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Efficient trading on a network with incomplete information

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Efficient trading on a network with incomplete information

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ABSTRACT: This paper considers a trading problem on a network with incomplete information. We consider a simple water trading problem in which three agents are located in a linear order along a river. Upper stream agents can sell some amount of the water to their downstream but not the other way around. The middle agent can be both a seller and a buyer. Agents have private information on their utility of water, which we assume is non-linear. We ask if there is an efficient trading mechanism for the allocation of water. We show that if agents have highly asymmetric initial endowments of water, incentive-compatible, individually rational, budget-balanced mechanisms exist that are also ex-post efficient.

JEL classification: C72, D82.
Keywords: Network; Incomplete Information; Water Trading; Mechanism Design

1 Introduction

Many real life trading problems are restricted not only to certain groups of agents but unidirectional within a group. In a supply chain of certain product, upstream firms sell their products to their downstream counterparts in one direction. In the allocation of water along a river, upstream agents sell water to their downstream neighboring agents. Even within a free trade union, for a given product, countries trade with each other along a value-added chain in one direction. In almost all practical trading, trading is often organized in certain order or structure.

In this paper, we focus on a specific trading problem on a linear network. We consider the problem of water trading along a river. A river flows through a number of regions (e.g., villages, municipalities, or countries, called agents hereafter). Due to various weather and geological conditions, some agents may receive more water (for example, from rainfall) than others. On the other hand, different agents may have different needs of water. Thus, it might be more efficient to allow upstream agents sell water to their downstream neighbors. Will this restricted trading always lead to an efficient allocation of water?

In the complete information version of the problem, Wang (2011) proposes a "downstream trading mechanism" that implements the unique efficient allocation of water and in the meantime generates a welfare distribution that is in the core of the associated game of the problem. In fact, there is a large literature on the efficient allocation of water along a river. Based on the game-theoretic approach, Ambec and Sprumont (2002) provide a different welfare distribution than Wang (2011) by the so-called "downstream incremental distribution". Ambec and Ehlers (2008a, b) provide another alternative "upstream incremental distribution". On the other hand, using the market-based approach, there are Young et al. (2000), Chong and Sunding (2006), Dinar and Wolf (1994), Giannias and Lekakis (1997), Lekakis (1998), and more.

The above two approaches all assume that agents' utility functions and the amount of water they each initially receive (and thus control) are common knowledge. While the latter is not an unrealistic assumption as one can always estimate accurately how much water or rainfall an agent (region) would receive in a given period of time, the former assumption of complete information on agents' utility functions may not be realistic because these functions are private information. Since the information on agents' utility functions is needed in determining the efficient allocation, the above two approaches that both depend on this information may be vulnerable to agents' misreporting the information. Now the question is under this incomplete information on agents' utility functions, can an efficient allocation of water still be achievable?

To address this question, we begin with a simple model. We assume that each agent's utility function is quadratic with the marginal utility at zero consumption being a privately known parameter called as this agent's type. This parameter uniquely determines the agent's peak demand (optimal consumption of water) for the family of quadratic functions (a subset of the set of all concave and single-peaked functions). We consider *direct* mechanisms (allocation rules) by which agents report simultaneously their types (or equivalently their peak demands) to a coordinator who then determines a feasible allocation of water as well as the monetary transfers between the agents.

Following the traditional approach in the mechanism design literature, we first provide a standard necessary and sufficient condition that guarantees the existence of such a mechanism (Theorem 1). Then, we focus on two cases, one with two agents and another with three agents, respectively. We show that if the initial distribution of water is *not too symmetric*, then there exists an incentive-compatible, individually rational, and budget-balanced direct mechanism that allocates the water efficiently (Propositions 1 and 2).

The key assumption of our model is that agents have quadratic utility functions. There are two main reasons for this assumption. First, this type of utility functions has been often used in the literature. For example, in Ambec and Sprumont (2002), agents' utility functions are assumed to be concave. In Ambec (2008), Ambec and Ehlers (2007, 2008),[1] agents' utility functions are assumed to be both concave and single-peaked. Single-peakedness implies that agents each have their optimal levels of consumption and may have negative marginal utilities as they consume more than their optimal amounts (e.g, assuming no free disposal as we do in this paper, too much water may cause flooding). Second, it allows us to derive our results analytically.

Our paper departs from the literature originated by Myerson and Sat-

[1] Ambec and Ehlers (2007, 2008) have mentioned the quadratic utility functions in their papers.

terthwaite (1983) in two aspects. First, there is a linear order relation on the agents that only upstream agents can sell water to their respective downstream agents and an agent in between two other agents can act both as a seller and a buyer. Second, in our model agents have (strictly) *nonlinear* utility functions.

This paper is an extension of an earlier paper by Lu and Wang (2010), in which a trading problem with nonlinear utility is considered. In that paper, no restriction on whom a trader can trade with is imposed. Here, trading are restricted by a linear network.

This paper is also related to Kranton and Minehart (2001) and Blume et al. (2009). In Kranton and Minehart, sellers each own one-unit of an indivisible good and buyers own none. In Blume et al., they also assume that goods are indivisible and that buyers and sellers can trade only through intermediaries (traders). In our paper, goods (water) are divisible and agents trade directly between each other. More importantly, agents have nonlinear utility.

2 The Problem

We consider essentially the same model proposed by Ambec and Sprumont (2002) except the following incomplete information assumptions on agents' benefit functions. A river follows through a finite number of countries (called agents hereafter), $N = \{1, 2, ..., n\}$, from upstream to downstream: $i < j$ means that i is upstream from j. Denote $Pi = \{1, ..., i\}$ and $P^0 i = Pi \setminus \{i\}$ the set of upstream agents and strict upstream agents of agent i, respectively. Denote $Si = \{i, ..., n\}$ and $S^0 i = Si \setminus \{i\}$ the set of downstream agents and strict downstream agents of agent i, respectively.

Assume that each agent $i \in N$ picks up e_i volume of water along the river (Figure 1). Assume that $e_1 > 0$. Agents consume certain amount of water and may sell some water to their downstream neighbors. For each agent, money is available in unbounded quantities to make payments. Agents value water and money. Agent i's utility from consuming x_i units of water and making payment t_i is

$$u_i(x_i, t_i, v_i) = v_i x_i - \frac{1}{2}\gamma x_i^2 - t_i, \text{[2]} \qquad (1)$$

[2] At certain levels of consumption agent may have negative marginal benefit, e.g., those

where v_i is agent i's type and assumed to be his private information, drawn independently from the distribution F with support $[\underline{v}, \overline{v}]$ and positive continuous density function f, and $\gamma > 0$ is a constant. Except $v_i, i \in N$, both F and γ are common knowledge.

It is easy to see that, for each agent $i \in N$, benefit function (1) has the following maximum (peak demand) that is uniquely determined by v_i.

$$\hat{x}_i = \frac{v_i}{\gamma}$$

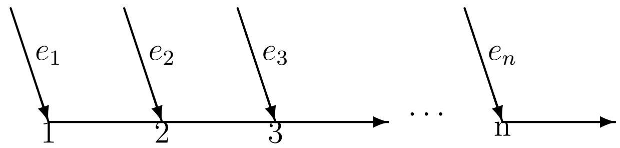

Figure 1. A Schematic Description of A Linear River Network.

A (water allocation) problem is a list (N, e, u), where $e = (e_1, ..., e_n)$ and $u = (u_1, ..., u_n)$. An allocation is a vector $x = (x_1, ..., x_n) \in R_+^N$.

In this paper, we are interested in trading mechanisms that will determine who will purchase additional amounts of water and who will sell some part of their initial endowments, and how much transfer from each agent should be made. The *Revelation Principle* (e.g., Myerson, 1979) allows us to focus on direct mechanisms in which agents simultaneously report their types $v = (v_1, v_2, \cdots, v_n)$ (or equivalently their peak demands) to a coordinator who then determines an allocation $x(v) = (x_1(v), \cdots, x_n(v))$ and a payment $t(v) = (t_1(v), \cdots, t_n(v))$, where $x_i(v)$ is agent i's consumption of water and $t_i(v)$ is the net money transfer from agent i. We call the pair of outcome functions $\{x, t\}$ a *trading mechanism.*

We require first that an allocation $x(v)$ be feasible:

$$\sum_{j=1}^{i} x_j(v) \leq \sum_{j=1}^{i} e_i, \text{for } i = 1, ..., n, \forall v \in [\underline{v}, \overline{v}]^n.$$

levels that are higher than the satiated level which may cause disutility such as flooding.

The rest of this section is standard in the mechanism design literature. For completeness we repeat it below. The reader can also see Lu and Wang (2010) for a similar model without a network structure.

Denote $-i = N\backslash\{i\}$. Denote $E_{-i}[\,\cdot\,]$ the expectation operator with respect to v_{-i}. Denote $X_i(v_i) = E_{-i}[x_i(v_i, v_{-i})]$. Denote agent i's expected payment as $T_i(v_i) = E_{-i}[t_i(v_i, v_{-i})]$. When agent i announces v_i as his type, his expected payoff is

$$U_i(v_i) = E_{-i}[u_i(x_i(v), t_i(v), v_i)] = E_{-i}[v_i x_i(v_i, v_{-i}) - \frac{1}{2}\gamma x_i^2(v_i, v_{-i})] - T_i(v_i).$$

The trading mechanism $\{x, t\}$ is called *incentive compatible* if it is optimal for each type of each agent to report his type truthfully when others report truthfully:

$$U_i(v_i) \geq E_{-i}[v_i x_i(\hat{v}_i, v_{-i}) - \frac{1}{2}\gamma x_i^2(\hat{v}_i, v_{-i})] - T_i(\hat{v}_i) \quad \forall i \in N \;\; \forall v_i, \hat{v}_i \in [\underline{v}, \overline{v}]. \quad (2)$$

The trading mechanism $\{x, t\}$ is called *individually rational* if all types of all agents are better off or no worse off by participating in the mechanism (in terms of their expected payoffs) than not participating and consuming their initial endowments:

$$U_i(v_i) \geq v_i e_i - \frac{1}{2}\gamma e_i^2 \quad \forall i \in N, \quad \forall v_i \in [\underline{v}, \overline{v}]. \quad (3)$$

An allocation x is *implementable* if there exists a transfer function t such that $\{x, t\}$ is an incentive compatible and individually rational mechanism. The following lemma characterizes the set of incentive compatible trading mechanisms. The standard proof of the lemma is omitted.

Lemma 1 *If the trading mechanism $\{x, t\}$ is incentive compatible, then for every $i \in N$*

$$U_i(v_i) - U_i(v_i^*) = \int_{v_i^*}^{v_i} E_{-i}[x_i(u, v_{-i})]\, du \quad \forall v_i, v_i^* \in [\underline{v}, \overline{v}]. \quad (4)$$

Given an incentive compatible mechanism $\{x, t\}$, equation (4) implies that the expected utility $U_i(v_i)$ is absolutely continuous and non-decreasing in v_i, with nonnegative derivative $E_{-i}[x_i(v_i, v_{-i})]$ almost everywhere.

The continuity of U_i implies that the expected net utility $U_i(v_i) - (v_i e_i - (1/2)\gamma e_i^2)$ has a minimum over $v_i \in [\underline{v}, \overline{v}]$.[3] Suppose that $U_i(v_i) - (v_i e_i - (1/2)\gamma e_i^2)$ is minimized at $v_i^* \in [\underline{v}, \overline{v}]$. Call v_i^* the *worst-off type* of agent i under allocation x. It is easy to see that if the individual rationality condition (3) is satisfied for v_i^*, then it is satisfied for all the other types as well. Thus the trading mechanism $\{x, t\}$ is individually rational if and only if

$$U_i(v_i^*) \geq v_i^* e_i - (1/2)\gamma e_i^2 \quad i = 1, ..., n. \tag{5}$$

Moreover, if $v_i^* \in (\underline{v}, \overline{v})$, then v_i^* must satisfy the first-order condition:

$$E_{-i}[x_i(v_i^*, v_{-i})] = e_i. \tag{6}$$

Equation (6) implies that the worst type agent just consumes what he gets e_i from his endowment.

Given an incentive compatible mechanism $\{x, t\}$ and a worst-off type v_i^*, let

$$\eta(v_i|v_i^*) = \begin{cases} v_i + \frac{F(v_i)}{f(v_i)} & \text{if } v_i < v_i^*, \\ v_i^* & \text{if } v_i = v_i^*, \\ v_i + \frac{F(v_i)-1}{f(v_i)} & \text{if } v_i > v_i^*. \end{cases} \tag{7}$$

We call $\eta(v_i|v_i^*)$ the *virtual valuation* under allocation x.[4] Note that, since $v_i + \frac{F(v_i)-1}{f(v_i)} < v_i < v_i + \frac{F(v_i)}{f(v_i)}$ for all v_i, an agent's virtual valuation is distorted downward (upward) to be below (above) his true valuation when his type v_i is higher (lower) than v_i^*.

For a given implementable allocation x, let R be the maximum expected revenue or gains from water trading from any incentive compatible and individually rational mechanism implementing x. Lemma 2 below characterizes implementable allocations.

Lemma 2 *For any implementable allocation x,*

$$R = \sum_{i=1}^{n} E[\eta(v_i|v_i^*)x_i - \frac{1}{2}\gamma x_i^2] - \sum_{i=1}^{n}[v_i^* e_i - \frac{1}{2}\gamma e_i^2], \tag{8}$$

[3]From (4), the utility function $U_i(v_i)$ is solely determined by the allocation x_i with the exception of a constant, hence the minimizer $v_i^* \in \arg\min_{v_i}\{U_i(v_i) - v_i e_i - \frac{1}{2}\gamma e_i^2\}$ only depends on x_i and e_i.

[4]Myerson (1981) introduced the concept of virtual valuation for ex ante identified traders. Lu and Robert (2001) extended this concept to the model with ex ante unidentified traders.

where v_i^* is the worst-off type for agent i under x.

Furthermore, for any allocation $x(v)$ such that $x_i(v_i, v_{-i})$ is non-decreasing in v_i for all $i \in N$, there exists a payment function $t(v)$ satisfying the budget balance condition $\sum_{i=1}^{n} t_i(v) = 0$ such that $\{x, t\}$ is incentive compatible and individually rational if and only if R defined in (8) is nonnegative.

The proof of the lemma can be found in Lu and Wang (2010) and is omitted.

We say that a trading mechanism $\{x, t\}$ is (ex post) *efficient* if for each vector of types $v = (v_1, \cdots, v_n)$ the outcome of the mechanism $\{x(v), t(v)\}$ is Pareto efficient. We call a trading mechanism an *efficient trading mechanism* if it is incentive-compatible, individually rational, budget balanced, and efficient.

The main question we address in this paper is: Is there an efficient trading mechanisms for the water allocation problem?

3 The Two-Agent Case

Assume that there are only two agents as shown in Figure 2 below.

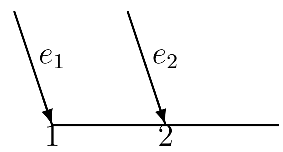

Figure 2. A Two-Agent Case.

An allocation of water is a vector $x = (x_1, x_2) \in R_+^2$. An allocation x is feasible if

$$\begin{aligned} x_1 &\leq e_1, \\ x_1 + x_2 &= e_1 + e_2. \end{aligned}$$

For simplicity, for the rest of the paper we assume that $\underline{v} = 1, \overline{v} = 2$ and that F is the uniform distribution on $[1, 2]$. We also assume that $e_1 + e_2 \geq 1/\gamma$.[5]

In the two-agent case, agent 1 must be a seller and agent 2 a buyer. Therefore, the worst type for agent 1 is $v_1^* = 2$ and for agent 2 $v_2^* = 1$.

Note that the total amount of water is e_1+e_2 units and that the maximum amount agent 1 can consume, is less than or equal to e_1 and the maximum amount agent 2 can consume, is less than or equal to $e_1 + e_2$. The efficient allocation is then given by

$$v_1 - \gamma x_1 = v_2 - \gamma x_2 \tag{9}$$

if

$$v_1 - \gamma e_1 \leq v_2 - \gamma e_2.$$

or

$$x_1 = e_1, x_2 = e_2,$$

if

$$v_1 - \gamma e_1 > v_2 - \gamma e_2.$$

It is easy to check that the efficient allocation is always feasible.

Lemma 3 *For any $v = (v_1, v_2)$, there exists a unique efficient allocation $x(v) = (x_1(v), x_2(v))$ where each $x_i(v_i, v_j)$ is strictly positive and non-decreasing in v_i for $i = 1, 2$.*

Proof. From the definition of efficient allocation, it is easy to see that

$$x_1 = \frac{e_1 + e_2}{2} + \frac{1}{2\gamma}(v_1 - v_2), \tag{10}$$

$$x_2 = \frac{e_1 + e_2}{2} + \frac{1}{2\gamma}(v_2 - v_1), \tag{11}$$

if

$$v_1 - \gamma e_1 \leq v_2 - \gamma e_2.$$

[5]This condition guarantees that agents consume nonnegative amounts in the efficient allocation in Lemma 3.

And

$$x_1 = e_1, x_2 = e_2, \tag{12}$$

if

$$v_1 - \gamma e_1 > v_2 - \gamma e_2.$$

Thus, each $x_i(v_i, v_j)$ is strictly positive and non-decreasing in v_i for $i = 1, 2$.

The lemma is proved. Q.E.D.

The following theorem characterizes the conditions under which an efficient trading mechanism exists.

Theorem 1 *There exists an efficient trading mechanism if and only if*

$$R = \sum_{i=1}^{2} E[\eta(v_i|v_i^*)x_i(v) - \frac{\gamma}{2}x_i^2(v)] - \sum_{i=1}^{2}(v_i^* e_i - \frac{\gamma}{2}e_i^2) \geq 0, \tag{13}$$

where x is the efficient allocation and v_i^* the worst-off type for agent i under x_i.

Proof : By Lemma 3, $x_i(v), i = 1, 2$ is non-decreasing in $v_i, i = 1, 2$. Invoking lemma 2, it follows that there exists an efficient trading mechanism if and only if the condition (13) holds. The theorem is proved. Q.E.D.

The following proposition shows that efficient trading mechanisms exist if the upstream agent has a relatively larger initial amount of water than the downstream agent.

Proposition 1 *Suppose that*

$$e_1 \geq e_2 + \frac{2}{\gamma}. \tag{14}$$

Then there exists an efficient trading mechanism.

Proof: (i) By assumption (14), we have $v_1 - \gamma e_1 \leq v_2 - \gamma e_2$. Therefore the efficient allocation is given by (10, 11). Now we calculate R in condition (13).

First, note that the variance of the distribution F is $1/12$. Use integration by parts we have

$$\begin{aligned}
E[\eta(v_i|v_i^*)x_i(v_i,v_j)] &= \int_1^2 (2v_i-1)\,X_i(v_i)\,dv_i - \int_{v_i^*}^2 X_i(v_i)\,dv_i \\
&= v_i^* X_i(v_i^*) - \int_1^2 v_i(v_i-1)\,dX_i(v_i) + \int_{v_i^*}^2 v_i\,dX_i(v_i) \\
&= v_i^* X_i(v_i^*) - \frac{1}{2\gamma}\int_1^2 v_i(v_i-1)\,dv_i + \frac{1}{2\gamma}\int_{v_i^*}^2 v_i\,dv_i \\
&= v_i^* X_i(v_i^*) - \frac{1}{4\gamma}v_i^{*2} + \frac{1}{4\gamma}((\frac{3}{2})^2 + \frac{1}{12}).
\end{aligned}$$

Secondly, by the independence of the distributions, we have

$$\begin{aligned}
& E[(x_i(v_i,v_{-i}))^2] \\
=\ & E\left[\frac{(e_1+e_2)^2}{4} + \frac{2(e_1+e_2)}{4\gamma}(v_i-v_j) + \frac{1}{4\gamma^2}(v_i-v_j)^2\right] \\
=\ & \frac{(e_1+e_2)^2}{4} + \frac{1}{4\gamma^2}E\left[v_i^2 - 2v_iv_j + v_j^2\right] \\
=\ & \frac{(e_1+e_2)^2}{4} + \frac{1}{2\gamma^2}\frac{1}{12}.
\end{aligned}$$

Thus, substituting into the right-hand side of equation (13), we obtain

$$R = \sum_{i=1}^{2}\left\{v_i^*(X_i(v_i^*) - e_i) + \frac{1}{4\gamma}((\frac{3}{2})^2 - v_i^{*2}) + \frac{\gamma}{2}\left(e_i^2 - \frac{(e_1+e_2)^2}{4}\right)\right\}. \quad (15)$$

Since

$$\begin{aligned}
X_1(v_1) = E_{v_2}x_1(v_1,v_2) &= \frac{e_1+e_2}{2} + \frac{1}{2\gamma}(v_1 - \frac{3}{2}), \\
X_2(v_2) = E_{v_1}x_2(v_1,v_2) &= \frac{e_1+e_2}{2} + \frac{1}{2\gamma}(v_2 - \frac{3}{2}),
\end{aligned}$$

and

$$v_1^* = 2, v_2^* = 1,$$

plugging in (15) we obtain that

$$R = \frac{1}{8\gamma} + \frac{\gamma(e_1-e_2)^2}{4} - \frac{e_1-e_2}{2}.$$

Thus, if $0 \leq e_1 - e_2 \leq \frac{4-2\sqrt{2}}{4\gamma}$ or $e_1 - e_2 \geq \frac{4+2\sqrt{2}}{4\gamma}$, then $R \geq 0$. But we assume that $e_1 - e_2 \geq 2/\gamma \geq \frac{4+2\sqrt{2}}{4\gamma}$. Thus, the proposition is proved. Q.E.D.

Example 1. Suppose that there are two agents $i = 1, 2$ with $\gamma = 1/2$. Suppose that $(e_1, e_2) = (4, 0)$. Then the efficient allocation is given by

$$x_1(v_1, v_2) = 2 + v_1 - v_2,$$

$$x_2(v_1, v_2) = 2 + v_2 - v_1.$$

Because $1/\gamma = 2 < e_1 + e_2 = 4$ and $e_1 = 4 \geq e_2 + 2/\gamma$, by Proposition 1, efficient trading mechanism exists. In fact, it is easy to calculate that the net transfer functions are as follows:[6]

$$t_1(v) = \frac{v_1^2 - v_2^2}{4} - \frac{v_1 - v_2}{4} - 1,$$

$$t_2(v) = -\frac{v_1^2 - v_2^2}{4} + \frac{v_1 - v_2}{4} + 1.$$

To see why, note that the marginal valuation of agent i for a quantity x is $v_i - \frac{x}{2}$. For example, when agent 1 receives $e_1 = 4$ and agent 2 receives $e_2 = 0$, the marginal benefit of agent 1 at e_1 is at most $2 - \frac{4}{2} = 0$, while the marginal benefit of agent 2 with $e_2 = 0$ is at least $1 - \frac{e_2}{2} = 1$, which is greater than 0. Thus, the expected gains from trade is positive, ensuring the existence of efficient trading mechanisms.

Suppose that $v_1 = 3/2, v_2 = 1$. Then, agent 1's peak amount is $v_1/\gamma = 3$ and agent 2's is $v_2/\gamma = 2$. The efficient allocation is $x_1 = 5/2, x_2 = 3/2$ and the monetary transfers are $t_1 = -13/16, t_2 = 13/16$. On the other hand, if $v_1 = 1, v_2 = 3/2$, then, agent 1's peak amount is $v_1/\gamma = 2$ and agent 2's is $v_2/\gamma = 3$. The efficient allocation is $x_1 = 3/2, x_2 = 5/2$ and the monetary transfers are $t_1 = -19/16, t_2 = 19/16$. If $v_1 = v_2$, then $x_1 = x_2 = 2$ and $t_1 = -1, t_2 = 1$.

If we suppose that $v_1 = v_2 = 2$, then

$$u_1(e_1, 0, v_1) = 2 \times 4 - \frac{1}{2} \times \frac{1}{2} \times 4^2 = 4,$$

[6] We relegate the calculation in the Appendix.

and

$$u_2(e_2, 0, v_2) = 0.$$

With the efficient allocation $(x_1, x_2) = (2, 2)$ and the transfers $(t_1, t_2) = (-1, 1)$, we have

$$u_1(2, -1, v_1) = 2 \times 2 - \frac{1}{2} \times \frac{1}{2} \times 2^2 + 1 = 4,$$

and

$$u_2(2, 1, v_2) = 2 \times 2 - \frac{1}{2} \times \frac{1}{2} \times 2^2 - 1 = 2.$$

Therefore agent 2 is strictly better off.

4 The Three-Agent Case

We maintain the assumptions that $[\underline{v}, \overline{v}] = [1, 2]$ and F is the uniform probability distribution on $[1, 2]$. Note that, in the three-agent case the second agent can be both a seller and a buyer.

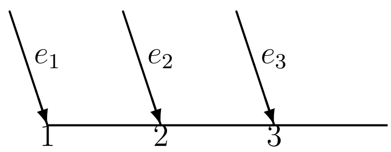

Figure 3. A Three-Agent Case.

A feasible allocation is a vector $x = (x_1, x_2, x_3) \in R^3_+$ satisfying

$$\begin{aligned} x_1 &\leq e_1, \\ x_1 + x_2 &\leq e_1 + e_2, \\ x_1 + x_2 + x_3 &= e_1 + e_2 + e_3. \end{aligned}$$

Under the assumption on the agents' benefit functions, it is easy to see that the efficient allocation x is given by the following three possibilities:

(1) Trading between all three agents $\{1,2,3\}$.

$$v_1 - \gamma x_1 = v_2 - \gamma x_2 = v_3 - \gamma x_3 \tag{16}$$

if

$$x_1 < e_1, x_3 > e_3, x_1 + x_2 + x_3 = e_1 + e_2 + e_3.$$

The efficient allocation is

$$x_1 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3\gamma}(2v_1 - v_2 - v_3),$$

$$x_2 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3\gamma}(2v_2 - v_1 - v_3),$$

$$x_3 = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3\gamma}(2v_3 - v_1 - v_2).$$

(2) Trading between agents 1 and 2 only.

$$v_1 - \gamma x_1 = v_2 - \gamma x_2 > v_3 - \gamma e_3 \tag{17}$$

if

$$x_1 < e_1, x_2 > e_2, x_1 + x_2 = e_1 + e_2, x_3 = e_3.$$

The efficient allocation is

$$x_1 = \frac{1}{2}(e_1 + e_2) + \frac{1}{2\gamma}(v_1 - v_2),$$

$$x_2 = \frac{1}{2}(e_1 + e_2) + \frac{1}{2\gamma}(v_2 - v_1),$$

$$x_3 = e_3.$$

(3) Trading between agents 2 and 3 only.

$$v_1 - \gamma e_1 > v_2 - \gamma x_2 = v_3 - \gamma x_3 \tag{18}$$

if

$$x_1 = e_1, x_2 < e_2, x_3 > e_3, x_2 + x_3 = e_2 + e_3.$$

The efficient allocation is

$$x_1 = e_1.$$

$$x_2 = \frac{1}{2}(e_2 + e_3) + \frac{1}{2\gamma}(v_2 - v_3),$$

$$x_3 = \frac{1}{2}(e_2 + e_3) + \frac{1}{2\gamma}(v_3 - v_2).$$

Lemma 4 *The efficient allocation $x_i(v)$ is non-decreasing in v_i for each $i = 1, 2, 3$. The worst-off type for trader i under x in the case that all three agents engage in trading is $v_i^* = 1$ when $e_i \le X_i(1)$, $v_i^* = X_i^{-1}(e_i) \in (1, 2)$ when $X_i(1) < e_i < X_i(2)$, and $v_i^* = 2$ when $e_i \ge X_i(2)$.*

Proof. It is easy to see that $x_i(v)$ is non-decreasing in v_i for each $i = 1, 2, 3$. Now we show the second part of the lemma. Note that agent i's expected net utility can be written as

$$\begin{aligned} U_i(v_i) - (v_i e_i - \frac{1}{2}\gamma e_i^2) &= U_i(v_i^*) + \int_{v_i^*}^{v_i} X_i(u)du - v_i e_i + \frac{1}{2}\gamma e_i^2 \\ &= U_i(v_i^*) - (v_i^* e_i - \frac{1}{2}\gamma e_i^2) + \int_{v_i^*}^{v_i} (X_i(u) - e_i)du. \end{aligned}$$

Since $X_i(v_i)$ is strictly increasing in v_i, then $U_i(v_i) - (v_i e_i - \frac{1}{2}\gamma e_i^2)$ is minimized at $v_i^* = 1$ when $e_i \le X_i(1)$, at $v_i^* = X_i^{-1}(e_i)$ when $X_i(1) < e_i < X_i(2)$, and at $v_i^* = 2$ when $e_i > X_i(2)$. Q.E.D.

Proposition 2 *In the following three cases, efficient trading exists:*

(1)

$$e_1 > \frac{e_2 + e_3}{2} + \frac{1}{\gamma}, e_3 < \frac{e_1 + e_2}{2} - \frac{1}{\gamma},$$

and

$$0 < \frac{e_1 + e_2 + e_3}{3} - \frac{1}{3\gamma} < e_2 < \frac{e_1 + e_2 + e_3}{3} + \frac{1}{3\gamma}.$$

(2)

$$e_1 \ge e_2 + \frac{2}{\gamma}, e_3 \ge \frac{e_1 + e_2}{2} + \frac{1}{\gamma}.$$

(3)

$$e_1 \leq \frac{e_2 + e_3}{2} - \frac{1}{\gamma}, e_2 \geq e_3 + \frac{2}{\gamma}.$$

Proof: We only prove the first case. The proofs for the other two cases are identical to the proof for Proposition 1 and are omitted.

By Lemma 2, we need to calculate R in (8), in which x is the efficient allocation.

Under the assumption on e_1, e_2 and e_3, we have for any $v_i \in [1,2], i = 1,2,3,$

$$\begin{aligned} x_1(v) &= \frac{1}{3}(e_1+e_2+e_3) + \frac{1}{3\gamma}(2v_1 - v_2 - v_3) \\ &\leq \frac{1}{3}(e_1+e_2+e_3) + \frac{1}{3\gamma}(2 \times 2 - 1 - 1) \\ &= \frac{1}{3}(e_1+e_2+e_3) + \frac{2}{3\gamma} \\ &= \frac{1}{3}[e_1+e_2+e_3+\frac{2}{\gamma}] \\ &< \frac{1}{3}[e_1+2e_1] \\ &= e_1, \end{aligned}$$

and similarly

$$x_3(v) > e_3.$$

Thus, in equilibrium allocation we must have

$$x_1 = \frac{1}{3}(e_1+e_2+e_3) + \frac{1}{3\gamma}(2v_1 - v_2 - v_3),$$

$$x_2 = \frac{1}{3}(e_1+e_2+e_3) + \frac{1}{3\gamma}(2v_2 - v_1 - v_3),$$

$$x_3 = \frac{1}{3}(e_1+e_2+e_3) + \frac{1}{3\gamma}(2v_3 - v_1 - v_2).$$

Now we compute

$$R = \sum_{i=1}^{3} E[\eta(v_i|v_i^*)x_i(v) - \frac{\gamma}{2}x_i^2(v)] - \sum_{i=1}^{3}(v_i^* e_i - \frac{\gamma}{2}e_i^2). \tag{19}$$

First, we compute

$$\begin{aligned}
E[\eta(v_i|v_i^*)x_i(v_i, v_j)] &= \int_1^2 (2v_i - 1)\, X_i(v_i)\, dv_i - \int_{v_i^*}^2 X_i(v_i)\, dv_i \\
&= v_i^* X_i(v_i^*) - \int_1^2 v_i(v_i - 1)\, dX_i(v_i) + \int_{v_i^*}^2 v_i\, dX_i(v_i) \\
&= v_i^* X_i(v_i^*) - \frac{2}{3\gamma}\int_1^2 v_i(v_i - 1)\, dv_i + \frac{2}{3\gamma}\int_{v_i^*}^2 v_i\, dv_i \\
&= v_i^* X_i(v_i^*) - \frac{1}{3\gamma}v_i^{*2} + \frac{7}{9\gamma}.
\end{aligned}$$

Second, by the independence of distributions, we can show that

$$E[(x_i(v_i, v_{-i}))^2] = \frac{(e_1 + e_2 + e_3)^2}{9} + \frac{1}{2\gamma^2}\frac{1}{9}.$$

Plugging into (19), we obtain

$$R = \sum_{i=1}^{3}\left\{v_i^*(X_i(v_i^*) - e_i) + \frac{1}{3\gamma}((\frac{3}{2})^2 - v_i^{*2}) + \frac{\gamma}{2}\left(e_i^2 - \frac{(e_1 + e_2 + e_3)^2}{9}\right)\right\} \tag{20}$$

Because

$$\begin{aligned}
e_1 &> \frac{e_2 + e_3}{2} + \frac{1}{\gamma} \\
&> \frac{e_2 + e_3}{2} + \frac{1}{2\gamma},
\end{aligned}$$

we have

$$2e_1 > e_2 + e_3 + \frac{1}{2},$$

or equivalently

$$3e_1 > e_1 + e_2 + e_3 + \frac{1}{\gamma},$$

i.e.,

$$e_1 > \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3\gamma} = X_1(2).$$

By Lemma 4, the last inequality implies $v_1^* = 2$.

Similarly, we can show that

$$e_3 < \frac{1}{3}(e_1 + e_2 + e_3) - \frac{1}{3\gamma} = X_3(1),$$

which, by Lemma 4, implies $v_3^* = 1$, and the assumption on e_2 implies

$$X_2(1) < e_2 < X_2(2),$$

therefore,

$$v^* = X_2^{-1}(e_2),$$

where

$$X_2(v_2) = \frac{1}{3}(e_1 + e_2 + e_3) + \frac{1}{3\gamma}(2v_2 - 3).$$

Plugging v_1^*, v_2^* and v_3^* into (20), we obtain that

$$\begin{aligned} R &= 2[\frac{e_1 + e_2 + e_3}{3} + \frac{1}{3\gamma}(4 - 3) - e_1] + \frac{1}{3\gamma}(\frac{9}{4} - 4) + \frac{\gamma}{2}(e_1^2 - \frac{(e_1 + e_2 + e_3)^2}{9}) \\ &\quad + \frac{1}{3\gamma}(\frac{9}{4} - \frac{3 - \gamma(e_1 + e_2 + e_3)}{2}) + \frac{\gamma}{2}(e_2^2 - \frac{(e_1 + e_2 + e_3)^2}{9}) \\ &\quad (\frac{e_1 + e_2 + e_3}{3} + \frac{1}{3\gamma}(2 - 3) - e_3) + \frac{1}{3\gamma}(\frac{9}{4} - 1) + \frac{\gamma}{2}(e_3^2 - \frac{(e_1 + e_2 + e_3)^2}{9}) \\ &= \frac{e_1 + 7e_2 + e_3}{6} + \frac{1}{12\gamma} + \frac{\gamma}{2}[e_1^2 + e_2^2 + e_3^2 - \frac{(e_1 + e_2 + e_3)^2}{3}] > 0 \end{aligned}$$

The proposition is proved, Q. E. D.

Example 2. Assume that $N = \{1, 2, 3\}$ and that $e = (e_1, e_2, e_3) = (6, 4, 2)$. Assume also that $\gamma = 1/2$. Then it is easy to check that e satisfies the efficiency assumption in Proposition 2. Thus, there is an efficient trading mechanism. For example, suppose that $v_1 = 1, v_2 = v_3 = 2$. Then

$$x_1(v) = 4 + \frac{2}{3}(2v_1 - v_2 - v_3) = 8/3,$$

$$x_2(v) = 4 + \frac{2}{3}(2v_2 - v_1 - v_3) = 14/3,$$
$$x_1(v) = 4 + \frac{2}{3}(2v_3 - v_1 - v_2) = 14/3.$$

Note that agents peak demands are $(v_1/\gamma, v_2/\gamma, v_3/\gamma) = (2, 4, 4)$ with the total amount of 10. Thus, in the efficient allocation, the three agents share equal amount of the extra 2 units. The calculation of the transfers are tedious and thus omitted. But we can see that in this oversupply of water case, agent 3 should be compensated by a negative transfer because the oversupply is caused by the upper stream agent 1.

The following figure describes the three possible configurations of initial amounts of water e that allow efficient trading of water.

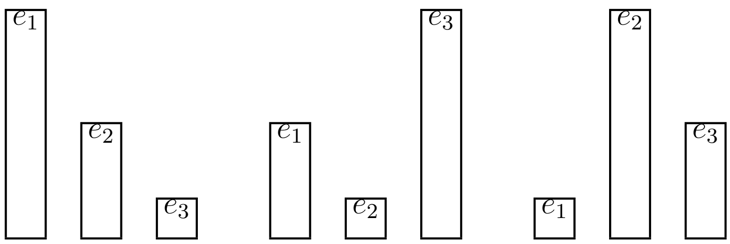

Figure 4. Three Possible Cases That Guarantee the Efficient Trading.

5 Concluding Remarks

In general, trading problems involving incomplete information often result in inefficiency. This was demonstrated in its simplest form in the seminal paper by Myerson and Satterthwaite (1983). However, in practice, trading often takes place in certain organized form. We impose a linear network structure on a trading problem with incomplete information. Specifically, we consider a water trading problem along a river. Under the assumption of concave and single-peaked benefit functions commonly used in water sharing problem and the assumption that agents have not too symmetric distribution of the initial amounts of water, we show that efficient trading mechanisms exist that are incentive-compatible, individually rational and budget balanced.

Our model can be extended to other trading network problems in which agents have single-peaked preferences and agents' peak levels are private information.

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Appendix

Transfers $(t_1(v), t_2(v))$ in **Example 1**:

Note that in the proof of Lemma 2, we have

$$t_i(v) = s_i(v_i) - s_j(v_j) - c_i, \tag{21}$$

where

$$s_i(v_i) = E_j[v_i x_i(v_i, v_j) - \frac{1}{2}\gamma x_i^2(v_i, v_j) - \int_{v_i^*}^{v_i} x_i(u, v_j)\, du]$$

and

$$c_i = \frac{1}{2}R + v_i^* e_i - \frac{1}{2}\gamma e_i^2 - E[\eta(v_j|v_j^*))x_j(v) - \frac{1}{2}\gamma x_j^2(v)].$$

First, we compute $s_1(v_1)$. Recall that $\gamma = 1/2, v_1^* = 2, v_2^* = 1$.

$$\begin{aligned}
s_1(v_1) &= E_2[v_1 x_1(v_1, v_2) - \frac{1}{2}\gamma x_1^2(v_1, v_2) - \int_{v_1^*}^{v_1} x_1(u, v_2)\, du] \\
&= v_1[\frac{e_1 + e_2}{2} + (v_1 - \frac{3}{2})] \\
&\quad -\frac{1}{4}E_2[\frac{(e_1 + e_2)^2}{4} + (e_1 + e_2)(v_1 - v_2) + (v_1 - v_2)^2] \\
&\quad -E_2[\int_2^{v_1}[\frac{e_1 + e_2}{2} + (u - v_2)]du] \\
&= \frac{e_1 + e_2}{2}v_1 + v_1^2 - \frac{3}{2}v_1 - \frac{1}{4}[\frac{(e_1 + e_2)^2}{4} \\
&\quad +(e_1 + e_2)\int_1^2 (v_1 - v_2)dv_2 + \int_1^2 (v_1 - v_2)^2 dv_2] \\
&\quad -\int_1^2 [\frac{e_1 + e_2}{2}(v_1 - 2) + \frac{1}{2}(u - v_2)^2|_2^{v_1}]dv_2 \\
&= \frac{1}{4}v_1^2 + \frac{3}{4}v_1 - \frac{(e_1 + e_2)^2}{16} + (e_1 + e_2)\frac{11 - 2v_1}{8} - \frac{19}{12}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
s_2(v_2) &= E_1[v_2 x_2(v_1, v_2) - \frac{1}{2}\gamma x_2^2(v_1, v_2) - \int_{v_2^*}^{v_2} x_2(v_1, u)\, du] \\
&= E_1[v_2 x_2(v_1, v_2) - \frac{1}{4}x_2^2(v_1, v_2) - \int_1^{v_2} x_2(v_1, u)\, du]
\end{aligned}$$

$$= \frac{1}{4}v_2^2 + \frac{3}{4}v_2 - \frac{(e_1+e_2)^2}{16} + (e_1+e_2)\frac{7-2v_2}{8} - \frac{19}{12}.$$

Thus,

$$s_1(v_1) - s_2(v_2) = \frac{v_1^2 - v_2^2}{4} + \frac{3(v_1 - v_2)}{4} + \frac{e_1+e_2}{2} - \frac{(e_1+e_2)(v_1-v_2)}{4}.$$

Now we compute c_1

$$c_1 = \frac{1}{2}R + v_1^* e_1 - \frac{1}{2}\gamma e_1^2 - E[\eta(v_2|v_2^*))x_2(v) - \frac{1}{2}\gamma x_2^2(v)].$$

In Proposition 1, we have calculated that

$$\begin{aligned} R &= \frac{1}{8\gamma} + \frac{\gamma(e_1-e_2)^2}{4} - \frac{e_1-e_2}{2} \\ &= \frac{1}{4} + \frac{(e_1-e_2)^2}{8} - \frac{e_1-e_2}{2} \end{aligned}$$

And

$$\begin{aligned} [E[\eta(v_2|v_2^*)x_2(v_1,v_2)] &= v_2^* X_2(v_2^*) - \frac{1}{2}v_2^{*2} + \frac{1}{2}((\frac{3}{2})^2 + \frac{1}{12}) \\ &= \frac{e_1+e_2}{2} + (1 - \frac{3}{2}) - \frac{1}{2} + \frac{7}{6} \\ &= \frac{e_1+e_2}{2} + \frac{1}{6} \end{aligned}$$

$$E[(x_2(v_1,v_2))^2] = \frac{(e_1+e_2)^2}{4} + \frac{1}{6}.$$

Thus,

$$\begin{aligned} c_1 &= \frac{(e_1-e_2)^2}{16} + \frac{(e_1+e_2)^2}{16} - \frac{e_1^2}{4} + \frac{5e_1-e_2}{4} \\ &= \frac{e_2^2 - e_1^2}{8} + \frac{5e_1-e_2}{4} \end{aligned}$$

Finally,

$$
\begin{aligned}
t_1(v) &= s_1(v_1) - s_2(v_2) - c_1 \\
&= \frac{v_1^2 - v_2^2}{4} + \frac{3(v_1 + v_2)}{4} + \frac{((e_1 + e_2)(2 + v_2 - v_1)}{4} \\
&\quad - \frac{e_2^2 - e_1^2}{8} - \frac{5e_1 - e_2}{4}.
\end{aligned}
$$

Therefore, if $e_1 = 4, e_2 = 0$, we have

$$t_1(v) = \frac{v_1^2 - v_2^2}{4} - \frac{v_1 - v_2}{4} - 1,$$

and thus

$$t_2(v) = -\frac{v_1^2 - v_2^2}{4} + \frac{v_1 - v_2}{4} + 1.$$