

However, this deviation is feasible only if the quantity sold is between 0 and 1. At the optimal deviation values, we have that $x_a^{2.1a} = \frac{P}{2\Delta v} + \frac{2\beta^2}{81\Delta v} + \frac{1}{2}$. We have that $0 \leq x_a^{2.1a} \leq 1$ if $-\frac{4\beta^2}{81} - \Delta v \leq P \leq -\frac{4\beta^2}{81} + \Delta v$.

Deviation 1a is thus impossible if we have

$$P \leq -\frac{4\beta^2}{81} - \Delta v \quad (2)$$

or

$$P \geq -\frac{4\beta^2}{81} + \Delta v \quad (3)$$

Step 2.1b: Firm a sells to nobody

It then chooses $p'_a = p_b + \Delta v = P + \Delta v$ and $y'_a = \frac{4\beta^2}{81}$. Its profits become $\Pi_a^{2.1b} = \frac{4\beta}{9} \left(\Delta v + \frac{2\beta}{9} \right) - \frac{4\beta^2}{81}$. It can be verified that $\Pi_a^{CC} - \Pi_a^{2.1b} \geq 0$ if

$$P \geq -\frac{4\beta^2}{81}.$$

We can see that this lower bound is incompatible with the upper bound (2). It is also easy to see that this lower bound is lower than the one found in (3).

Step 2.1c: Firm b captures the market

It then chooses $p'_b = p_a - \Delta v = P - \Delta v$. Its profits become $\Pi_b^{2.1c} = P - \Delta v + \frac{\beta}{9} \left(\Delta v + \frac{2\sqrt{2}\beta}{9} \right)$. It can be verified that $\Pi_b^{CC} - \Pi_b^{2.1c} \geq 0$ if

$$P \leq -\frac{(4 - 2\sqrt{2})\beta^2}{81} + \Delta v. \quad (4)$$

Step 2.1d: firm b shares the market

Firm b 's profits becomes $\Pi_b^{2.1d} = p_b \left(\frac{P-p_b}{\Delta v} \right) + \frac{\beta}{9} \left(\Delta v + \sqrt{\frac{8\beta^2}{81}} \sqrt{\frac{2\Delta v+p_b-P}{\Delta v}} \right)$. FOC wrt p_b is $\frac{P-2p_b}{\Delta v} + \frac{\sqrt{2}\beta^2}{81\sqrt{\Delta v}\sqrt{2\Delta v+p_b-P}}$. Verifying that the deviation is non-profitable is equivalent to verifying that this FOC, evaluated at $p_b = P$, is non-negative, meaning that firm b finds it unprofitable to decrease its price and to start selling some units. The FOC wrt p_b evaluated at $p_b = P$ is $-\frac{P}{\Delta v} + \frac{\beta^2}{81\Delta v}$. The FOC is non-negative if and only if

$$P \leq \frac{\beta^2}{81}. \quad (5)$$

We can easily see that the upper bound is smaller than the one found in (4).

Step 2.2: verifying that firm b has no incentives to take the lead

Step 2.2a: Firm b takes the lead and captures the market.

It then chooses $p'_b = p_a - \Delta v = P - \Delta v$ and $y'_b = \frac{8\beta^2}{81}$. Its profits become $\Pi_b^{2.2a} = P - \Delta v + \frac{4\beta}{9} \left(-\Delta v + \frac{4\beta}{9} - \sqrt{\frac{8\beta^2}{81}} \right) - \sqrt{\frac{8\beta^2}{81}}$. It can be verified that $\Pi_b^{CC} - \Pi_b^{2.2a} \geq 0$ if

$$P \leq \Delta v + \frac{5\beta \Delta v}{9} + \frac{(8\sqrt{2} - 4)\beta^2}{81}$$

This upper bound is larger than the one we found in (5).

Step 2.2b: Firm b takes the lead and shares the market

Firm b 's profits becomes $\Pi_b^{2.2b} = p_b \left(\frac{P-p_b}{\Delta v} \right) + \frac{4\beta}{9} \left(-\Delta v - \sqrt{\frac{8\beta^2}{81}} \sqrt{\frac{2\Delta v+p_b-P}{\Delta v}} + \sqrt{y_b} \sqrt{\frac{\Delta v+P-p_b}{\Delta v}} \right) - y_b$. We obtain that the optimal investment is $y_b = \frac{4\beta^2}{81} \frac{\Delta v+P-p_b}{\Delta v}$. Replacing, we obtain that the FOC wrt p_b is $\frac{P-2p_b}{\Delta v} - \frac{4\beta^2}{81} \left(\frac{\sqrt{2}}{\sqrt{\Delta v} \sqrt{2\Delta-P+p_b}} + \frac{1}{\Delta v} \right)$. A sufficient condition for the deviation to be non-profitable is for this FOC, evaluated at $p_b = P - \Delta v$ to be non-positive. In that case, firm b , if it takes the lead, prefers to capture the market than to share it, and it suffices to make sure than deviation 2.2a is not credible. We have that the FOC wrt p_b , evaluated at $p_b = P - \Delta v$, is $\frac{-P+2\Delta v}{\Delta v} - \frac{4\beta^2}{81\Delta v} (\sqrt{2} + 1)$. It is non-positive if and only

$$P \geq 2 \Delta v - \frac{4\beta^2}{81} (\sqrt{2} + 1).$$

It is easy to verify that the lower bound (3) is larger for the values of Δv and β in the relevant intervals.

Therefore, we have that P must be above the lower bound found in (3) and below the upper bound found in (5). The interval is non-empty if and only if $\Delta v \leq \frac{5\beta^2}{81}$.

3) leapfrogging equilibrium with market split

We need to show that: 1) at the announced equilibrium, we have that $v'_a - v'_b < 0$ (firm b takes the lead) and that $0 \leq x_b^{LS} \leq 1$, 2) firms a and b have no incentives to deviate by moving to a corner solution and 3) firms a and b have no incentives to deviate by changing their objective with respect to the lead (Firm a has no incentives to take the lead, firm b has no incentives to give it up).

Step 3.1a: verifying that $v'_a - v'_b < 0$

We have that $v'_a - v'_b = \Delta v + \sqrt{y_a} \sqrt{\frac{2\Delta v+p_b-p_a}{\Delta v}} - \sqrt{y_b} \sqrt{\frac{\Delta v+p_a-p_b}{\Delta v}}$. At the equilibrium values, it simplifies to $\Delta v - \frac{8\beta}{27} - \frac{10\beta^3}{2187\Delta v}$. Solving for Δv and keeping the non-negative root (as $\Delta v \geq 0$) we obtain the following condition:

$$\Delta v \leq \frac{1}{27} \left(4\beta + \frac{\beta\sqrt{144+30\beta}}{3} \right) \quad (6)$$

Step 3.1b: verifying that $0 \leq x_b^{LS} \leq 1$

We have that $x_b^{LS} = \frac{p_a-p_b}{\Delta v}$. Substituting the equilibrium values, we obtain $x_b^{LS} = \frac{1}{3} + \frac{5\beta^2}{243\Delta v}$. It is easy to see that $x_b^{LS} \geq 0$ is trivially satisfied. We have that $x_b^{LS} \leq 1$ if

$$\Delta v \geq \frac{5\beta^2}{162} \quad (7)$$

Step 3.2: verifying that firms have no incentives to move to a corner solution in the first period.

At the equilibrium values, we have that

$$\begin{aligned} \Pi_a^{LS} &= \frac{\Delta v}{9} (4 - \beta) + \frac{10\beta^2}{729} + \frac{40\beta^4}{59049 \Delta v} \\ \Pi_b^{LS} &= \frac{\Delta v}{9} (1 - 4\beta) + \frac{46\beta^2}{729} + \frac{25\beta^4}{59049 \Delta v} \end{aligned}$$

Step 3.2a: Firm a has no sales

It then chooses $p'_a = p_b + \Delta v$. Its profits become $\Pi_a^{3.2a} = \frac{\beta}{9} \left(-\Delta v + \sqrt{\frac{32\beta^2}{243} + \frac{40\beta^4}{19683\Delta v}} \right)$. It can be verified that $\Pi_a^{LS} - \Pi_a^{3.2a} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 3.2a.

Step 3.2b: Firm a captures the market

It then chooses $p'_a = p_b$. Its profits become $\Pi_a^{3.2b} = \frac{\Delta v}{3} - \frac{7\beta^2}{243} + \frac{\beta}{9} \left(-\Delta v + \sqrt{\frac{16\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}} \right)$. It can be verified that $\Pi_a^{LS} - \Pi_a^{3.2b} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 3.2b.

Step 3.2c: Firm b captures the market

It then chooses $p'_b = p_a - \Delta v$ and $y'_b = \frac{8\beta^2}{81}$. Its profits become $\Pi_b^{3.2c} = \left(-\frac{\Delta v}{3} - \frac{26\beta^2}{243}\right) + \frac{4\beta}{9} \left(-\Delta v + \frac{4\beta}{9}\right) - \frac{8\beta^2}{81}$. It can be verified that $\Pi_b^{LS} - \Pi_b^{3.2c} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 3.2c.

Step 3.2d: Firm b has no sales

It then chooses $p'_b = p_a$ and $y'_b = \frac{4\beta^2}{81}$. Its profits become $\Pi_b^{3.2d} = \frac{4\beta}{9} \left(-\Delta v + \frac{2\beta}{9}\right) - \frac{4\beta^2}{81}$. It can be verified that $\Pi_b^{LS} - \Pi_b^{3.2d} \geq 0$ for all possible values of β and Δv . There are no beneficial deviations of type 3.2d.

Step 3.3: verifying that firms have no incentives to change their objective with respect to the lead

Step 3.3a: Firm b gives up lead and has no sales

It then chooses $p'_b = p_a$ and $y'_b = 0$. Its profits become $\Pi_b^{3.3a} = \frac{\beta\Delta v}{9}$. It can be verified that $\Pi_b^{LS} - \Pi_b^{3.3a} \geq 0$ for values of β and Δv satisfying (6) and (7). There are no beneficial deviations of type 3.3a.

Step 3.3b: Firm b gives up lead and shares the market

It then chooses $p'_b = \frac{p_a}{2} = \frac{\Delta v}{3} - \frac{\beta^2}{243}$ and $y'_b = 0$. Its profits become $\Pi_b^{3.3b} = \left(\frac{\Delta v}{3} - \frac{\beta^2}{243}\right) \left(\frac{1}{3} - \frac{\beta^2}{243\Delta v}\right) + \frac{\beta\Delta v}{9}$. We have that $\Pi_b^{LS} - \Pi_b^{3.3b} \geq 0$ if and only if we have the following condition on Δv :

$$\Delta v \leq \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15}\right) \quad (8)$$

This upper-bound is lower than the one we had in (6).

Step 3.3c: Firm b gives up lead and captures the market

It then chooses $p'_b = p_a - \Delta v$ and $y'_b = 0$. Its profits become $\Pi_b^{3.3c} = \left(-\frac{\Delta v}{3} - \frac{2\beta^2}{243}\right) + \frac{\beta\Delta v}{9}$. It can be verified that $\Pi_b^{LS} - \Pi_b^{3.3c} \geq 0$ for values of β and Δv satisfying (8) and (7). There are no beneficial deviations of type 3.3c.

Step 3.3d: Firm a takes the lead and captures the market

It then chooses $p'_a = p_b$ and $y'_a = \frac{8\beta^2}{81}$. Its profits are $\Pi_a^{3.3d} = \frac{\Delta v}{3} - \frac{7\beta^2}{243} + \frac{4\beta}{9} \left(\Delta v + \frac{4\beta}{9} - \sqrt{\frac{16\beta^2}{243} + \frac{20\beta^4}{19683\Delta v}}\right) - \frac{8\beta^2}{81}$. It can be verified that $\Pi_a^{LS} - \Pi_a^{3.3d} \geq 0$ for values of β and Δv satisfying (8) and (7). There are no beneficial deviations of type 3.3d.

Step 3.3e: Firm a takes the lead and shares the market

The profit becomes

$$\Pi_a^{3.3e} = p_a \left(\frac{\Delta v + p_b - p_a}{\Delta v}\right) + \frac{4\beta}{9} \left(\Delta v + \sqrt{y_a} \sqrt{\frac{2v + p_b - p_a}{\Delta v}} - \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}}\right) - y_a$$

It then chooses p'_a and y'_a , higher order polynomials that can take complex values. However, for $\Delta v \leq 0.12166\beta^2$, it can be shown that the FOC wrt to price evaluated at the optimal price and investment of the deviation where it takes the lead and captures the market (step 3.3d) is negative, meaning that Firm a has no incentives to raise its price and move to deviation 3e.

In the non-empty region $0.12166\beta^2 \leq \Delta v \leq \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15}\right)$, the upper bound found in (8), the analysis is complicated by the fact that the (global) maximum can involve complex numbers. Numerical analysis allows us to conclude (by finding the correct local maximum) that in the region there are no profitable deviations of type 3.3e.

Step 3.3f: Firm a takes the lead and has no sales

It would then choose $p'_a = p_b + \Delta v$ and $y'_a = \frac{8\beta^2}{81}$. Its profit can be shown to be dominated by those of deviation 3.3e, by evaluating the FOC wrt to p_a at $p_a = p_b + \Delta v$ and $y_a = \frac{4\beta^2}{81}$ which is always negative, meaning that Firm a can increase its profits by lowering its price, and thus moving into deviation 3.3e.

Combining the results above, the leapfrogging equilibrium with market split is feasible if and only $\frac{5\beta^2}{162} \leq v \leq \frac{2\beta}{27} \left(\frac{4}{5} + \frac{\sqrt{144+30\beta}}{15}\right)$.

4) leapfrogging equilibria with market capture

We need to show three results: 1) That at the announced equilibrium, we have that $\Delta v' < 0$ (firm b takes the lead), 2) that firms a and b have no incentives to deviate by moving away from the corner solution and 3) firms a and b have no incentives to deviate by changing their objective with respect to the lead (Firm a has no incentives to take the lead, firm b has no incentives to give it up).

Step 4.1: verifying that $v'_a - v'_b < 0$

We have that $v'_a - v'_b = \Delta v + \sqrt{y_a} \sqrt{\frac{2\Delta v + p_b - p_a}{\Delta v}} - \sqrt{y_b} \sqrt{\frac{\Delta v + p_a - p_b}{\Delta v}}$. At the equilibrium values, it simplifies to $v'_a - v'_b = \Delta v - \frac{4\beta}{9}$. We obtain the following condition

$$\Delta v \leq \frac{4\beta}{9} \quad (9)$$

Step 4.2: verifying that firms have no incentives to move away from the corner solution in the first period.
At the equilibrium values, we have that

$$\begin{aligned} \Pi_a^{LC} &= -\frac{\beta \Delta v}{9} + \frac{4\beta^2}{81} \\ \Pi_b^{LC} &= P - \Delta v(1 + \frac{4\beta}{9}) + \frac{8\beta^2}{81} \end{aligned}$$

Step 4.2a: Firm a shares the market

The profits of Firm a become $\Pi_a^{4.2a} = p_a \left(\frac{P-p_a}{\Delta v} \right) + \frac{\beta}{9} \left(-\Delta v + \frac{2\sqrt{2}\beta}{9} \sqrt{\frac{2\Delta v + p_a - P}{\Delta v}} \right)$. FOC wrt to p_a is $\frac{P-2p_a}{\Delta v} + \frac{\beta^2}{81} \frac{\sqrt{2}}{\sqrt{\Delta v} \sqrt{2\Delta v + p_a - P}}$. Verifying that the deviation is non-profitable is equivalent to verifying that this FOC, evaluated at $p_a = P$, is non-negative, meaning that firm a finds it unprofitable to decrease its price and to start selling some units. The FOC wrt p_a evaluated at $p_a = P$, is $-\frac{P}{\Delta v} + \frac{\beta^2}{81\Delta v}$. It is non-negative if and only if

$$P \leq \frac{\beta^2}{81}. \quad (10)$$

Step 4.2b: Firm a captures the market

It then chooses $p'_a = p_b = P - \Delta v$. Its profits become $\Pi_a^{2b} = P - \Delta v + \frac{\beta}{9} \left(-\Delta v + \frac{2\sqrt{2}\beta}{9} \right)$. It can be verified that $\Pi_a^{LC} - \Pi_a^{4.2b} \geq 0$ if

$$P \leq \frac{(4 - 2\sqrt{2})\beta^2}{81} + \Delta v.$$

It is obvious that this upper bound is larger than (10) found in 4.2a.

Step 4.2c: Firm b shares the market

It then chooses $p'_b = \frac{P}{2} - \frac{2\beta^2}{81}$ and $y_b = \frac{24\beta}{81v} (\Delta v + \frac{P}{2} + \frac{2\beta^2}{81})$. Its profits become $\Pi_b^{4.2c} = \frac{16\beta^4 + 648P\beta^2 + 1296\beta^2\Delta v - 11664\beta\Delta v^2 + 6561P^2}{26244\Delta v}$

It can be shown that we always have $\Pi_b^{LC} - \Pi_b^{4.2c} < 0$. However, this deviation is feasible only if the quantity sold is between 0 and 1 and $v'_a - v'_b \geq 0$.

At the optimal deviation values, we have that $x_b^{4.2c} = \frac{P}{2\Delta v} + \frac{2\beta^2}{81\Delta v}$. We have that $0 \leq x_b^{4.2c} \leq 1$ if $-\frac{4\beta^2}{81} \leq P \leq -\frac{4\beta^2}{81} + 2\Delta v$.

Similarly, we have that $v'_a - v'_b = \frac{1}{729\Delta v} (4\beta^3 + 81P\beta + 162\beta\Delta v - 729\Delta v^2)$. We have that $v'_a - v'_b \geq 0$ if $P \geq -\frac{1}{81\beta} (4\beta^3 + 162\beta\Delta v - 729\Delta v^2)$.

In the interval $\Delta v \leq \frac{4\beta}{9}$ we always have that $-\frac{1}{81\beta} (4\beta^3 + 162\beta\Delta v - 729\Delta v^2) \leq -\frac{4\beta^2}{81}$.

Combining, deviation 4.2c is profitable and possible if $-\frac{4\beta^2}{81} \leq P \leq -\frac{4\beta^2}{81} + 2\Delta v$.

Deviation 2c is thus impossible if we have

$$P \leq -\frac{4\beta^2}{81} \quad (11)$$

or

$$P \geq -\frac{4\beta^2}{81} + 2\Delta v \quad (12)$$

Step 4.2d: Firm b has no sales

It then chooses $p'_b = p_a = P$ and $y'_b = \frac{4\beta^2}{81}$. Its profits become $\Pi_b^{4.2d} = \frac{4\beta}{9} \left(-\Delta v + \frac{2\beta}{9} \right) - \frac{4\beta^2}{81}$.

It can be shown that $\Pi_b^{LC} - \Pi_b^{4.2d} \geq 0$ if $\Delta v - \frac{4\beta^2}{81} \leq P$. Notice also that this deviation is possible only if $v'_a - v'_b \leq 0$, or $\Delta v \leq \frac{2\beta}{9}$.

Deviation 4.2d is thus non-profitable and/or impossible if $\Delta v > \frac{2\beta}{9}$ or if

$$\begin{aligned} P &\geq -\frac{4\beta^2}{81} + \Delta v \\ \Delta v &\leq \frac{2\beta}{9} \end{aligned} \quad (13)$$

Step 4.3: verifying that firms have no incentives to change their objective with respect to the lead

Step 4.3a: Firm b gives up lead and captures the market

It then chooses $p'_b = p_a - \Delta v = P - \Delta v$ and $y'_b = 0$. Its profits become $\Pi_b^{4.3a} = (P - \Delta v) + \frac{\beta\Delta v}{9}$. It can be verified that $\Pi_b^{LC} - \Pi_b^{4.3a} \geq 0$ if

$$\Delta v \leq \frac{8\beta}{45}$$

This condition allows to simplify the results above. First, see that in step 4.2d, the condition $\Delta v \leq \frac{2\beta}{9}$ becomes unfeasible. The condition (13) thus becomes necessary. Second, notice that condition (13) is incompatible with (11) from step 4.2c. The condition (12) thus also becomes a necessary condition. Thirdly, notice that the lower bound (12) is larger than the lower bound (13). The only way to make deviations 4.2a-4.2d and 4.3a simultaneously unprofitable is to satisfy constraints (12) and (10). Notice that the price interval is non-empty if and only if

$$\Delta v \leq \frac{5\beta^2}{162}, \quad (14)$$

a more restrictive condition on Δv than the one found before.

Step 4.3b: Firm b gives up lead and shares the market

It then chooses $p'_b = \frac{p_a}{2} = \frac{P}{2}$ and $y'_b = 0$. Its profits become $\Pi_b^{4.3b} = \frac{P^2}{4\Delta v} + \frac{\beta\Delta v}{9}$. We can verify that $\Pi_b^{LC} - \Pi_b^{4.3b} \geq 0$ if $2\Delta v - \frac{2}{9} \left(\sqrt{\beta\Delta v(8\beta - 45\Delta v)} \right) \leq P \leq 2\Delta v + \frac{2}{9} \left(\sqrt{\beta\Delta v(8\beta - 45\Delta v)} \right)$. Notice that by condition (14) this bounds are always real numbers. Notice also that the deviation is feasible only if the quantity sold is between 0 and 1, which is not the case if $P \leq 0$ or $P \geq 2\Delta v$. Notice that since $2\Delta v + \frac{2}{9} \left(\sqrt{\beta\Delta v(8\beta - 45\Delta v)} \right) \geq 2\Delta v$, the upper bound is not a concern: either P is low enough that the deviation is unprofitable or high enough that the deviation is unfeasible. If $\Delta v > \frac{8\beta^2}{81+45\beta}$, the lower bound above is negative, which means that again it is not a concern: any price P makes deviation 4.3b either unprofitable or unfeasible. If $\Delta v \leq \frac{8\beta^2}{81+45\beta}$, we have a potential profitable deviation if $0 \leq P \leq 2\Delta v - \frac{2}{9} \left(\sqrt{\beta\Delta v(8\beta - 45\Delta v)} \right)$. However, given condition (14) on Δv , we have that $2\Delta v - \frac{4\beta^2}{81} \geq 2\Delta v - \frac{2}{9} \left(\sqrt{\beta\Delta v(8\beta - 45\Delta v)} \right)$ and thus the lower bound of condition (12) is a stricter condition.

Step 4.3c: Firm b gives up lead and has no sales

It then chooses $p'_b = p_a = P$ and $y'_b = 0$. Its profits become $\Pi_b^{4.3c} = \frac{\beta\Delta v}{9}$. It can be verified that $\Pi_b^{LC} - \Pi_b^{4.3c} \geq 0$ if

$$P \geq \Delta v + \frac{5\beta\Delta v}{9} - \frac{8\beta^2}{81}.$$

Given condition (14) on Δv , the lower bound of condition (12) is a stricter condition.

Step 4.3d: Firm a takes the lead and captures the market

It then chooses $p'_a = p_b = P - \Delta v$ and $y'_a = \frac{8\beta^2}{81}$. Its profits $\Pi_a^{4.3d} = P - \Delta v + \frac{4\beta}{9} \left(\Delta v + \frac{(4-2\sqrt{2})\beta}{9} \right) - \frac{8\beta^2}{81}$. It can be verified that $\Pi_a^{LC} - \Pi_a^{4.3d} \geq 0$ if

$$P \leq \frac{(8\sqrt{2}-4)\beta^2}{81} + \frac{(9-5\beta)\Delta v}{9} \quad (15)$$

Condition (10) found in step 4.2a is more restrictive.

Step 4.3e: Firm a takes the lead and shares the market

The profit becomes

$$\Pi_a^{4.3e} = p_a \left(\frac{P - p_a}{\Delta v} \right) + \frac{4\beta}{9} \left(\Delta v + \sqrt{y_a} \sqrt{\frac{\Delta v + P - p_a}{\Delta v}} - \sqrt{\frac{8\beta^2}{81}} \sqrt{\frac{2\Delta v + p_a - P}{\Delta v}} \right) - y_a$$

It then chooses p_a and y_a , higher order polynomials that can take complex values. Numerical analysis allows us to conclude that within region defined by conditions (12), (10) and (14) there are no profitable deviations of type 4.3e.

Step 4.3f: Firm a takes the lead and has no sales

It would then choose $p'_a = p_b + \Delta v = P$ and $y'_a = \frac{4\beta^2}{81}$. Its profit then becomes $\Pi_a^{4.3f} = \frac{4\beta}{9} \left(\Delta v - \frac{2\beta}{9} \right) - \frac{4\beta^2}{81}$. It can be verified that $\Pi_b^{LC} - \Pi_b^{4.3a} \geq 0$ if $\Delta v \leq \frac{16\beta}{45}$. Condition (14) from step 4.2a is more restrictive.

Therefore, we have leapfrogging equilibria with market capture and with P between the bounds found in (12) and (10). The interval is non-empty if $0 \leq \Delta v < \frac{5\beta^2}{162}$.