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Stable cost sharing in production allocation games

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Abstract

Suppose that a group of agents have demands for some good. Each one of them owns a technology allowing to produce the good, with these technologies varying in their effectiveness. We consider technologies exhibiting either increasing return to scale (IRS) or decreasing returns to scale (DRS). In each case, we solve the issue of the efficient allocation of the production between the agents. In the case of IRS, we prove that it is always efficient to centralize the production of the good, whereas efficiency in the case of DRS typically requires to spread the production. We then show that there exist stable cost sharing mechanisms whether we have IRS or DRS. Finally, we characterize a family of stable mechanisms exhibiting no price discrimination (agents are charged the same price for each unit demanded). Under some specific circumstances, our method generates the full core of the problem.

JEL classification numbers: C71, D63

Keywords: cost sharing, efficiency, stability, production allocation, returns to scale

1 Introduction

We examine the cooperative games (with transferable costs) that arise in the context where multiple agents have distinct technologies allowing to produce homogeneous goods (e.g., autonomous regions in the same country produce electricity using fossil fuels, hydropower, nuclear power, etc.; with each region having its own technology and demand for electricity. Thus, efficient regions will produce more than their own demands in order to sell electricity to less efficient ones). We use the phrase **production allocation game (PAG)** to refer to any such problem. Other interesting applications (besides the production of utilities) of our model include the cases of: a multinational firm trying to allocate its production between its plants over the world; family members/colleagues/neighbors dividing tasks among themselves. Two interesting questions arise in any PAG. First, one needs to determine the cost-minimizing allocation of the production. Secondly, the participants need to share that minimum cost, with notions of fairness and stability imposing restrictions on how to operate.

The structure of the available technologies greatly influences how these two underlying issues may be resolved. On the production side, if technologies exhibit decreasing returns to scale (DRS), the optimal plan typically spreads the production by having many producers contribute small quantities. In contrast, in the case of increasing returns to scale (IRS), we prove that it is always advantageous to centralize the production. For the cost sharing issue, the procedure allowing to find sensible rules also varies depending on the returns to scale, although in both cases we are concerned with compensating agents who produce for others with less efficient technologies.

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In both cases, we define a set of stable allocations that are very natural and intuitive: we consider a unit price p, and any agents who consume units that they have not produced have to pay that price p for any such unit. On the other hand, each agent producing units they do not consume is awarded the amount p for any such unit. Finally, agents who produce exactly their demand simply have to pay their stand-alone cost. We define lower and upper bounds for this p by using the stability requirement. These definitions differ if we are in the increasing or decreasing returns to scale case. In the DRS case, if demands are in $\{0,1\}$, our method generates the full core and allows to solve for the general equilibria of the economy induced by the PAG.

The model shares similarities with the literature on cost sharing with technological cooperation (Trudeau (2009a), Bahel and Trudeau (2013)), where agents put not only their demands but also their technologies in common when cooperating. That literature considers very general models, where the non-vacuity of the core is not even guaranteed. The model considered here is much more structured, and it can be viewed as a type of network flow problems (Quant et al. (2006), Trudeau (2009b)). In network flow problems, we have to deliver various quantities of a good to agents located at different points in space. There is a cost function for each link (between agents or between an agent and the source) that describes how much it costs to transport any quantities on the given link. If all of these functions are convex, the core is always non-empty (Quant et al. (2006)), a result that carries over to our PAGs under DRS. If all functions are concave, the core of a network flow problem may be empty. Interestingly, given the simplified structure of our PAGs under IRS, we are able to prove that they always exhibit a non-empty core.[1]

The paper is structured as follows. Section 2 presents the Production Allocation Games. PAGs with decreasing returns to scale are studied in Section 3. In Section 4, we look at PAGs with increasing returns to scale. Some additional results are presented in the Appendix.

2 The model

Let $N = \{1, ..., n\}$ be the finite set of agents, with $n \geq 2$. Each agent $i \in N$ demands the amount $x_i \in \mathbb{N}$ of the same good. The demand profile is thus $x \equiv (x_i)_{i \in N}$ and, for any coalition $S \subseteq N$, the aggregate demand can be written as $X_S \equiv \sum_{i \in S} x_i$. We define $X \equiv X_N$.

Each agent can produce any quantity q of the good at cost $C_i(q)$, with the function $C_i : \mathbb{N} \to \mathbb{R}_+$ being increasing and satisfying $C_i(0) = 0$. We denote by $\mathcal{C}$ the set of all such cost functions C_i. Let $C = (C_i)_{i \in N}$.

It will be convenient at times to use the marginal cost functions, which describe the incremental costs of the agents. Agent i's marginal cost function, c_i, is defined by $c_i(q) \equiv C_i(q) - C_i(q-1)$, for all $q = 1, 2, ...$ In the following definition and throughout the paper, we assume that the set of agents N is fixed. For any demand profile $x \in \mathbb{N}^N$ and any coalition $S \subseteq N$, let $\Delta(x, S) \equiv \{q \in \mathbb{N}^S : \sum_{i \in S} q_i = X_S\}$. A vector $q \in \Delta(x, S)$ is a production plan that allows the production of the aggregate demand of the members of S within the facilities of its members.

Definition 1
Let $x \in \mathbb{N}^N$ and $C \in \mathcal{C}^N$. We call ***production allocation game*** *(PAG) associated with (x, C) the cooperative cost game with player set N and characteristic cost function $\tilde{C}^x(\cdot)$ defined by:*

$$\tilde{C}^x(S) = \min_{q \in \Delta(x,S)} \sum_{i \in S} C_i(q_i), \text{ for all } S \subseteq N.$$

The above definition means that, in a PAG, the best action available to every coalition S is to split the production of its aggregate demand X_S between the plants it owns so as to minimize the total cost

[1] Another well-known member of the family of network flow problems is the shortest path problem, where we have either a single demander or linear cost functions. Rosenthal (2013) and Bahel and Trudeau (2014) study the issues of fairness and stability in shortest path problems.

of producing X_S. We will often abuse terminology by referring to a pair (x, C) as a PAG. There is no ambiguity, since N is fixed and each pair (x, C) induces a unique PAG in the sense of Definition 1.

For every PAG, there are two underlying issues that a mechanism designer ought to address. The first one is about efficiency: the optimal allocation of the whole demand X between the plants must minimize the total cost of production. The second one has to do with incentives: if possible, the cost of producing X has to be split between the agents in a way that prevents any subgroup of players from defecting. In the following definition, we introduce the mechanisms used to split the cost between the agents in N.

Definition 2
A Cost Sharing Method (CSM) is a mapping $y : \mathbb{N}^N \times \mathcal{C}^N \to \mathbb{R}^N$ *such that, for every PAG* (x, C), $\sum_{i \in N} y_i(x, C) = \tilde{C}^x(N) = \min\limits_{q \in \Delta(x,N)} \sum_{i \in N} C_i(q_i)$.

Note from Definition 2 that we allow for negative cost shares. Indeed, some agents may be compensated if (for example) they have a demand of zero and their plant is used to produce the other's demands. Also observe that we only consider efficient mechanisms; the designer hence needs to know the solution to the efficiency problem before assigning shares to the agents. An example of a cost sharing method is the proportional rule, y^{pr}, which splits the minimum cost of producing X in proportion to the agents' demands. Precisely, the proportional rule is defined by $y_i^{pr}(x, C) = \frac{x_i}{X} \tilde{C}^x(N)$, for any $i \in N$ and $x \in \mathbb{N}^N$ s.t. $X > 0$.[2]

Throughout the paper, we are interested in finding core allocations. The notion of the core is crucial in game theory and goes back to Gillies (1953). It contains the set of allocations satisfying the property that no group of agents can do better by splitting away from the grand coalition. Formally,

$$Core(x, C) = \left\{ y \in \mathbb{R}^N \,\middle|\, y(S) \leq \tilde{C}^x(S) \text{ for all } S \subset N \text{ and } y(N) = \tilde{C}^x(N) \right\}.$$

We say that a CSM $y(x, C)$ satisfies core selection if $y(x, C) \in Core(x, C)$ for all PAG (x, C). In addition we will say that $y(x, C)$ is individually rational if, for any $i \in N$, $y_i(x, C) \leq C_i(x_i)$ always holds. Obviously, core selection implies individual rationality.

3 PAGs with convex cost functions (DRS)

In this section we examine the case of PAGs (x, C) such that all the C_i are (weakly) convex. That is to say, for all $i \in N$ and $q \geq 1$, we have $c_i(q) \leq c_i(q + 1)$. For all such PAG, we discuss the issues of efficiency and stability.

A convex PAG is illustrated in the following example.

Example 1 *Consider the PAG described in Figure 1.*

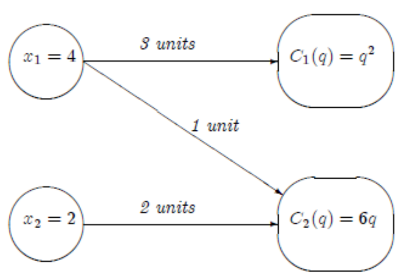

Figure 1: Convex PAG example

[2]Efficiency and individual rationality require that $y_i^{pr}(x, C) = 0$ whenever $X = 0$.

In this case where $N = \{1,2\}$, one can see that efficiency requires to split the total amount to produce $[X = 6]$ equally between the two plants —as indicated in Figure 1. That is to say, the lowest production cost is $\tilde{C}^x(N) = C_1(3) + C_2(3) = 9 + 18 = 27$. The cost shares associated with the proportional rule, defined in the previous section, are thus $y_1^{pr}(x, C) = 18$ and $y_2^{pr}(x, C) = 9$.

3.1 Finding the optimal production plan

Given the particular structure of our problem, we first show that we can assign production in an incremental manner. We use the following notation: For any $i \in N$ let $e^i \in \mathbb{R}_+^N$ be such that $e_j^i = 1$ if $i = j$ and 0 otherwise.

Lemma 1 *Consider a PAG (x, C) s.t. C_i is convex for all $i \in N$ and $X \geq 1$. Pick j s.t $C_j(1) = \min_{i \in N} C_i(1)$ and k s.t. $x_k \geq 1$; and let $C'_j(l) = C_j(l+1) - C_j(1)$ for $l \in \mathbb{N}$, $C' = \left(C'_j, C_{\{-j\}}\right)$, $x' = x - e^k$.*

Then we have: $q^* \in \arg\max_{q \in \Delta(x', N)} \sum_{i \in N} C'_i(q_i)$ implies that $q^* + e^j \in \arg\max_{q \in \Delta(x, N)} \sum_{i \in N} C_i(q_i)$.

Proof. We proceed by contradiction. Suppose that $q^* \in \arg\max_{q \in \Delta(x', N)} \sum_{i \in N} C'_i(q_i)$ but that $q^* + e^j \notin \arg\max_{q \in \Delta(x, N)} \sum_{i \in N} C_i(q_i)$. Then, there exists $\mathring{q} \in \arg\max_{q \in \Delta(x, N)} \sum_{i \in N} C_i(q_i)$ such that

$$\begin{aligned}
\sum_{i \in N} C_i\left(\mathring{q}_i\right) &< \sum_{i \in N \setminus j} C_i\left(q_i^*\right) + C_j(q_j^* + 1) \\
&< \sum_{i \in N \setminus j} C'_i\left(q_i^*\right) + C'_j(q_j^*) + C_j(1)
\end{aligned}$$

It is easy to see that we must have that $\mathring{q}_j > 0$. Suppose otherwise. Then given the facts that $C_j(1) = \min_{i \in N} C_i(1)$ and that all C_i are convex, we have that the production plan $\mathring{q} - e^i + e^j$ is less expensive than $\mathring{q}$, for any i s.t. $\mathring{q}_i > 0$.[3] Thus, we can rewrite the above inequality as

$$\begin{aligned}
\sum_{i \in N \setminus j} C_i\left(\mathring{q}_i\right) + C_j\left(\mathring{q}_j\right) &< \sum_{i \in N \setminus j} C'_i\left(q_i^*\right) + C'_j(q_j^*) + C_j(1) \\
\sum_{i \in N \setminus j} C_i\left(\mathring{q}_i\right) + C'_j(\mathring{q}_j - 1) + C_j(1) &< \sum_{i \in N \setminus j} C'_i\left(q_i^*\right) + C'_j(q_j^*) + C_j(1) \\
\sum_{i \in N \setminus j} C'_i\left(\mathring{q}_i\right) + C'_j(\mathring{q}_j - 1) &< \sum_{i \in N \setminus j} C'_i\left(q_i^*\right) + C'_j(q_j^*)
\end{aligned}$$

which contradicts the fact that q^* solves the cost minimization of (x', C') ■

Applied multiple times, the above Lemma implies that once we have found how to optimally produce k units, we do not need to start over to find how to produce $k' > k$ units. Finding how to allocate the remaining $k' - k$ units is sufficient.

Inspired by this, we describe an algorithm allowing to find an optimal allocation of the production $X > 0$ between the agents in N.

Stage 1. Pick $i_1 \in \underset{i \in N}{\text{argmin}}\ c_i(1)$[4] and write $C^*(N, 1) = c_{i_1}$. Next, let $\hat{q}^1 = e^{i_1}$ and update the cost functions: $\hat{c}_{i_1}^2 = c_{i_1}(2); \hat{c}_i^2 = c_i(1)$, for all $i \in N \setminus \{i_1\}$. Proceed to stage 2 if $X > 1$.

Stage $k \geq 2$. Pick $i_k \in \underset{i \in N}{\text{argmin}}\ \hat{c}_i^k$, and let $\hat{q}^k = \hat{q}^{k-1} + e^{i_k}$. Write the cost of producing the first k units of demand as $C^*(N, k) = \hat{c}_{i_k}^k + C^*(N, k-1)$. The cost functions are then updated as follows: $\hat{c}_i^{k+1} = c_i(\hat{q}_i^k + 1)$, for all $i \in N$. Proceed to stage $k+1$ as long as $k < X$.

[3] Note that there exists $i \in N \setminus j$ such that $\mathring{q}_i > 0$ since $\sum_{i \in N} \mathring{q}_i = X \geq 1$.

[4] In case many agents i have the lowest marginal cost, we pick the one with the lowest label.

The procedure ends at step $k = X$, that is to say, when the amount X to produce has been fully allocated between the plants in N; and the allocation of the production is then given by the profile $\hat{q}^X$. In addition, the production cost associated with the procedure is given by $C^*(N, X)$. As stated by the following result, $\hat{q}^X$ is an efficient allocation of the production between the agents.

Theorem 1 *For any PAG $(x, C) \in \mathbb{N}^N \times \mathcal{C}^N$, the allocation of the production $\hat{q}^X$ described by the algorithm solves the problem*

$$\min_{q \in \Delta(x,N)} \sum_{i \in N} C_i(q_i);$$

and the minimum cost to produce X, given the technologies $(C_i)_{i \in N}$, is $C^*(N, X)$.

Proof. For $k = 1$, it is clear that $\hat{q}^1$ is an optimal way to produce one unit. It follows from Lemma 1 that if $\hat{q}^k$ is an optimal way to produce k units, $\hat{q}^{k+1}$ is an optimal way to produce $k+1$ units, for any $k \in \mathbb{N}$. ■

Efficiency requires that $\sum_{i \in N} y_i(x, C) = C^*(N, X)$ and the algorithm can be used to determine its value.

It is worth pointing out that our method is an adaptation to our discrete setting of the natural method used in the presence of continuous demands, which allows to find the optimal production plan by equalizing the marginal costs of all agents. It is also important to note that, at the end of the algorithm ($k = X$), the vector $\left(\hat{c}_i^X\right)_{i \in N \setminus i_X}$ gives the amount it would cost to produce an additional unit by some agent other than i_X. We illustrate the method with an example.

Example 2 *Suppose that $N = \{1, 2, 3, 4, 5\}$ and the demand profile is $x = (0, 0, 1, 1, 2)$. In addition, consider the cost functions given in the following table (for the first three units produced).*

k	$C_1(k)$	$C_2(k)$	$C_3(k)$	$C_4(k)$	$C_5(k)$
1	*5*	*9*	*6*	*10*	*10*
2	*13*	*19*	*16*	*20*	*20*
3	*23*	*29*	*26*	*30*	*30*

The results obtained with the algorithm are summarized in the following table.

	c_1	c_2	c_3	c_4	c_5	i_k	$\hat{q}^k$	$C^*(N,k)$
Stage 1	*5*	*9*	*6*	*10*	*10*	*1*	*(1,0,0,0,0)*	*5*
Stage 2	*8*	*9*	*6*	*10*	*10*	*3*	*(1,0,1,0,0)*	*11*
Stage 3	*8*	*9*	*10*	*10*	*10*	*1*	*(2,0,1,0,0)*	*19*
Stage 4	*10*	*9*	*10*	*10*	*10*	*2*	*(2,1,1,0,0)*	*28*

3.2 Finding stable allocations

We can use a result of Quant et al. (2006) to show that for any convex PAG, the core is always non-empty.

Theorem 2 *For all PAG (x, C) with C_i convex for all $i \in N$, $Core(x, C)$ is non-empty.*

Proof. Quant et al. (2006) prove that the core of a convex network flow problem is always non-empty. PAGs can be modelled as network flow problems where $C_i(t)$ is the cost to send t units of goods from a source to agent i, and where there is no shipping cost to send units from an agent i to an agent j. Given that the functions $C_i(t)$ are all convex, convex PAGs are convex network flow problems (and thus have a non-empty core). ■

The proof of Quant et al. (2006) is not constructive and does not allow us to gain any insight on the allocations in the core. In the following, we introduce some methods that satisfy core selection.

Lemma 2 *Consider a PAG $(x, C) \in \mathbb{N}^N \times \mathcal{C}^N$ and suppose that $\hat{q}_i^X = x_i$, for all $i \in N$. Then, for any CSM y that satisfies core selection, we have:*

$$y_i(x, C) = C_i(x_i), \text{ for all } i \in N.$$

Proof. The result follows directly from individual rationality and efficiency. ■

Thus, the interesting case is when some agents produce more (less) units than they demand, that is to say, $\hat{q}_i^X > x_i$ for some $i \in N$. In this case, we provide a non-empty set containing stable cost shares.

For any $p \geq 0$, define the cost allocation y^p by:

$$y_i^p(x, C) = C_i\left(\hat{q}_i^X\right) - p\left(\hat{q}_i^X - x_i\right). \tag{1}$$

It is easy to check that the share profile $y^p(x, C)$ satisfies efficiency. In addition, compute the two (nonnegative) numbers

$$\bar{p}_L \equiv \min_{i \in N} \hat{c}_i^X \text{ and } \bar{p}_H \equiv \min_{i \in N \setminus i_X} \hat{c}_i^X.$$

They are the marginal costs of (respectively) the last unit (produced by i_X) and a hypothetical extra unit (produced by some agent other than i_X). By the properties of $\hat{c}_i^k$, it is easy to see that $\bar{p}_L \leq \bar{p}_H$, allowing us to define a continuum of cost allocations as follows:

$$\mathcal{Y}^D(x, C) \equiv \{y^p(x, C) : p \in [\bar{p}_L, \bar{p}_H]\}.$$

One can then state the following result.

Theorem 3 *Consider a convex PAG (x, C). Then we have:*
a) $\emptyset \neq \mathcal{Y}^D(x, C) \subseteq Core(x, C)$. That is to say, $\mathcal{Y}^D(x, C)$ is a non-empty set of stable cost allocations.
b) If $x_i \in \{0, 1\}$ for any $i \in N$ and $|\{i : \hat{q}_i^X > x_i\}| > 1$, then

$$\mathcal{Y}^D(x, C) = Core(x, C).$$

Proof. Let (x, C) be a convex PAG.

a) Given the problem (x, C), it is obvious that $\mathcal{Y}^D(x, C)$ is nonempty, since $\bar{p}_H \geq \bar{p}_L$. If $\hat{q}_i^X = x_i$, for all $i \in N$, it is easy to see from Eq. (1) that $\mathcal{Y}^D(x, C)$ consists of a single allocation which is the unique element of the core (by Lemma 2). Otherwise, we have coalitions that produce more than they consume (net producers) and coalitions that consume more than they produce (net buyers). If a coalition is a net producer $\left(\sum_{i \in S} \hat{q}_i^X > X_S\right)$, the coalition cannot benefit from seceding, as the extra units are sold at a price no smaller than their marginal cost of production (as $p \geq \bar{p}_L = \min_{i \in N} \hat{c}_i^X$). If a coalition is a net buyer $\left(\sum_{i \in S} \hat{q}_i^X < X_S\right)$, the coalition cannot benefit from seceding, as the extra units are bought at a price no larger than their marginal cost to produce them internally $\left(\text{as } p \leq \bar{p}_H = \min_{i \in N} \hat{c}_i^{X+1}\right)$. Agents that consume as much as they produce have trivially no incentives to deviate.

b) Suppose that (x, C) is such that $x_i \in \{0, 1\}$ for any $i \in N$ and $|\{i : \hat{q}_i^X > x_i\}| > 1$.

A priori, not necessarily all units sold by a producer have the same price. Suppose that one unit sold by j (to some t) has a price p and another unit sold by k (to some s) has a price $p' > p$. Then, letting B_j be the (possibly empty) set containing all agents buying from j (other that t), the coalition $\{j, s\} \cup B_j$ can deviate by setting a price p'' s.t. $p' > p'' > p$ for the unit sold by j to s (with unchanged prices for the units sold to agents in B_j). Therefore, in a stable allocation, any two distinct producers must sell each of their units at the same price. And as a consequence, since we have at least two net producers, any two units sold by the same producer must have the same price as well. Let p be that unique price.

If $p < \bar{p}_L = \min_{i \in N} \hat{c}_i^X$, the producer of the last unit would like to deviate by kicking a net demander of the last unit out, as she loses money on the production of that last unit.

If $p > \bar{p}_H = \min_{i \in N \setminus i_X} \hat{c}_i^X$, the agent that can produce an extra unit at cost $\bar{p}_H = \min_{i \in N \setminus i_X} \hat{c}_i^X$ could deviate with a net demander by selling that unit at a price p' such that $\bar{p}_H \leq p' < p$. ■

We thus have three groups of agents: those that produce their demand and nothing else, those that produce more than they demand (the producers) and those that produce less than they demand (the buyers). We have a unique price at which the producers sell to the buyers. That price cannot be below the marginal cost of the last unit of demand, while competition between producers keeps the price below (or equal to) the marginal cost of an extra unit of demand.

Example 3 *We revisit Example 2. The allocation* y^p *is* $(13-2p, 9-p, 6, p, 2p)$.

We have that $\bar{p}_L = \min\limits_{i\in N} \hat{c}_i^X = 9$ *and* $\bar{p}_H = \min\limits_{i\in N\setminus i_X} \hat{c}_i^X = 10$. *One thus obtain that* $y^{\bar{p}_L} = (-5, 0, 6, 9, 18)$ *and* $y^{\bar{p}_H} = (-7, -1, 6, 10, 20)$.

In the Appendix we describe the core for cases where we have a single producer. Note that the combination of Theorem 3-b and Theorem A.1 (in the Appendix) gives the description of the entire core for all convex PAGs where each agent demands at most one unit.

3.3 Interpretation as market equilibrium

The simple allocations y^p (defined by Eq. (1)) can be used to find the competitive equilibria of an economy with 2 commodities, where the technologies are given by the n cost functions. These equilibria forbid any form of price or quantity discrimination. We illustrate this through an example. The first commodity (X) is the good that the agents demand, and the second commodity (Y) is the numeraire, in units of which the production cost will be expressed. Let x_i be the (discrete) quantity consumed of good X and y_i be the expenditure in terms of the numeraire Y. Suppose that each agent has a quasi-linear utility function $U_i(x_i, y_i) = u_i(x_i) - y_i$, with u_i (weakly) increasing and concave. We assume that each agent has an endowment of zero of the consumption good X, and an amount of the numeraire Y that is sufficient for them to purchase their demand of X.

The preferences (and the efficient technology) allow us to determine the demands $\hat{x}$. We then show that supply and demand for the consumption good are in equilibrium if and only if the price p for good X is such that $\bar{p}_L \leq p \leq \bar{p}_H$.

Example 4 *Recall the 5 agents (and their marginal costs) of Example 2and suppose that the utility functions* u_i *(for 3 units or less) are given by*

x_i	$u_1(x_1)$	$u_2(x_2)$	$u_3(x_3)$	$u_4(x_4)$	$u_5(x_5)$
1	*8*	*7*	*20*	*12*	*13*
2	*16*	*14*	*28*	*20*	*24*
3	*24*	*21*	*36*	*28*	*32*

In the following table we slightly modify the procedure of the algorithm in Subsection 3.1 by assigning each unit produced at stage k *to the agent* (j_k) *whose marginal utility at that stage is maximal. The variable* $\hat{q}_i$ $(\hat{x}_i)$ *denotes the number of units produced (consumed) by* i *at the* end *of stage* k. *Note that* $\hat{x}^k$ *is the vector of assigned consumptions at the end of stage* k *and* u_i' *stands for* i*'s marginal utility (of consumption* $\hat{x}_i$*) at the* beginning *of stage* k.

k	c_1, u_1'	c_2, u_2'	c_3, u_3'	c_4, u_4'	c_5, u_5'	i_k, j_k	$\hat{q}^k$	$\hat{x}^k$
1	$\underline{5}$, *8*	*9, 7*	*6,* $\overline{20}$	*10, 12*	*10, 13*	*1,3*	e^1	e^3
2	*8,8*	*9,7*	$\underline{6}$,*8*	*10,12*	*10,*$\overline{13}$	*3,5*	e^1+e^3	e^3+e^5
3	$\underline{8}$,*8*	*9,7*	*10,8*	*10,*$\overline{12}$	*10,11*	*1,4*	$2e^1+e^3$	$e^3+e^4+e^5$
4	*10,8*	$\underline{9}$,*7*	*10,8*	*10,8*	*10,*$\overline{11}$	*2,5*	$2e^1+e^2+e^3$	$e^3+e^4+2e^5$
5	*10,8*	*10,7*	*10,8*	*10,8*	*10,8*	-	-	-

At the end of Stage 4 $(k = 4)$*, it is easy to see that the efficient consumption profile is* $\hat{x}^4 = e^3 + e^4 + 2e^5 = (0, 0, 1, 1, 2)$ *and the efficient production profile is* $\hat{q}^4 = 2e^1 + e^2 + e^3 = (2, 1, 1, 0, 0)$. *Indeed, at Stage 5, it is not efficient to produce a fifth unit, since the minimum marginal cost (10) is higher than the maximum marginal utility (8).*

It is then easy to see from the above table that our upper and lower prices are $\bar{p}_L = 9$ *and* $\bar{p}_H = 10$. *At any price* $p \in [\bar{p}_L, \bar{p}_H]$*, agents 1,2,3 maximize their profits by producing the respective amounts 2, 1, 1; and agent 4,5 each maximize their profits by producing nothing. On the demand side, at any such price p, agents 1 and 2 will obviously not consume any unit of X, while agents 3,4,5 will respectively* *consume 1,1 and 2. Hence, supply and demand are both equal to 4; and all agents are optimizing given the fixed price p. Therefore, the tuples* $((\hat{q}^4, \hat{x}^4), y^p), p)$, *with* $p \in [\bar{p}_L, \bar{p}_H]$, *are general equilibria; and one can check that there are no other equilibria. It is easy to see that our algorithm (with some minor adjustments) allows to find the general equilibria for any such economy.*

4 PAGs with concave cost functions (IRS)

In this section we examine the case of *concave PAGs*, that is to say, the case of PAGs (x, C) such that the cost functions C_i are all (weakly) concave:[5] For all $i \in N$ and $q \geq 1$, we have $c_i(q) \leq c_i(q+1)$. In other words, the marginal cost of each agent is a decreasing function of the amount to produce. We first state a few preliminary results that follow from the concavity of the respective cost functions.

Lemma 3
Suppose that $C = (C_i)_{i\in N}$ *is a collection of concave cost functions. Then, for each agent* $i \in N$, *we have:*

$$\frac{C_i(q)}{q} \geq \frac{C_i(q+1)}{q+1} \geq c_i(q+1), \text{ for all } q \in \{1, 2, 3, \ldots\}.$$

Proof. Let $i \in N$ and $q \geq 1$. We can write:

$$\frac{C_i(q+1)}{q+1} = \frac{\overbrace{c_i(1) + \ldots + c_i(q)}^{C_i(q)} + c_i(q+1)}{q+1} = \frac{q\frac{C_i(q)}{q} + c_i(q+1)}{q+1} \tag{2}$$

Since C_i is concave, we have: $c_i(k) \geq c_i(q+1)$ for $k = 1, \ldots, q$; and it follows that $\frac{C_i(q)}{q} = \frac{c_i(1)+\ldots+c_i(q)}{q} \geq c_i(q+1)$. We can thus conclude that any weighted average $\left[\omega\frac{C_i(q)}{q} + (1-\omega)c_i(q)\right]$ of the two numbers $\left[\frac{C_i(q)}{q} \text{ and } c_i(q))\right]$ satisfies: $\frac{C_i(q)}{q} \geq \omega\frac{C_i(q)}{q} + (1 - \omega c_i(q)) \geq c_i(q)$, for all $\omega \in [0, 1]$. Taking $\omega = \frac{q}{q+1}$ and recalling (2), one can write:

$$\frac{C_i(q)}{q} \geq \frac{C_i(q+1)}{q+1} = \overbrace{\frac{q}{q+1}}^{\omega}\frac{C_i(q)}{q} + \overbrace{\frac{1}{q+1}}^{1-\omega}c_i(q+1) \geq c_i(q+1).$$

■

The above lemma states the known result that, for any concave cost function, the associated average cost function is decreasing and everywhere above the marginal cost function. In the remainder of this section, we consider a **fixed** concave PAG (x, C) s.t. $x \neq 0_N$.

4.1 Finding the optimal production plan

For any $k \geq 1$, let $i_k^* \in \arg\min_{i\in N} C_i(k)$ be (one of) the agent(s) whose cost is lowest when the total demand to produce is k units.[6] In addition, we will use the notation $i^* \equiv i_X^*$, that is, agent i^* minimizes the cost of producing the total demand for the problem (x, C),

It is not difficult to show that, in the case of concave PAGs, it is always optimal to centralize the production in one of the facilities so as to take advantage of the increasing returns to scale. This result is formally stated as follows.

[5] Recall that the notation c_i stands for agent i's marginal cost function, that is to say, $c_i(q) = C_i(q) - C_i(q-1)$, for any $q = 1, 2, \ldots$.

[6] Such an agent always exists because N is a finite set. And even though we may have many such agents, our results apply regardless of which one is picked.

Theorem 4 *For the concave PAG (x, C), the cost of production is minimized by the production plan $\bar{q} \in \mathbb{N}^N$ s.t. $\bar{q}_{i^*} = X$ and $\bar{q}_i = 0$, for all $i \neq i^*$. That is to say,*

$$C_{i^*}(X) = \min_{q \in \Delta(x,N)} \sum_{i \in N} C_i(q_i).$$

Proof. Given the fixed concave PAG (x, C), let $\bar{q} \in \Delta(x, N) \equiv \{q \in \mathbb{N}^N : \sum_{i \in N} q_i = X\}$. Since $C_i(0) = 0$ for any agent i, one can write:

$$\begin{aligned} \sum_{i \in N} C_i(\bar{q}_i) = & \sum_{i \in N : \bar{q}_i > 0} C_i(q_i) \\ = & X \sum_{i \in N : \bar{q}_i > 0} \frac{\bar{q}_i}{X} \frac{C_i(\bar{q}_i)}{q_i} \\ = & X \sum_{i \in N : \bar{q}_i > 0} \omega_i \frac{C_i(\bar{q}_i)}{\bar{q}_i}, \end{aligned} \tag{3}$$

where $\omega_i = \frac{\bar{q}_i}{X} \in [0, 1]$. Note that we have $\sum_{i \in N} \omega_i = 1$, given that $\bar{q} \in \Delta(x, N)$. Since $\bar{q}_i \leq X$ for any $i \in N$, it follows from Lemma 3 that:

$$\frac{C_i(\bar{q}_i)}{q_i} \geq \frac{C_i(X)}{X} \geq \frac{C_{i^*}(X)}{X} \equiv \min_{i' \in N} \frac{C_{i'}(X)}{X}, \tag{4}$$

for any $i \in N$ s.t. $\bar{q}_i > 0$. Plugging (4) into (3) then gives

$$\sum_{i \in N} C_i(\bar{q}_i) = X \sum_{i \in N : \bar{q}_i > 0} \omega_i \frac{C_i(\bar{q}_i)}{\bar{q}_i} \geq X \sum_{i \in N : \bar{q}_i > 0} \omega_i \frac{C_{i^*}(X)}{X} = C_{i^*}(X).$$

In other words, $C_{i^*}(X) = \min_{q \in \Delta(x,N)} \sum_{i \in N} C_i(q_i)$. ■

Interestingly, Theorem 4 means that, in order to efficiently allocate the production X, one just needs to compute and compare the n numbers corresponding to the respective costs of producing the total demand in each of the different plants. This is in contrast with the case of convex cost functions where one typically needs to spread the production over the different plants.

4.2 Finding stable allocations

We assume in what follows that $X - x_{i^*} > 0$ —at least one player other than i^* has a positive demand. Otherwise, there is only one (trivial) core cost allocation where all players other than i^* pay zero. For any $p \geq 0$, define the cost allocation y^p by:[7]

$$\begin{aligned} y^p_{i^*} &= C_{i^*}(X) - p(X - x_{i^*}); \\ y^p_i &= p x_i, \text{ for all } i \neq i^*. \end{aligned}$$

Notice that the expressions above are simplifications of the formula for y^p given in the DRS case of Section 3. These simplifications are possible under IRS because we always have a single producer, as stated in Theorem 4.

In addition, compute the two (nonnegative) numbers

$$\tilde{p}_L \equiv \frac{C_{i^*}(X)}{X} \text{ and } \tilde{p}_H \equiv \frac{\min_{i \neq i^*} C_i(X - x_{i^*})}{X - x_{i^*}}.$$

[7] Note that y^p is a well-defined allocation. Indeed, since $\sum_{i \in N} y^p_i = C_{i^*}(X)$, it satisfies efficiency.

Using Lemma 3, it is easy to see that we have $\tilde{p}_L \leq \tilde{p}_H$. This observation allows us to define a continuum of cost allocations as follows.

$$\mathcal{Y}^I(x, C) \equiv \{y^p : p \in [\tilde{p}_L, \tilde{p}_H]\}.$$

One can then state the following result.

Theorem 5 *Given the concave PAG (x, C), we have $\emptyset \neq \mathcal{Y}^I(x, C) \subseteq Core(x, C)$. That is to say, $\mathcal{Y}^I(x, C)$ is a nonempty set of stable cost allocations.*

Proof. Given the concave problem (x, C), it is obvious that $\mathcal{Y}^I(x, C)$ is nonempty, since $\tilde{p}_L \leq \tilde{p}_H$. To prove that $\mathcal{Y}^I(x, C) \subseteq Core(x, C)$, it is enough to show that both $y^{\tilde{p}_L}$ and $y^{\tilde{p}_H}$ are stable (recall that the core is a convex set).

Fix a nonempty coalition $S \subset N$ and let $i_S^* \in \arg\min_{i \in S} C_i(X_S)$. We know from Theorem 4 that $C_{i_S^*}(X_S) = \min\limits_{q \in \Delta(x,S)} \sum\limits_{i \in S} C_i(q_i)$; and therefore the stand-alone cost of the coalition S is given by $C_{i_S^*}(X_S)$.

1- Stability of $y^{\tilde{p}_L}$

Under the allocation $y^{\tilde{p}_L}$, each player is charged $\tilde{p}_L = \frac{C_{i^*}(X)}{X}$ per unit demanded. Therefore, we have

$$\sum_{i \in S} y_i^{\tilde{p}_L} = \sum_{i \in S} \frac{C_{i^*}(X)}{X} x_i = \frac{C_{i^*}(X)}{X} \sum_{i \in S} x_i = \frac{C_{i^*}(X)}{X} X_S.$$

Using Lemma 3, we can write $\frac{C_{i^*}(X)}{X} \leq \frac{C_i(X_S)}{X_S}$, for all $i \in S$; and hence we have:

$$\sum_{i \in S} y_i^{\tilde{p}_L} = \frac{C_{i^*}(X)}{X} X_S \leq \left(\min_{i \in S} \frac{C_i(X_S)}{X_S}\right) X_S = \min_{i \in S} C_i(X_S) \equiv C_{i_S^*}(X_S).$$

Thus, no coalition S jointly pays more than its stand-alone cost; and hence $y^{\tilde{p}_L}$ is stable.

2- Stability of $y^{\tilde{p}_H}$

Under the allocation $y^{\tilde{p}_H}$, each agent **other than** i^* is charged a unit price of $\tilde{p}_H = \frac{\min\limits_{i \neq i^*} C_i(X - x_{i^*})}{X - x_{i^*}}$. We will therefore distinguish 2 cases: (a) $i^* \notin S$; and (b) $i^* \in S$.

(a) Suppose that $i^* \notin S$. Then we have $i^* \neq i_S^* \in S$ and $X_S \leq X - x_{i^*}$. It thus follows from Lemma 3 that

$$\sum_{i \in S} y_i^{\tilde{p}_H} = \frac{\min\limits_{i \neq i^*} C_i(X - x_{i^*})}{X - x_{i^*}} X_S \leq \frac{C_{i_S^*}(X_S)}{X_S} X_S = C_{i_S^*}(X_S)$$

Thus, no such coalition S will object to $y^{\tilde{p}_H}$.

(b) Suppose now that $i^* \in S$. In this case, since $\tilde{p}_H \geq \tilde{p}_L$, we have:

$$\begin{aligned}
\sum_{i \in S} y_i^{\tilde{p}_H} &= C_{i^*}(X) - \tilde{p}_H \sum_{i \neq i^*} x_i + \tilde{p}_H \sum_{i \notin S} x_i \\
&= C_{i^*}(X) - \tilde{p}_H \sum_{i \in S \setminus \{i^*\}} x_i \\
&\leq C_{i^*}(X) - \tilde{p}_L \sum_{i \in S \setminus \{i^*\}} x_i = \sum_{i \in S} y_i^{\tilde{p}_L} \leq C_{i_S^*}(X_S).
\end{aligned}$$

Note that the last inequality comes from the stability of $y^{\tilde{p}_L}$ (which was shown earlier). In conclusion, $y^{\tilde{p}_H}$ is stable. ■

It is interesting to note that $y^{\tilde{p}_L}$ is the *proportional cost allocation* —which charges $y_i^{\tilde{p}_L}(x,C) = \frac{x_i}{X}C_{i^*}(X) = \tilde{p}_L x_i$ to any $i \in N$. Using the fact that $y^{\tilde{p}_L}$ belongs to the set $\mathcal{Y}^I(x,C)$, we hence conclude that the proportional allocation is always stable in a concave PAG. This is another noteworthy difference with the case of convex PAGs —where the proportional allocation is typically not stable.[8]

Example 5 *Let $N = \{1,2,3\}$ and $x = (0,1,2)$. The cost functions for agents 1 and 2 are as follows.*

k	$C_1(k)$	$C_2(k)$
1	*6*	*4*
2	*6*	*7*
3	*6*	*9*

Agent 3 has a very inefficient technology of production (which is why we do not give her cost function). The efficient production plan is to have agent 1 produce all 3 units.

We have that $\tilde{p}_L = \frac{C_1(3)}{3} = 2$ and $\tilde{p}_H = \frac{C_2(3-0)}{3-0} = 3$. Thus, we have that $y^{\tilde{p}_L} = (0,2,4)$ and $y^{\tilde{p}_H} = (-3,3,6)$. It is easy to verify that these allocations are stable, as stated by Theorem 5.

5 Conclusion

Production Allocation Games are extremely natural, as they cover multiple cases where agents need to efficiently organize the production of goods or services, before sharing their costs. While production is centralized if we have increasing returns to scale and (usually) decentralized with decreasing returns to scale, we propose for both cases similar families of simple core allocations. These allocations are obtained by defining the value of a unit being produced by agents who have efficient technologies for others who do not. The respective expressions of the minimum and maximum prices differ in the IRS and DRS cases. However, in the particular case of constant returns to scale (i.e., linear cost functions) our two approaches both apply and, in fact, they each generate the entire core of the problem.

Interestingly, in the case of decreasing returns to scale, the allocations we propose are closely related to the general equilibria of the economy where the two commodities are the output and input of the production allocation game, and whose producers and consumers are the players.

In terms of cost sharing methods, it is to expect that in addition to the allocations corresponding to the upper and lower bounds (respectively the most advantageous allocations for net producers and net buyers), an interesting allocation would be obtained by using the average of these two bounds as the unit price. This particular allocation could be viewed as one where the benefits from cooperation are equally divided between net producers and net buyers in a production allocation game. Studying the properties of such allocation methods is left for subsequent studies.

We conclude this work by pointing out that our analysis can be used to solve the problem of stable and efficient allocation of multiple (divisible and independent) tasks between coworkers, firms in a partnership, and so on. A simple adjustment to our framework allows to include each such problem as a sum of independent production allocation games (to which our results apply).

References

Bahel, E., Trudeau, C., 2013. A discrete cost sharing model with technological cooperation. International Journal of Game Theory 42, 439–460.

Bahel, E., Trudeau, C., 2014. Stable lexicographic rules for shortest path games. Economics Letters, forthcoming.

[8]Another difference is that the interpretation of allocations y^p as market allocations in the case of concave PAGs is muddled by the same difficulties affecting general equilibrium analysis in economies with non-decreasing returns to scale. In fact, for such economies, there may exist no general equilibrium at all, which is why we do not provide a counterpart for Example 4.

Gillies, D. B., 1953. Some theorems on n-person games. Ph.D. Thesis, Department of Mathematics, Princeton University.

Quant, M., Borm, P., Reijnierse, H., 2006. Congestion network problems and related games. European Journal of Operational Research 172, 919–930.

Rosenthal, E. C., 2013. Shortest path games. European Journal of Operational Research 224 (1), 132 – 140.

Trudeau, C., 2009a. Cost sharing with multiple technologies. Games and Economic Behavior 67, 695–707.

Trudeau, C., 2009b. Network flow problems and permutationally concave games. Mathematical Social Sciences 58, 121 – 131.

A Appendix: PAGs with decreasing returns and a single producer

We describe the core in the (rare) cases where even though we have decreasing returns to scale, an agent has a superior technology that makes him the single producer.

Consider a fixed PAG (x,C) such that $\{i \in N \setminus j : \hat{q}_i^X > 0\} = \{j\}$; and let

$$\begin{aligned} A &\equiv \{i \in N \setminus j : x_i = 0\}; \\ B &\equiv \{i \in N \setminus j : x_i = 1 \text{ and } c_i(1) \leq \min_{k \neq i,j} c_k(x_k+1)\}; \\ B' &\equiv \{i \in N : x_i = 1 \text{ and } c_i(1) > \min_{k \neq i,j} c_k(x_k+1)\}. \end{aligned}$$

Theorem A.1
Suppose $x_i \in \{0,1\}$ for any $i \in N$ and $\{i \in N : \hat{q}_i^X > 0\} = \{j\}$, then a cost allocation y is stable iff

$$\begin{aligned} y_i &= 0 \text{ if } i \in A; && (5) \\ y_i &\in [c_j(X), c_i(1)] \text{ if } i \in B; && (6) \\ y_i &\in \left[c_j(X), \min\left\{\min_{k \in A \cup B} C_k(x_k+1) - y_k, c_i(1)\right\}\right] \text{ if } i \in B'; && (7) \\ y_j &= C_j(X) - \sum_{i \neq j} y_i. && (8) \end{aligned}$$

Proof. Only if. It is not difficult to see that any stable allocation must satisfy the conditions of the theorem.

If. Suppose now that y is an allocation satisfying the conditions of the theorem. Then we have the following.

(a) No single-player coalition can do better on its own.

Obviously, agents in A pay their stand-alone cost of zero. An agent $i \in B \cup B'$ pays less than $c_i(1)$, his stand-alone cost. We have that $y_j \leq C_j(X) - (X - x_j)\, c_j(X)$. If $x_j = 0$, by the convexity of the cost function, $y_j \leq 0$, her stand-alone cost. If $x_j = 1$, $y_j \leq C_j(1)$, her stand-alone cost.

(b) No other coalition can do better on its own.

It is clear by the upper bounds of what agents in $B \cup B'$ can pay that any coalition not containing j has no incentives to deviate. We consider a coalition S such that $j \in S$. We have

$$
\begin{aligned}
y(S) &= \sum_{q=1}^{X} c_j(q) - y(N\backslash S) \\
&= C_j(X_S) + \sum_{q=X_S+1}^{X} c_j(q) - y(N\backslash S) \\
&= C_j(X_S) + \sum_{q=X_S+1}^{X} c_j(q) - \sum_{i\in N\backslash(S\cup A)} y_i
\end{aligned}
$$

We have that $|N\backslash(S\cup A)| = X - X_S$. Since $y_i \leq c_j(X)$ for all $i \in B \cup B'$,

$$
\sum_{i\in N\backslash(S\cup A)} y_i \geq \sum_{q=X_S+1}^{X} c_j(q)
$$

and thus $y(S) \leq C_j(X_S)$, the stand-alone of coalition S. ■

In that case, the (unique) producer can price discriminate and charge different prices to the buyers. Notably, the maximum price it can charge to agent i decreases with the efficiency of agent i's technology.