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ABSTRACT: This paper considers the cost sharing problem on a fixed tree network. It provides a characterization of the family of cost sharing methods satisfying the axioms of Additivity and the Independence of Irrelevant Costs. Additivity is a classical axiom. The Independence of Irrelevant Costs axiom is new and replaces the traditional Dummy axiom to capture the network structure of the model.

JEL classification: C71, D70.

Keywords: Cost sharing; Tree network.

1 Introduction

Recently, there has been a renewed interest in the cost sharing problem on a fixed network. The problem can be traced back to the well-known airport landing fee problem (Littlechild and Owen, 1973). Koster et al. (2001) have considered the core allocations for cost sharing problems on a tree network. Moulin and Laigret (2011) and Moulin and Hougaard (2012) have considered cost sharing problems on more general networks.

This paper reconsiders the cost sharing problem on a fixed tree network. A finite number of agents are located on a tree network with a source. Each link (edge) has a cost and the total costs of all links must be shared among the agents.

We consider two axioms on cost sharing methods. We retain the classical axiom of Additivity (Shapley, 1953; Wang, 1999; Moulin, 2002) and introduce a new axiom called the Independence of Irrelevance Costs (IIC). The IIC requires that an agent's cost share should not be affected by those links' costs that the agent is not "related to". To be precise, call an agent a predecessor of another agent if he is on the path that connects the source to the latter. Call an agent a follower of another agent if he has the latter as a predecessor. For a given agent and a given link that connects a pair of nodes (agents), if both nodes are the predecessors or followers of the agent, we say that the link (and the associated cost) is *related to* the agent (or the agent is *relevant* to the link as we interchangeably call). The IIC says that an agent should not be responsible for the costs of those links that are irrelevant to him.[1]

A family of simple methods, called *the location labeled methods*, satisfy Additivity and the IIC. A location labeled method simply allocates each link's cost to an agent who is related to the link. Thus, by a location labeled method, an agent is only responsible for those link costs that he is related to but not beyond. Apparently, any convex combination of these methods satisfies Additivity and the IIC axioms.

The main result of this paper shows that no other methods than the above family satisfy Additivity and the IIC axioms. Precisely, *the set of additive*

[1]Think of the following example. A network of oil pipelines connects a number of countries. If some part of the pipelines needs repair, the associated cost should be shared among the countries that are related to the repaired part. Other countries that are not related to the part shouldn't bear any that cost. This axiom rules out the *equal division* of the total costs.

methods that satisfy the Independence of Irrelevant Costs axiom is the set of all convex combinations of the location labeled methods.

The next step is to introduce additional axioms that are relevant to the particular network cost sharing problems we are concerned with. We will not pursue that here. Instead, we conclude the paper by reconsidering the airport landing fee problem and proposing a new method for the problem. The method is chosen from the family of methods characterized above and it has the property that an airline pays a relatively less proportion for a distant cost component.

2 The Model

Consider a graph $T = \{N \cup \{0\}, E\}$ where $N = \{1, 2, ..., n\}$ ($n > 0$ is a positive integer) is the set of nodes each representing an agent and the node $\{0\}$ is a source that provides certain service to all the agents, and $E = \{e = \{i, j\} | i, j \in N\}$ is the set of links that connect all the agents to the source, either directly or indirectly, without forming cycles. The graph T is called a tree. Assume that there is cost associated with each link in E. Formally, a cost function (on T) is a mapping $C : E \to R_+$. Note that the number of links in E, denoted as $|E|$, is n. We denote $C(T) = \sum_{e \in E} C(e)$ the total cost of T.

A *cost sharing problem* (on a fixed tree network) is a triple (N, T, C). A *solution* to a problem (N, T, C) is a vector $x = (x_1, ..., x_n) \in R^N_+$ such that $\sum_i x_i = C(T)$, where x_i is the cost share assigned to agent i. A *method* is a mapping x that assigns to each problem (N, T, C) a solution $x(N, T, C)$. When N and T are fixed, a cost sharing problem can be written as a cost vector $C \in R^N_+$ and a solution as $x(C)$.

For any given T and any node $i \in N$, there is a unique path $P = \{i_0 = 0, i_1, ..., i_{k-1}, i_k = i\}$ in T connecting the source 0 and i in which $(i_m, i_{m+1}) \in E$ for $m = 0, 1, ..., k-1$. Call the nodes $0, i_1, ..., i_{k-1}$ predecessors of i and denote the set of predecessors of i as $P(i)$. Call j a follower of i if i is a predecessor of j. Denote the set of followers of i as $F(i)$. Denote $A(i) = \{i\} \cup P(i) \cup F(i)$. Note that any agent in $A(i)$ is either a predecessor or a follower of i and is considered relevant or related to agent i. Since each agent $i \in N$ has a unique predecessor i_{-1}, denote $c_i = C(\{i_{-1}, i\})$.

Now we introduce the following two axioms on cost sharing methods.

Additivity: For any $C^1, C^2 \in R_+^N$, $x(C^1 + C^2) = x(C^1) + x(C^2)$.

Independence of Irrelevant Costs (IIC): For any given T, any $k \in N$ and any $c_k \in R_+$, if

$$C^k = (0, \ldots, 0, c_k, 0, \ldots, 0),$$

then

$$\sum_{j \in A(k)} x_j(C^k) = c_k.$$

or equivalently,

$$x_j(C^k) = 0, j \notin A(k).$$

The IIC says that the cost of a link should only be shared among its relevant or related agents. In other words, no agent should be charged for any cost that is irrelevant to him. Note that in this paper, the IIC plays a similar role as the traditional Dummy axiom (Moulin, 2002).

3 The Representation Theorem

First, we need the following definition.

Definition 1 *For a given problem (N, T, C), a labeling L is a mapping from N to N such that $L(i) \in A(i)$ for each $i \in N$. Given a labeling L, define a cost sharing method, called a location labeled method, as follows.*

$$\phi^L(C) = \sum_{k \in N} c_k \cdot e^{L(k)} \tag{1}$$

where $e^{L(k)}$ is an n-dimensional unit vector whose $L(k)$-th component is 1 and all other components are 0.

It is easy to check that a location labeled method is additive and satisfies the IIC. Denote by $conv(CL)$ the set of all convex combinations of the location labeled methods. It is easy to see that all methods in $conv(CL)$ are also additive and satisfy the IIC.

Denote by Φ the set of all additive methods that satisfy the IIC. We now state our main theorem.

Theorem 1 *The set of additive methods that satisfy the Independence of Irrelevant Costs axiom is the set of all convex combinations of the location labeled methods, i.e.,*

$$\Phi = conv(CL). \tag{2}$$

The proof of Theorem 1 relies on the following two lemmas. The proofs of the two lemmas are omitted and available from the authors upon request.

Lemma 1 *If x satisfies Additivity and Independence of Irrelevant Costs, then x has the following representation:*

$$x_i(C) = \sum_{k \in A(i)} w_k^i c_k, i = 1, ..., n, \tag{3}$$

where $0 \leq w_k^i \leq 1$, and

$$\sum_{i \in A(k)} w_k^i = 1, k = 1, ..., n. \tag{4}$$

A *column stochastic matrix* is a matrix that all entries are nonnegative and that the sum of the entries in each column is one. A *unit column stochastic matrices* is a column stochastic matrix that in each column there is one nonzero entry that is equal to 1. The following lemma shows that any column stochastic matrix can be decomposed as a convex combination of unit column stochastic matrices.[2]

Lemma 2 *Any column stochastic matrix can be written as a convex combination of the unit column stochastic matrices.*

Now we are ready to prove the main theorem.

Proof of Theorem 1. We have already known that $\Phi \supseteq conv(CL)$. We now prove that $\Phi \subseteq conv(CL)$.

[2]The reader may immediately recall the well-known *double stochastic matrix*. A matrix is double stochastic if it is both column stochastic and row stochastic (the entries in each row of the matrix are nonnegative and sum to one). The well-known *Birkhoff-von Neumann Theorem* (Birkhoff, 1946) states that any double stochastic matrix can be written as a convex combination of the *permutation matrices*. A permutation matrix has a single nonzero entry, equal to 1, in each row *and* column. Note that Lemma 2 implies the Birkhoff-von Neumann Theorem but not the other way around.

Let $\phi \in \Phi$. From Lemma 1, ϕ corresponds to a weight system:

$$((w_k^1)_{k\in A(1)}; (w_k^2)_{k\in A(2)}; \ldots; (w_k^j)_{k\in A(j)}; \ldots; (w_k^n)_{k\in A(n)})$$

satisfying $\sum_{j\in A(k)} w_k^j = 1$ for all $k \in N$. Rearrange it as

$$W \equiv ((w_1^j)_{j\in A(1)}; (w_2^j)_{j\in A(2)}; \ldots; (w_k^j)_{j\in A(k)}; \ldots; (w_n^j)_{j\in A(n)}). \tag{5}$$

It is clear that the rearrangement is unique.

Extend the weight system W into an $n \times n$ matrix, also denoted as, W. This is done by expanding each column into an n-vector:

$$W_{jk} = \begin{cases} w_k^j, & \text{if} j \in A(k) \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

Now the matrix W is a column stochastic matrix. By Lemma 2, it can be written as a convex combination of the unit column stochastic matrices, $B_{i_1 i_2 \cdots i_n}$, where in column j the i_j'th entry is equal to 1 and $i_j \in A(j)$. But each unit column stochastic matrix $B_{i_1 i_2 \cdots i_n}$ corresponds to a location labeled method associated with the labeling $L(j) = i_j, j = 1, ..., n$. Therefore, ϕ can be written as a convex combination of the location labeled methods, i.e., $\Phi \subseteq conv(CL)$.

The theorem is proved. Q.E.D.

We conclude the paper by reconsidering the well-known *airport landing fee problem* (Littlechild and Owen, 1973; Littlechild and Thompson, 1977). We use the following simple example.

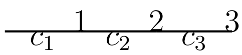

Figure 1. An Airport Problem

The following method is well-known.

$$\begin{aligned} x_1^A(C) &= \frac{1}{3}c_1, \\ x_2^A(C) &= \frac{1}{3}c_1 + \frac{1}{2}c_2, \\ x_3^A(C) &= \frac{1}{3}c_1 + \frac{1}{2}c_2 + c_3, \end{aligned} \tag{7}$$

Now, we propose the following method. It is easy to check that it is a convex combination of the location-labeled methods.

$$
\begin{aligned}
x_1(C) &= \frac{6}{11}c_1, \\
x_2(C) &= \frac{3}{11}c_1 + \frac{3}{5}c_2, \\
x_3(C) &= \frac{2}{11}c_1 + \frac{2}{5}c_2 + c_3,
\end{aligned} \tag{8}
$$

Note that in (8) agent 1 pays more and agent 3 pays less compared to their corresponding cost shares in (7). Indeed, each agent pays a relatively larger share of his own cost with the entire cost as an upper bound, and less proportions for distant cost components. This reflects the consideration that agents should pay more for the (e.g., relatively intensive) use of their local network.[3]

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