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Independence of dummy units and Shapley-Shubik methods in cost sharing problems with technological cooperation

Eric Bahel (Virginia Tech)
Christian Trudeau (University of Windsor)

Working paper 13 - 04

Independence of dummy units and Shapley-Shubik methods in cost sharing problems with technological cooperation

Eric Bahel* Christian Trudeau†

This draft: April 2013

Abstract

In the discrete cost sharing model with technological cooperation (Bahel and Trudeau (IJGT, 2013)), we study the implications of a number of properties that strengthen the well-known Dummy axiom. Our main axiom, which requires that costless units of demands do not affect the cost shares, is used to characterize two classes of rules. Combined with anonymity and a specific stability property, this requirement picks up sharing methods that allow the full compensation of at most one technological contribution. If instead we strengthen the well-known Dummy property to include agents whose technological contribution is offset by the cost of their demand, we are left with an adaptation of the Shapley-Shubik method that treats technologies as private and rewards their contributions. Our results provide two interesting axiomatizations for the adaptations of the Shapley-Shubik rule to our framework.

1 Introduction

We consider the cost sharing problem where the cost of a project is to be split between the participating agents and where agents have their own technology of production. The model, which was featured in Bahel and Trudeau (2012), is a generalization of the discrete cost sharing model that was introduced in Moulin (1995). In traditional cost sharing models, the gains (losses) from cooperation come exclusively from the increasing (respectively, decreasing) returns to scale exhibited by the fixed technology. We instead allow every coalition of agents to have their own technology. Technology is meant in a broad way and can include patents (R&D), know-how (joint venture), information, negotiation power and location (e.g. network formation). We are interested in methods allowing to share the joint costs of production and particularly in the new aspects related to technology, such as how we should incorporate the savings generated by technological improvements in the cost shares.

In the traditional discrete cost sharing model, Wang (1999) showed that the classic properties of additivity and dummy characterize the so-called flow methods, which are represented by a unit flow from the origin to the vector representing the full demand to produce. The share of an agent is then a weighted average of his incremental costs, with the flows providing the weights. Bahel and Trudeau (2012) provide a similar result in the more general context where technological cooperation is allowed. They show that flow methods are the only cost sharing rules satisfying additivity, strong dummy (an adaptation of the dummy axiom) and monotonicity with respect to demand increments (i.e., the share of an agent should not decrease if her demand increments become costlier, with everything else kept constant).

The Shapley-Shubik method (Shubik (1962)) is an adaptation of the Shapley value to the case where the agents demand different quantities of (possibly heterogeneous) goods. While well studied in

*Department of Economics, Virginia Polytechnic Institute and State University
Blacksburg, VA 24061, USA. Email: erbahel@vt.edu
†Department of Economics, University of Windsor
401 Sunset Avenue, Windsor, Ontario, Canada. Email: trudeauc@uwindsor.ca

the model with continuous demands, it has received less attention in the discrete case. Moulin (1995) characterizes the method as part of a larger family (that also includes the serial and pseudo average cost methods). In Moulin and Sprumont (2007), it is characterized using the following property: removing any costless units of demand (regardless of the agent they belong to) should not affect the cost shares. That property is called ordinality and linked to the property of the same name used in the continuous model (Sprumont (1998)): as we can add costless "half-units" without affecting shares, the choice of units to compare the asymmetric goods that are demanded does not affect the cost shares. In this paper we follow Sprumont (2008), who uses a similar property which only applies to the last units, and call the property independence of dummy units. We use the requirement to adapt the characterization of the Shapley-Shubik method to our framework. Since our model extends the traditional one, it is no surprise that we find many anonymous methods that satisfy independence of dummy units, as there are many ways to adapt the Shapley-Shubik method to our context. The various adaptations differ in the way they allocate savings coming from the improvement of the technology. One of the contributions of the present paper is to introduce fairness and stability properties as to the allocation of those savings; these properties are combined with the axiom independence of dummy units to characterize some remarkable rules.

The first property requires that the agents with positive demands benefit from the technological cooperation of the others (fair rent), which leaves us with two noticeable methods (and their weighted averages). The first one is the so-called public Shapley-Shubik method, which completely discards technological contributions. The second one also has a Shapley-Shubik flavor but treats the technology differently. Given an arbitrary ordering of the n agents, the first $n-1$ share their technologies and the incremental costs are computed (according to the chosen ordering). The final agent is then assigned the incremental benefit of his technology and the incremental cost of his demand.

We also examine the property "dummy over total cost" which states that an agent whose technological contributions are always offset by the increments due to his demand should pay nothing. Combining this property with independence of dummy units, we find the private Shapley-Shubik method that considers the effects of both the technology and the demand when computing incremental costs.

The paper is structured as follows: Section 2 formally describes the model and the basic properties needed to obtain the unit-flow representation. Section 3 describes and illustrates different ways of adapting the Shapley-Shubik method to our model. Section 4 provide the main characterization results. Some discussion is provided in Section 5.

2 Preliminaries

2.1 The model

The framework we use is that of Bahel and Trudeau (2012). Let $N = \{1, ..., n\}$ be the set of agents, with $n \geq 2$ being the number of agents. Each agent is endowed with a technology and has a demand $x_i \in \mathbb{N}$ for some specific good. Technology influences the cost and, when cooperating, the agents put together not only their demands but also their technologies; as a consequence, each coalition possesses a specific technology. Technologies are aggregated in a manner that we do not observe; we only observe the effects of that aggregation on the costs.

Throughout the paper, we use the following convention for vector inequalities:

- $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) \leq \tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)$ iff $\bar{x}_i \leq \tilde{x}_i$, for every $i = 1, ..., m$;
- $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) < \tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)$ iff $[\bar{x} \neq \tilde{x}$ and $\bar{x}_i \leq \tilde{x}_i,$ for every $i = 1, ..., m]$.

The cost function $C(S, \cdot)$ is defined over the set $\mathbb{N}^S$. For any $x_S \in \mathbb{N}^S$, $C(S, x_S) \in \mathbb{R}_+$ represents the cost of supplying the demand x_S when all the agents in S cooperate on its production. Note that some agents in S might demand zero while cooperating to produce the positive demands of the other agents in S. We make the following assumptions on these cost functions:

1. $C\left(S_1, x_{S_1}\right) \geq C\left(S_2, \left(x_{S_1}, 0_{S_2 \setminus S_1}\right)\right)$, where $S_1 \subseteq S_2 \subseteq N$, $x_{S_1} \in \mathbb{N}^{S_1}$ and $\left(x_{S_1}, 0_{S_2 \setminus S_1}\right) \in \mathbb{N}^{S_2}$ (more agents cooperating technologically can only improve the technology; therefore, for a fixed demand profile to be produced, the cost cannot increase).

2. $C(S, \bar{x}_S) \leq C(S, \tilde{x}_S)$ for any $\bar{x}_S$, $\tilde{x}_S \in \mathbb{N}^S$ s.t. $\bar{x}_S \leq \tilde{x}_S$ (for a fixed technology, the cost is non-decreasing in the demand profile).

3. $C(S, 0_S) = 0$ for any $S \subseteq N$ (for any technology, the cost of producing nothing is zero).

Let Γ be the set of cost functions $C(\cdot,\cdot)$ that satisfy conditions 1, 2 and 3. A cost sharing problem is (C, x) such that $C \in \Gamma$ and $x \in \mathbb{N}^N$. Denote the set of problems as $\Gamma \times \mathbb{N}^N$.

Definition 1 *A cost sharing method (CSM) y is a mapping defined from $\Gamma \times \mathbb{N}^N$ to $\mathbb{R}^N$ such that $\sum_{i \in N} y_i(C, x) = C(N, x)$, for any $(C, x) \in \Gamma \times \mathbb{N}^N$.*

A CSM is thus a mechanism which, for each cost sharing problem, assigns a cost share to each of the agents, with the requirement that the shares sum up to the cost of producing the demand x when all the agents are cooperating.

2.2 Basic properties and unit-flow representation

We start by recalling the results of Bahel and Trudeau (2012), on which we will build to define and characterize new CSMs. Let us first introduce additivity and dummy, two of the most basic properties in the cost sharing literature. Additivity says that whenever the cost function can be divided into multiple subfunctions, the level of aggregation of the costs should not affect the shares.

Additivity*:* A CSM y meets additivity if $y(C_1 + C_2, x) = y(C_1, x) + y(C_2, x)$, for any $C_1, C_2 \in \Gamma$ and any $x \in \mathbb{N}^N$.

The notion of "dummy agent" is well known in the literature and indicates that an agent has no impact on the cost function. We extend the concept to agents for which neither the demand nor the technological cooperation impacts the cost function.

Definition 2 *For the cost function $C \in \Gamma$, we say that agent i is a*

- demand-dummy *if $C(S \cup \{i\}, (t, a)) - C(S \cup \{i\}, (t, a-1)) = 0$ for all $S \subseteq N \setminus \{i\}$, $t \in N^S$ and $a = 1, 2, ...$;*
- technology-dummy *if $C(S \cup \{i\}, \{t, 0\}) - C(S, t) = 0$ for all $S \subseteq N \setminus \{i\}$ and $t \in N^S$;*
- dummy *if she is both demand-dummy and technology-dummy.*

The dummy axiom is a basic fairness requirement as it links an agent to the cost for which she is responsible. If an agent does not have any impact on the cost, she should not be assigned any part of it. In the present model, an agent is dummy if both her demand and her technology are inconsequential to the cost.

Dummy: A CSM y meets dummy if, for any $C \in \Gamma$ we have: $y_i(C, x) = 0$, for any agent i who is dummy for C.

In our context, the following (stronger) requirement is natural.

Strong Dummy: A CSM y meets strong dummy if, for any $C \in \Gamma$, we have the following properties:

i) $y_i(C, x) \leq 0$, for any agent i who is demand-dummy for C;

ii) $y_i(C, x) \geq 0$, for any agent i who is technology-dummy for C.

Next, we introduce a requirement which (along with additivity and strong dummy) is necessary and sufficient to characterize a well-defined family of CSM.

Monotonicity with respect to Demand-increment Costs (MDC): A CSM y meets MDC if, for all $x \in \mathbb{N}^N$, $\hat{C}, \bar{C} \in \Gamma$ such that $\hat{C}(N, x) \geq \bar{C}(N, x)$ and any agent i, we have: $y_i(\hat{C}, x) \geq y_i(\bar{C}, x)$ whenever

(i) $\hat{C}(S, x+e^i_S) - \hat{C}(S,x) \geq \bar{C}(S, x+e^i_S) - \bar{C}(S,x)$
for all S s.t. $i \in S \subseteq N$ and all $x \in N^S$;

(ii) $\hat{C}(S \cup \{i\}, (x,0)) - \hat{C}(S,x) = \bar{C}(S \cup \{i\}, (x,0)) - \bar{C}(S,x)$
for all S s.t. $i \notin S \subseteq N$ and all $x \in N^S$.

MDC is a natural requirement: it means that the cost share of an agent should not decrease if the increases in her demand become costlier while the cost reductions due to her technological cooperation remain unchanged.

In the standard discrete model, Wang (1999) provided a useful characterization by showing that every method satisfying additivity and dummy can be represented by a system of non-negative weights associated with the incremental cost vectors. Furthermore, the author showed that this system of weights can be represented by a unit-flow system. To allow for a unit-flow representation of cost sharing rules, we re-express every cost function in Γ as follows. We define a vector z that encompasses both technological cooperation and demand: the first unit of z_i represents agent i's technological cooperation, the following units represent agent i's demand. One can interpret any $z \in \mathbb{N}^N \setminus \{0_N\}$ as follows: if $z_i > 0$, then agent i cooperates in production; her demand (whether or not she cooperates) is $x_i = \max(0, z_i - 1)$.

Let $e^S \in \mathbb{N}^N$ be such that $e^S_j = 1$ if $j \in S$ and $e^S_j = 0$ otherwise. For $i \in N$, we will often write e^i instead of $e^{\{i\}}$. Note that the mapping Φ which, to any (S,x) s.t. $\emptyset \neq S \subseteq N$ and $x \in \mathbb{N}^S$, assigns $z = (x, 0_{N\setminus S}) + e^S$ is one-to-one. Indeed, denoting $N(z) = \{i \in N \mid z_i > 0\}$, the (unique) inverse image of any $z \in \mathbb{N}^N \setminus \{0_N\}$ is given by (S,x) s.t. $S = N(z)$ and $x = (z - e^{N(z)})_{N(z)}$.

We then have a cost function C^* with a single argument z that accounts for both demand and technology. Observe that the domain of C^* is $\mathbb{N}^N \setminus \{0^N\}$.

For any $z \in \mathbb{N}^N$ and any $S \subseteq N$, define $z^S \in \mathbb{N}^N$ by $(z^S)_i = z_i$ if $i \in S$ and $(z^S)_i = 0$ if $i \in N \setminus S$. We define z_x as $x + e^N$.

Definition 3 *A **flow** to z_x is a mapping $f(z_x, \cdot)$ from $[0, z_x]$ to $[0,1]^N$ that has the following properties:*
i) $\sum_{i \in N(r)} f_i(z_x, r) = \sum_{i \in N:\, r_i < z_i} f_i(z_x, r + e^i)$ *for all* $r \in [0^N, z] \setminus [0^N, e^N]$*;*
ii) $\sum_{i \in N} \sum_{t \in [e^i, e^N]} f_i(z_x, t + e^i) = \sum_{i \in N} f_i(z_x, z_x) = 1$*;*
iii) $f_i(z_x, t) \geq 0$ *for all* $t \in [e^i, z_x]$*.*

We interpret $f_i(z_x, t)$ as the flow from node $t - e^i$ to node t, where $t_i > 0$, and z_x as the final destination of these flows. Property i) is the flow-conservation property guaranteeing that the inflow is equal to the outflow at each node. Property ii) states that there is a flow of one coming out of the origin and arriving at z_x. Condition iii) rules out negative flows. As stated in the following definition, any arbitrary flow system can be used to define a specific CSM.

Definition 4 *A CSM $y(\cdot,\cdot)$ is a **flow method** if for all $(C,x) \in \Gamma \times \mathbb{N}^N$ there exists a flow $f(z_x, \cdot)$ to z_x such that, for all $i \in N$, we have: $y_i(C,x) = \sum_{t \in [e^i, z_x]} f_i(z_x, t) \partial_i C^*(t)$, where $\partial_i C^*(t) = C^*(t) - C^*(t - e^i)$.*

The following result characterizes the set of flow methods in cost sharing models with technological cooperation.

Theorem 1 *(Bahel and Trudeau (2012)) A CSM satisfies additivity, strong dummy and MDC if and only if it is a flow method.*

As flow methods are characterized by the basic properties of additivity, strong dummy and MDC, we always assume those in the remainder of the paper. We add to these three properties to characterize noticeable subsets of flow methods.

3 Extending the Shapley-Shubik method

In traditional cost sharing models where the cost is a function of the vector of demands, the Shapley-Shubik method is the Shapley value of the game $V(\cdot)$ such that $V(S) = C(x_S)$ is the cost of meeting the demands of S. Formally,

$$SS_i = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(V(S \cup \{i\}) - V(S)\right).$$

There are many ways to adapt this rule to our framework, depending on how one treats the technology. One possible adaptation, that we call the public Shapley-Shubik method, simply treats technologies as public properties and assumes that everybody has access to the best technology (which, in our case, is always the technology of the largest group, N). Therefore, the stand-alone cost for coalition S is $C(N, (x_S, 0_{N\setminus S}))$. The formula for the public Shapley-Shubik method is therefore

$$SS_i^{pub}(C,x) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C(N, (x_{S\cup\{i\}}, 0_{N\setminus(S\cup\{i\})})) - C(N, (x_S, 0_{N\setminus S}))\right).$$

We can write it in a more succinct way using C^* :

$$SS_i^{pub}(C,x) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C^*\left(z^{S\cup\{i\}} + e^{N\setminus(S\cup\{i\})}\right) - C^*\left(z^S + e^{N\setminus S}\right)\right).$$

At the opposite end of the spectrum, we can have situations where technologies are private properties, meaning that the technology of agent i can only be used by a coalition that includes i. Therefore, when computing the stand-alone cost of coalition S, we assume that it can only use the technologies of its members. As a result, the stand-alone cost for coalition S is $C(S, x_S)$. Applying the Shapley value to the resulting stand-alone game generates the private Shapley-Shubik method, formally defined as follows:

$$SS_i^{priv}(C,x) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C(S\cup\{i\}, x_{S\cup\{i\}}) - C(S, x_S)\right). \quad (1)$$

If we use C^* instead, we obtain:

$$SS_i^{priv}(C,x) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(C^*(z^{S\cup\{i\}}) - C^*(z^S)\right).$$

Example 1 *Consider the cost sharing problem with 3 players where $x = (2,1,0)$ and the cost function C^* illustrated in Figure 1. Notice that all nodes on the cube that contains the origin must have a cost of zero, as it corresponds to the vectors $z \leq e^N$, where there is no demand. Figure 2 presents the respective flows associated with the public and private Shapley-Shubik methods.*

We obtain the following allocations:

$$\begin{aligned} SS^{pub} &= (3,1,0) \\ SS^{priv} &= (5,2,-3). \end{aligned}$$

4 Characterization results

Suppose that an agent has a unit of demand that can always be provided at zero cost. Then, this unit is not relevant and it should be possible to remove it from the problem without affecting the cost shares. For example, if an agent gets his good from a supplier that offers a free unit for every 10 units bought (of this particular good), the 11th, 22nd, 33rd, etc. units will always be free and can be removed from the problem. Similarly, if the agent can always get the first unit for free but needs to pay for subsequent units, we can remove that first unit from the problem. For any flow method, a unit of demand can only generate a non-negative cost to its owner; removing that costless unit is hence to the benefit of its owner. Notice also that if removing a costless unit changes the cost shares, manipulations of the demand become a possibility in cases where agents can add or hide such costless units.

Example 2 *It is easy to see that in Example 1, the first unit of agent 1 is always free. Therefore, we should be able to remove that unit from the problem without changing the cost shares. Let* $x' = (1, 1, 0)$ *and* C'^* *be illustrated in Figure 3.*

Consider now the problem (C'', x) *with* C''^* *illustrated in Figure 4. Observe that now it is the second unit of agent 1 that is always free. If we remove it, we also obtain the problem* (C', x').

The following axiom formalizes the idea expressed above.

Independence of dummy units: A CSM y meets independence of dummy units if, $y(C, x) = y(C', (x_{-i}, x_i - 1))$ whenever $C(S \cup i, (t, a)) = C(S \cup i, (t, a + 1))$ for some $a \in \{0, ..., x_i - 1\}$ and all $S \subseteq N \backslash i$, $t \in N^S$, and where $C'(S \cup i, (t, b)) = C(S \cup i, (t, b + 1)$ if $b \geq a + 1$, $C'(T, r) = C(T, r)$ otherwise.

The changes from C to C' are illustrated in Figure 5. If C contains a flat portion between a and $a + 1$ for all coalitions S and all production levels, then we can build and use C' that eliminates this $(a + 1)^{th}$ unit. Therefore, what was originally the $(a + 2)^{th}$ unit in the original problem becomes the $(a + 1)^{th}$ unit in C'.

Independence of dummy units restricts considerably the set of flow methods. It is satisfied only by a subset of fixed-flow methods (Moulin and Sprumont (2005)), which have a particularly nice interpretation. Suppose that the demand is x and that a CSM is represented by the flow $f(z_x, \cdot)$. If the demand decreases to $y \leq x$ and the CSM is a fixed-flow method, then it is represented by a projection of the flow $f(z_x, \cdot)$ onto the box $\left[0^N, z_y\right]$.[1]

We show in what follows that requiring independence of dummy units leaves us with fixed-flow methods for which a great number of edges are crossed by a flow of zero. There will be a strictly positive flow in direction i at point t if all other agents are either not cooperating ($t_j = 0$), only lending their technology ($t_j = 1$), or sharing their technology and asking for their full demand ($t_j = x_j + 1$). As a result, it suffices to define flows to $2e^N$, as it fully characterizes flows to any other z.

Lemma 1 *A flow method that satisfies independence of dummy units*

i) is a fixed-flow method.

ii) is such that for any $x \in \mathbb{N}^N$ *and* $i \in N$, $f_i(z_x, t) = 0$ *for all* $t \in \left[e^i, z_x\right]$ *if there exists* $j \in N \backslash \{i\}$ *such that* $t_j \notin \{0, 1, x_j + 1\}$

iii) is uniquely characterized by the flows to $z = 2e^N$.

Proof. See Appendix. ■

Example 3 *Recall Example 2. By independence of dummy units, we must have that* $y(C, x) = y(C', x') = y(C'', x)$. *The results of Lemma 1 are illustrated in Figure 6. If a flow in the figure*

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[1] Formally, let $f(z_x, \cdot)$ be a flow to z_x and $x' \leq x$. A projection of $f(z_x, \cdot)$ on $[0, z_{x'}[$, denoted $p_{x'} f(z_x, \cdot)$ is defined as follows: for any $i \in N$ and $t \in [0, z_{x'}[$ write $M = \left\{ j \in N \middle| t_j = x'_j + 1 \right\}$ and let

$$\begin{aligned} p_{x'} f_i(z_x, t) &= 0 \text{ if } i \in M \\ &= \sum_{w_M \in \left[z^M_{x'}, z^M_x\right]} f_i(z_x, t^{N \backslash M} + w_M) \text{ otherwise,} \end{aligned}$$

with the convention that the sum is simply $f_i(z_x, t)$ if $M = \emptyset$. Then, a cost sharing method y is a fixed-flow method if for $x' \leq x$, there is a flow $f(z_x, \cdot)$ representing $y(\cdot, x)$ such that $p_{x'} f(z_x, \cdot)$ represents $y(\cdot, x')$.

on the left has the same pattern as a flow on the right figure, these flows must be equal. We can see that with independence of dummy units, when we add a unit of demand for agent 1, the flows on the edges of the cube $[0, z_x]$ *are the same as those on the edges of the cube* $[0, z_{x'}]$. *The only difference is that, for the case with* x, *we have inserted a series of null flows between the edges* t *such that* $t_1 = 2$.

Next, we examine the implications of the fair rent property defined in Bahel and Trudeau (2012). Fair rent requires that the agents with positive demands be always willing to pay the required rent to the others (who each demand zero) in exchange for their technology.

Fair rent: A CSM y meets fair rent if we have $\sum_{i \in N(x)} y_i(C, x) \leq C(N(x), x_{N(x)})$, for any $(C, x) \in \Gamma \times \mathbb{N}^N$.

Fair rent is in the vein of core selection: the coalition formed by the agents with positive demands will accept to cooperate with the other agents (who each demand zero) only if it is not worse off from this cooperation. In other words, the rent paid for the technological cooperation should not exceed the savings brought to the coalition.

We also use the anonymity property which says that the identity of the agents should not matter, just their relevant characteristics in terms of demand, technology and costs.

Denote by $\Pi(N)$ the set of bijections from N into itself and let $\pi \in \Pi(N)$. If $z \in \mathbb{R}_+^N$, define $\pi z \in \mathbb{R}_+^N$ by $(\pi z)_{\pi(i)} = z_i$ for all $i \in N$. If $C \in \Gamma$, define $\pi C \in \Gamma$ by $\pi C(\pi z) = C(z)$ for all $z \in \mathbb{N}^N$

Anonymity: A CSM y meets anonymity if $y(\pi C, \pi x) = \pi y(C, x)$ for all $\pi \in \Pi(N)$, $C \in \Gamma$, and $x \in \mathbb{N}^N$.

We need the following notation: for each $\pi \in \Pi(N)$, let PUB^π be the *Public Technology path method* corresponding to the permutation π. It is such that agents are added in the order π and pay their incremental cost, while using the technology of the whole group. Formally, let

$$PUB^\pi_{\pi_i} = C(N, (x_{[\pi_i]}, 0_{N \backslash [\pi_i]})) - C(N, (x_{[\pi_i - 1]}, 0_{N \backslash [\pi_i - 1]}))$$

for $i = 1, ..., n$, with $[\pi_i] = \{\pi_1, ..., \pi_i\}$ and $[\pi_0] = \emptyset$. We have that

$$SS^{pub} = \frac{1}{n!} \sum_{\pi \in \Pi(N)} PUB^\pi,$$

i.e. that the public Shapley-Shubik method is the average of Public Technology path methods.

For each $\pi \in \Pi(N)$, let SR^π be the *Single Rent path method* corresponding to the permutation π. It is such that for the given order π, the first $n-1$ players aggregate their technologies and each pay the incremental cost of their demand in that order, using that aggregate technology. The last agent then arrives and receives the cost difference due to his demand and his technology. Formally, let

$$SR^\pi_{\pi_i} = C\left(N \backslash \pi_n, \left(x_{[\pi_i]}, 0_{N \backslash ([\pi_i] \cup \pi_n)}\right)\right) - C\left(N \backslash \pi_n, \left(x_{[\pi_i - 1]}, 0_{N \backslash ([\pi_i - 1] \cup \pi_n)}\right)\right)$$

for $i = 1, ..., n-1$ and $SR^\pi_{\pi_n} = C(N, x) - C\left(N \backslash \pi_n, \left(x_{N \backslash \pi_n}, 0_{\pi_n}\right)\right)$. Let the *Average Single Rent method* be defined as

$$ASR = \frac{1}{n!} \sum_{\pi \in \Pi(N)} SR^\pi.$$

We can also define the Average Single Rent method in the following way. Let $\tilde{C}(\cdot, x)$ be such that

$$\tilde{C}(S, x) = \begin{cases} \sum_{i \in N \backslash S} \frac{C(N \backslash \{i\}, (x_S, 0_{N \backslash (S \cup \{i\})}))}{|N| - |S|} & \text{if } |S| \leq |N| - 2 \\ C(S, x_S) & \text{else} \end{cases}.$$

Then, the Average Single Rent method is the Shapley value of $\tilde{C}$. Notice that we modify the cost function in such a way that the cost of supplying the demand of coalition S is computed as the average of the costs when it has access to the technologies of all but one member of $N \backslash S$.

Example 4 *Recall the problem (C,x) of Example 1. Figure 7 illustrates the flows associated to the Average Single Rent method. Notice that while there was no strictly positive flows in the direction of agent 3 for the public Shapley-Shubik method, here we have one. The private Shapley-Shubik method has more (see Figure 2), but fair rent only allows us to provide a rent to agent 3 for the cost savings generated when her technology is applied to the full demand vector of agents 1 and 2.*

Applying it to the cost function C, we obtain a cost allocation $ASR(C,x) = (4,2,-2)$.

We find that adding fair rent further limits the possibilities and yields the convex combinations of Public Technology path methods and Single Rent path methods.

Theorem 2 *A flow method satisfies independence of dummy units and fair rent if and only if it is in the convex hull of the set $\left\{SR^{\pi},_{\pi\in\Pi(N)}\right\} \cup \left\{PUB^{\pi},_{\pi\in\Pi(N)}\right\}$.*

Proof. By Lemma 1, any flow method that satisfies independence of dummy units is uniquely characterized by its flows to $2e^N$.

Let $x = 2e^N$. We first show that for a flow method that satisfies independence of dummy units and fair rent, we have that $f_i(z_x, 2e^i + e^{N\backslash(S\cup i)}) = 0$ for all $S \subseteq N\backslash i$, with $|S| > 1$. By contradiction, suppose that $f_i(z_x, 2e^i + e^{N\backslash(S\cup i)}) = a > 0$. Bahel and Trudeau (2012) show that by fair rent, we only put strictly positive weight to paths where the technology of an agent with no demand is activated when the demand (of others) is null or when everybody else has their full demand. Since, by Lemma 1, a flow method that satisfies independence of dummy units is a fixed-flow method, the result extends here to all agents. Therefore, we have that $f_j(z_x, 2e^i + e^{N\backslash(S\cup i)}) = 0$ for all $j \in N\backslash(S\cup i)$ and $f_k(z_x, 2e^i + e^{N\backslash(S\cup i)} + e^k) = 0$ for all $k \in S$. By flow conservation, this strictly positive flow that enters at $2e^i + e^{N\backslash(S\cup i)}$ has to be moved towards z_x. Notice that we will never be able to move it in directions S, as we would then fail to meet fair rent. Our only possibility is to keep moving it in directions $N\backslash(S\cup i)$. However, this won't allow us to reach z_x. We will eventually reach the point $2e^{N\backslash S}$ where it will be impossible to move the flows anymore. Therefore, we must have that $a = 0$.

It is easy to see that any flow method such that $f_i(z_x, 2e^i + e^{N\backslash(S\cup i)}) = 0$ for all $S \subseteq N\backslash i$, with $|S| > 1$ is in the convex hull of the set $\left\{SR^{\pi},_{\pi\in\Pi(N)}\right\} \cup \left\{PUB^{\pi},_{\pi\in\Pi(N)}\right\}$ and that these methods satisfy all properties. ■

Adding anonymity leaves us with the convex combinations of the public Shapley-Shubik method and the Average Single Rent method.

Theorem 3 *A flow method satisfies independence of dummy units, anonymity and fair rent if and only if it is a convex combination of SS^{pub} and ASR.*

Proof. By Theorem 2, we have that any flow method that satisfies independence of dummy units and fair rent is uniquely characterized by its flows to $2e^N$ and that $f_i(z_x, 2e^i + e^{N\backslash(S\cup i)}) = 0$ for all $S \subseteq N\backslash i$, with $|S| > 1$.

Let $x = 2e^N$. Fix $k \in N\backslash i$ and let $\alpha \equiv f_i(z_x, 2e^i + e^{N\backslash i,k})$. By anonymity, we must have that $f_i(z_x, 2e^i + e^{N\backslash i,j}) = \alpha \geq 0$ for all $j \in N\backslash i$. By anonymity and the properties of a flow, we must have $\sum_{t\in[e^i,e^N]} f_i(z_x, t + e^i) = \frac{1}{n}$. Therefore, we must have that $f_i(z_x, 2e^i + e^{N\backslash i}) = \frac{1}{n} - (n-1)\alpha$.

Next, we show that each value of α fully determines flows to z_x.

For any t, define $N^1(t) = \{j \in N \,|t_j = 1\}$. Suppose that $\sum_{j\in N(t)} f_j(z_x, t) > 0$. By fair rent, if $|N(t)| < n-1$, we have that $f_k(z_x, t + e^k) = 0$ for all k such that $t_k = 0$. By anonymity and flow conservation, we have $f_k(z_x, t + e^k) = \frac{\sum_{j\in N(t)} f_j(z_x,t)}{|N^1(t)|}$ for all $k \in N^1(t)$. If $N(t) = N\backslash k$, fair rent puts no restrictions. By flow conservation we have that $f_k(z_x, t + e^k) = \sum_{j\in N(t)} f_j(z_x, t)$.

Since we have defined above all of the flows that come out of $\left]0, e^N\right]$, we have everything to define flows for all points in $]0, z_x]$.

By the non-negativity of flows, we must have that $\frac{1}{n} - (n-1)\alpha \geq 0$ or $\alpha \leq \frac{1}{n(n-1)}$ as well as $\alpha \geq 0$. Therefore, any $\alpha \in \left[0, \frac{1}{n(n-1)}\right]$ generates flows $f^{\alpha}(z_x, \cdot)$ such that the corresponding flow method satisfies independence of dummy units, anonymity and fair rent. It remains to show that any member of that family is a convex combination of SS^{pub} and ASR.

We can verify that both SS^{pub} and ASR satisfy the properties. Let $f^{SS^{pub}}(z_x,\cdot)$ and $f^{ASR}(z_x,\cdot)$ be the flows generating respectively SS^{pub} and ASR. Clearly, $f_i^{SS^{pub}}(z_x, 2e^i + e^{N\backslash i}) = \frac{1}{n}$, implying that $f^{SS^{pub}} = f^0$. Also, we have that $f_i^{ASR}(z_x, 2e^i + e^{N\backslash i}) = 0$, as this is when agent 1 adds his demand only after everybody has contributed technologically but nobody else has added their demand. This never happens in our definition of ASR. In a given order, if i is in the first $n-1$ agents, his demand is added when the technology of the last agent is missing; if i is the last agent, his demand is added when all other demands have been added. This implies that $f^{ASR} = f^{\frac{1}{n(n-1)}}$. It is easy to see that the convex combination of SS^{pub} and ASR generates the whole family of rules represented by the flows $f^{\alpha}(z_x,\cdot)$, with $\alpha \in \left[0, \frac{1}{n(n-1)}\right]$. ■

It is obvious from the previous theorem that, among this set of methods, the Average Single Rent method maximizes the rent paid by the agents with positive demands to those that have no demand. By contrast, the public Shapley-Shubik method minimizes this rent, setting it to zero.

These methods thus make very little use of the technological contributions: the public Shapley-Shubik method ignores them completely while the Average Single Rent method uses only the contribution of an agent's technology when she joins the whole group and their full demand vector. A natural way to proceed if we want to consider technological contributions is to use Private Technology path methods: let $PRIV^{\pi}$ be the *Private Technology path method* corresponding to the permutation π. It is such that agents are added in the order π and pay their incremental cost, with agents coming in with their technology and their demand when they join the group. Formally, let $PRIV^{\pi}_{\pi_i} = C([\pi_i], x^{[\pi_i]}) - C([\pi_{i-1}], x^{[\pi_{i-1}]})$ for $i = 1, ..., n$. Recalling the private Shapley-Shubik rule introduced in Equation (1), one can see that $SS^{priv} = Sh(C) = \frac{1}{n!}\sum_{\pi\in\Pi(N)} PRIV^{\pi}$, i.e. that the private Shapley-Shubik method is the average of Private Technology path methods. To characterize it we replace fair rent by the following requirement:

Dummy over total cost: A CSM y meets dummy over total cost if $y_i(C,x) = 0$ whenever $C(S,r) = C(S\cup\{i\},(r,x_i))$ for all $S \subseteq N\backslash i$ and all $r \in \left[0, x^S\right]$.

The property has a nice interpretation: if an agent can always be added at no extra cost, he should be assigned a share of zero. Notice that this can be because the agent is a dummy, but it could also be because the savings generated by his technology are always exactly compensated by the extra cost generated by his demand. In that sense, dummy over total cost imposes that we value in the same manner one dollar of savings coming from i's technology and one dollar of cost generated by her demand.

Theorem 4 *A flow method satisfies independence of dummy units and dummy over total cost if and only if it is a convex combination of private technology path methods.*

Proof. By Theorem 2, we have that any flow method that satisfies independence of dummy units is uniquely characterized by its flows to $2e^N$. Let $x = 2e^N$. It it easy to see that dummy over total cost implies that

$$f_i(z_x, z^S + e^i) = f_i(z_x, z^S + 2e^i) \text{ for all } \emptyset \neq S \subseteq N\backslash i. \tag{2}$$

Next, we show that $f_i(z_x, z^S + e^T + ae^i) = 0$ for all $S \subset N\backslash i$, $\emptyset \neq T \subseteq N\backslash(S\cup i)$ and $a = 1, 2$.

Suppose by contradiction that $f_i(z_x, z^S + e^T + 2e^i) = \epsilon > 0$ for some $S \subset N\backslash i$, $\emptyset \neq T \subseteq N\backslash(S\cup i)$.

For any $j \in T$, we must have, by (2), that $f_j(z_x, z^S + e^T + 2e^i) = f_j(z_x, z^S + e^T + 2e^i + e^j)$. This means that the flow $f_i(z_x, z^S + e^T + 2e^i) = \epsilon$ cannot be moved in direction j, for any $j \in T$. It also cannot be moved in direction k for any $k \in S$, as all those agents already have their complete demand. Therefore, the flow can only be moved in direction l, for $l \in N\backslash(S\cup T\cup i)$. By flow conservation, this flow eventually reaches the point $z^{N\backslash T} + e^T$. We would need to move it to $z^{N\backslash T} + e^T + e^R$ for some $R \subseteq T$, but we can't, as (2) implies that $f_j(z_x, z^{N\backslash T} + e^T) = f_j(z_x, z^{N\backslash T} + e^T + e^j)$ for all $j \in T$. Therefore, we can conclude that $f_i(z_x, z^S + e^T + 2e^i) = 0$ for all $S \subset N\backslash i$, $\emptyset \neq T \subseteq N\backslash(S\cup i)$.

We can proceed in the same manner to show that $f_i(z_x, z^S + e^T + e^i) = 0$ for all $S \subset N\backslash i$, $\emptyset \neq T \subseteq N\backslash(S \cup i)$, now that we know that $f_i(z_x, z^S + e^T + 2e^i) = 0$, and thus that any positive flow would not be movable in direction i.

Remaining flow methods are such that $f_i(z_x, z^S + e^i) = f_i(z_x, z^S + 2e^i) = \alpha_i^S$ for all $i \in N$, $S \subseteq N\backslash i$ and $f_i(z_x, t + e^i) = f_i(z_x, t + 2e^i) = 0$ if $t_j = 1$ for some $j \in N\backslash i$. It is easy to see that all such flow methods are convex combinations of private technology path methods. ■

Adding anonymity trivially yields the average of private technology path methods, which is the private Shapley-Shubik method.

Theorem 5 *The private Shapley-Shubik method SS^{priv} is the unique flow method that satisfies independence of dummy units, anonymity and dummy over total cost.*

Thus, if we believe that savings from technological improvements and incremental costs caused by additional units of demand should be weighted equally, the only anonymous flow method left is the private Shapley-Shubik method. By Theorem 2, it fails to meet fair rent. In a similar manner, the previous theorem implies that both the public Shapley-Shubik and the Average Single Rent methods fail to meet dummy over total cost, as they do not give the same weight to technological improvements and demand increments.

5 Discussion

Given that flow methods satisfying independence of dummy units are characterized by their flow to $2e^N$ (see Lemma 1), we illustrate these flows for the three methods discussed in the paper for the case of 3 agents depicted in Figure 8. Notice that the methods cover the cases where all positive flows come out of the cube $\left[0, e^N\right]$ when (a) only one agent (private Shapley-Shubik), (b) all but one (Average Single Rent), (c) all agents (public Shapley-Shubik) have shared their technologies before we start introducing demands. For $n > 3$ agents, we have different symmetric methods where positive flows come out of cube $\left[0, e^N\right]$ when $1 < k < n - 1$ agents have shared their technologies before we start introducing demands. However, these methods fail both fair rent and dummy over total cost.

Section 4 shows that in a context where we have technological cooperation as well as production cooperation, stability is an issue: the property of fair rent, which is only a weak version of core selection, is quite restrictive. Combined with independence of dummy units and anonymity, it leaves us with methods that barely reward any technological contributions. We thus have to choose between (more fully) rewarding agents for their technologies and stability requirements. This choice is not trivial and should depend on the characteristics of the problem in hand. Obviously, additional constraints on the eligible cost functions (coming from the set of problems considered) could potentially allow for more possibilities to combine these conflicting properties.

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A Appendix: Proof of Lemma 1

Proof. Let y be a flow method represented by the flow f that satisfies independence of dummy units.

i) Showing that y is a fixed-flow method.

To show that we have a fixed-flow method, it is sufficient to show that for x and x' such that $x_i > 0$ and $x' = (x_{-i}, x_i - 1)$, we have:

a) $f_j(z_{x'}, t + x_i e^i) = f_j(z_x, t + x_i e^i) + f_j(z_x, t + (x_i + 1)e^i)$ for all $j \in N \backslash i$, $t \in \left[e^j, z_x^{N \backslash i}\right]$,

b) $f_j(z_{x'}, t) = f_j(z_x, t)$ for all $j \in N \backslash i$, $t \in \left[e^j, z_{x'} - e^i\right]$ and

c) $f_i(z_{x'}, t) = f_i(z_x, t)$ for all $t \in \left[e^i, z_{x'}\right]$.

The three conditions impose restrictions on the changes in the flows if we remove one unit of demand for agent i. Condition a) tells us how the projection affects the flows in directions $j \neq i$ for edges where i gets his full demand. Conditions b) and c) say that other flows are unchanged.

To show a), we need to show that $f_j(z_{x'}, t + x_i e^i + a e^j) = f_j(z_x, t + x_i e^i + a e^j) + f_j(z_x, t + (x_i + 1)e^i + a e^j)$ for all $j \in N \backslash i$, $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$ and $a = 1, ..., x_j + 1$. Because of the restrictions on what constitutes an admissible cost function, we use different methods, depending on the set of agents for which $t_k = 1$, as the cost function cannot be increasing when we go from $t_k = 0$ to $t_k = 1$ and cannot be decreasing when we go from $t_k = 1$ to $t_k = 2$.

Define $N_0(t) = \{k \in N \backslash \{i,j\} \,|t_k = 0\}$, $N_1(t) = \{k \in N \backslash \{i,j\} \,|t_k = 1\}$ and $N_d(t) = \{k \in N \backslash \{i,j\} \,|t_k \geq 2\}$.

We first suppose that $x_i \geq 2$.

Case 1: $a \geq 2$.

Define C such that $C^*(r) = 1$ if $r_i \geq x_i$, $r_j \geq a$, $r^{N_d(t)} \geq t^{N_d(t)}$ and $r^{N_0(t)} = 0^{N_0(t)}$, with $C^*(r) = 0$ otherwise. Let $\hat{C}$ be such that $\hat{C}^*(r) = 0$ if $r^{N_1(t)} = e^{N_1(t)}$ and $\hat{C}^*(r) = C^*(r)$ otherwise. If $N_1(t) \neq \emptyset$, it follows from Theorem 1 that

$$y_j(C, x) - y_j(\hat{C}, x) = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} \left(\begin{array}{c} f_j(z_x, r + e^{N_1(t)} + x_i e^i + a e^j) + \\ f_j(z_x, r + e^{N_1(t)} + (x_i + 1)e^i + a e^j) \end{array} \right).$$

Observe that there is never any extra cost to provide i's last unit in C and $\hat{C}$. We can apply independence of dummy units to remove agent i's last unit and we must have that $y(C, x) = y(C', x')$, $y(\hat{C}, x) = y(\hat{C}', x')$ where $C'^*(r) = C^*(r)$ and $\hat{C}'^*(r) = \hat{C}^*(r)$ for all $r \in [0, z_{x'}]$. We have that $y_j(C', x') - y_j(\hat{C}', x') = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + e^{N_1(t)} + x_i e^i + a e^j)$. We thus have that

$$\sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} \left(\begin{array}{c} f_j(z_x, r + e^{N_1(t)} + x_i e^i + a e^j) + \\ f_j(z_x, r + (x_i + 1)e^i + a e^j) \end{array} \right) = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + e^{N_1(t)} + x_i e^i + a e^j) \tag{3}$$

Let $t = z_x^{N_d(t)}$ and see that we have that $f_j(z_x, z_x^{N_d(t)} + e^{N_1(t)} + x_i e^i + a e^j) + f_j(z_x, z_x^{N_d(t)} + e^{N_1(t)} + (x_i + 1)e^i + a e^j) = f_j(z_{x'}, z_x^{N_d(t)} + e^{N_1(t)} + x_i e^i + a e^j)$. Define a sequence $s^1, ..., s^K$, with $K = \sum_{k \in N_d(t)} (z_k - t_k)$, $s^1 = z^{N_d(t)}$, $s^K = t^{N_d(t)}$ and $s^l = s^{l-1} - e^k$ for some $k \in N_d(t)$. Proceeding recursively, by replacing t by s^k in the chosen sequence, yields that $f_j(z_x, t + x_i e^i + a e^j) + f_j(z_x, t + (x_i + 1)e^i + a e^j) = f_j(z_{x'}, t + x_i e^i + a e^j)$ for all $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$ and $2 \leq a \leq x_j + 1$.

If $N_1(t) = \emptyset$, we have that $y_j(C, x) = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} \begin{pmatrix} f_j(z_x, r + x_i e^i + a e^j) \\ + f_j(z_x, r + (x_i + 1)e^i + a e^j) \end{pmatrix}$. We can apply independence of dummy units to remove agent i's last unit and we must have that $y(C, x) = y(C', x')$. Since $y_j(C', x') = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + x_i e^i + a e^j)$, we obtain Equation (3) once again (with $e^{N_1(t)} = 0^N$). Using the same recursive procedure, we obtain $f_j(z_x, t + x_i e^i + a e^j) + f_j(z_x, t + (x_i + 1)e^i + a e^j) = f_j(z_{x'}, t + x_i e^i + a e^j)$ for all $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$ and $2 \leq a \leq x_j + 1$.

Case 2: $a = 1$.

Define the cost function D such that $D^*(r) = 1$ if $r_i \geq x_i$, $r^{N_d(t)} \geq t^{N_d(t)}$, $r^{N_0(t)} = 0^{N_0(t)}$ and there exists $l \in N_1(t) \cup \{j\}$ such that $r_l = 0$, with $D^*(r) = 0$ otherwise. Notice that agent j makes a non-zero contribution only when he adds his technology to those of agents in $N_1(t)$ and agent i demands at least $x_i - 1$, agents in $N_d(t)$ demand at least $t^{N_d(t)} - e^{N_d(t)}$. We thus obtain that $y_j(D, x) = -\sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} \left(f_j(z_x, r + e^{N_1(t)} + x_i e^i + e^j) + f_j(z_x, r + e^{N_1(t)} + (x_i + 1)e^i + e^j)\right)$. Observe that there is never any extra cost to provide i's last unit in D. We can apply independence of dummy units to remove agent i's last unit and we must have that $y(D, x) = y(D', x')$ where $D'^*(r) = D^*(r)$ for all $r \in [0, z_{x'}]$. Since $y_j(D', x') = -\sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + e^{N^1(t)} + x_i e^i + e^j)$, we obtain

$$\sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} \begin{pmatrix} f_j(z_x, r + e^{N_1(t)} + x_i e^i + e^j) + \\ f_j(z_x, r + e^{N_1(t)} + (x_i + 1)e^i + e^j) \end{pmatrix} = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + e^{N^1(t)} + x_i e^i + e^j).$$

Applying the same recursive procedure as above allows us to conclude that $f_j(z_x, t + x_i e^i + a e^j) + f_j(z_x, t + (x_i + 1)e^i + a e^j) = f_j(z_{x'}, t + x_i e^i + a e^j)$ for all $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$ and $a = 1$.

We now discuss the case where $x_i = 1$. Repeat the proof above by adapting the definitions of C and D: instead of using the condition that $r_i \geq x_i$ to assign a cost of one to r, use instead $r_i \geq 0$. This gives us $f_j(z_x, t + a e^j) + f_j(z_x, t + e^i + a e^j) + f_j(z_x, t + 2e^i + a e^j) = f_j(z_{x'}, t + a e^j) + f_j(z_{x'}, t + e^i + a e^j)$. We then need to change once again the construction of C and D, using now the condition that $r_i = 0$ to assign a cost of one to r. This gives us $f_j(z_x, t + a e^j) = f_j(z_{x'}, t + a e^j)$. Combining with the previous result, we obtain $f_j(z_x, t + x_i e^i + a e^j) + f_j(z_x, t + (x_i + 1)e^i + a e^j) = f_j(z_{x'}, t + x_i e^i + a e^j)$ for all $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$, $x_i = 1$ and $a = 1, ..., x_j + 1$.

To show b), we need to show that $f_j(z_{x'}, t + b e^i + a e^j) = f_j(z_x, t + b e^i + a e^j)$ for all $j \in N \backslash i$, $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$, $a = 1, ..., x_j + 1$ and $b = 0, ..., x_i - 1$.

For $b \geq 2$, repeat the first part of the proof of a) (for $x_i \geq 2$) by adapting the definitions of C and D: instead of using the condition that $r_i \geq x_i$ to assign a cost of one to r, use instead $r_i \geq b$. We then obtain equations such as $\sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_x, r + b e^i + a e^j) = \sum_{r \in \left[t^{N_d(t)}, z_x^{N_d(t)}\right]} f_j(z_{x'}, r + b e^i + a e^j)$. Using the same recursive procedure allows us to conclude that $f_j(z_{x'}, t + b e^i + a e^j) = f_j(z_x, t + b e^i + a e^j)$ for all $j \in N \backslash i$, $t \in \left[0, z_x^{N \backslash \{i,j\}}\right]$, $a = 1, ..., x_j + 1$ and $b = 2, ..., x_i - 1$.

For $b \leq 1$, repeat the second part of the proof of a) (for $x_i = 1$).

Part c) follows by flow conservation.

ii) Showing that $f_i(z_x,t)=0$ for all $t\in\left[e^i,z_x\right]$ if there exists $j\in N\backslash\{i\}$ such that $t_j\notin\{0,1,x_j+1\}$.

Fix $j\in N\backslash\{i\}$, $x\in\mathbb{R}_+^N$, $t\in\left[2e^{N\backslash\{i,j\}},z_x^{N\backslash\{i,j\}}\right]$ and $a\in\{2,...,x_i+1\}$. Define C_1 such that

$$C_1^*(r)=\begin{cases}1 \text{ if } r^{N\backslash\{i,j\}}\geq t,\ r_j\geq 2 \text{ and } r_i\geq a\\ 0 \text{ otherwise}\end{cases}$$

We have that $y_i(C_1,x)=\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}\sum_{m=2}^{x_j+1}f_i(z_x,r+me^j+ae^i)$. Observe that there is never any extra cost to provide j's last x_j-1 units in C_1. By independence of dummy units, we can remove agent j's last x_j-1 units of demand and we must have that $y(C_1,x)=y(C_1',x^j)$, with C_1' such that $C_1'^*(r)=C_1^*(r)$ for all $r\in[0,z_{xj}]$ and $x^j=(x_{-j},1)$

We have that $y_i(C_1',x^j)=\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}f_i(z_x',r+e^j+ae^i)$.

Define C_2 such that

$$C_2^*(r)=\begin{cases}1 \text{ if } r^{N\backslash\{i,j\}}\geq t,\ r_j=x_j+1 \text{ and } r_i\geq a\\ 0 \text{ otherwise}\end{cases}$$

We have that $y_i(C_2,x)=\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}f_i(z_x,r+(x_j+1)e^j+ae^i)$. Observe that there is never any extra cost to provide j's first x_j-1 units in C_2. By independence of dummy units, we can remove agent j's first x_j-1 units of demand and we must have that $y(C_2,x)=y(C_2',x^j)$, with C_2' such that $C_2'^*(r)=C_2^*(r)$ if $r_j\leq 1$ and $C_2'^*(r)=C_2^*(r+(x_j+1)e^j)$ otherwise. Observe that $C_1'=C_2'$.[2]

We have that $y_i(C_2',x^j)=\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}f_i(z_{x'},r+e^j+ae^i)$.

Combining the two results above, we must have that

$$\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}\sum_{m=2}^{x_j+1}f_i(z_x,r+me^j+ae^i)=\sum_{r\in\left[t,z_x^{N\backslash\{i,j\}}\right]}f_i(z_x,r+(x_j+1)e^j+ae^i)$$

Let $t=z_x^{N\backslash i,j}$ and see that it implies that $f_i(z_x,z_x^{N\backslash i,j}+me^j+ae^i)=0$ for $m=2,...,x_j$. Define a sequence $s^1,...,s^K$, with $K=\sum_{k\in N\backslash i,j}(z_k-s_k)$, $s^1=z^{N\backslash i,j}$, $s^K=s$ and $s^l=s^{l-1}-e^k$ for some $k\in N\backslash i,j$. Proceeding recursively, by replacing t by s^k in the chosen sequence, yields that $f_i(z_x,t+me^j+ae^i)=0$ for $m=2,...,x_j$ and any $t\in\left[2e^{N\backslash\{i,j\}},z_x^{N\backslash\{i,j\}}\right]$. Using the same method as in the proof of part a), we can extend to any $t\in\left[0,z_x^{N\backslash\{i,j\}}\right]$ and $a=1,...,x_i+1$. Since it holds for all $i,j\in N$ and $t\in\left[0,z_x^{N\backslash\{i,j\}}\right]$, we can conclude that $f_i(z_x,t)=0$ for all $t\in\left[e^i,x\right]$ if there exists $j\in N\backslash\{i\}$ such that $t_j\notin\{0,1,x_j+1\}$.

iii) Showing that the CSM is uniquely defined by the flows to $2e^N$.

First, observe that by the flow conservation property, result ii) also implies that $f_i(z_x,t+ae^i)=f_i(z_x,t+(a+1)e^i)$ for all $t\in\left[0,z_x^{N\backslash\{i\}}\right]$ and $a=1,...,x_i$.

Take any $x\in\mathbb{N}^N$ such that $x\geq 2e^N$. We show that if the flows to z_x are well defined, then so are the flows to z_x+e^i and $z_x-x_ie^i$, i.e. problems where we add to the demand of agent i and where we remove all units of demand of agent i.

[2] Starting from C_2 and removing the first x_j-1 units of demand of agent j leaves us with the same cost function as if we start with C_1 and remove the last x_j-1 units of demand of agent j.

For any $j \in N \backslash i$, $s \in \left[0, z_x^{N \backslash i,j}\right]$ and $b \in \{1, ..., x_j + 1\}$, we must have, by i) and ii), that $f_j(z_x + e^i, s + (x_i + 1)e^i + be^j) = 0$, $f_j(z_x + e^i, s + (x_i + 2)e^i + be^j) = f_j(z_x, s + (x_i + 1)e^i + be^j)$. We must also have that $f_j(z_x + e^i, s + ae^i + be^j) = f_j(z_x, s + ae^i + be^j)$ for all $a = 0, 1, ..., x_i$. For all $s \in \left[0, z_x^{N \backslash i}\right]$ and $a = 0, 1, ..., x_i + 1$, we have that $f_i(z_x + e^i, s + ae^i) = f_i(z_x, s + ae^i)$ and $f_i(z_x + e^i, s + (x_i + 2)e^i) = f_i(z_x, s + (x_i + 1)e^i)$. We have thus fully defined the flows to $z_x + e^i$.

For any $j \in N \backslash i$, $s \in \left[0, z_x^{N \backslash i,j}\right]$ and $b \in \{1, ..., x_j + 1\}$ we must have by the fact that the CSM is a fixed-flow method that $f_j(z_x - x_i e^i, s + e^i + be^j) = \sum_{k=1}^{x_{i+1}} f_j(z_x, s + ke^i + be^j)$ while $f_j(z_x - x_i e^i, s + be^j) = f_j(z_x, s + be^j)$. Also, for all $s \in \left[0, z_x^{N \backslash i}\right]$ we have that $f_i(z_x - x_i e^i, s + e^i) = f_i(z_x, s + e^i)$. We have thus fully defined the flows to $z_x - x_i e^i$.

Therefore, if the flow to $2e^N$ is well defined, it implies a unique flow to any $z \in \mathbb{N}^N$. ■

Figure 1: Cost function C^* from Example 1

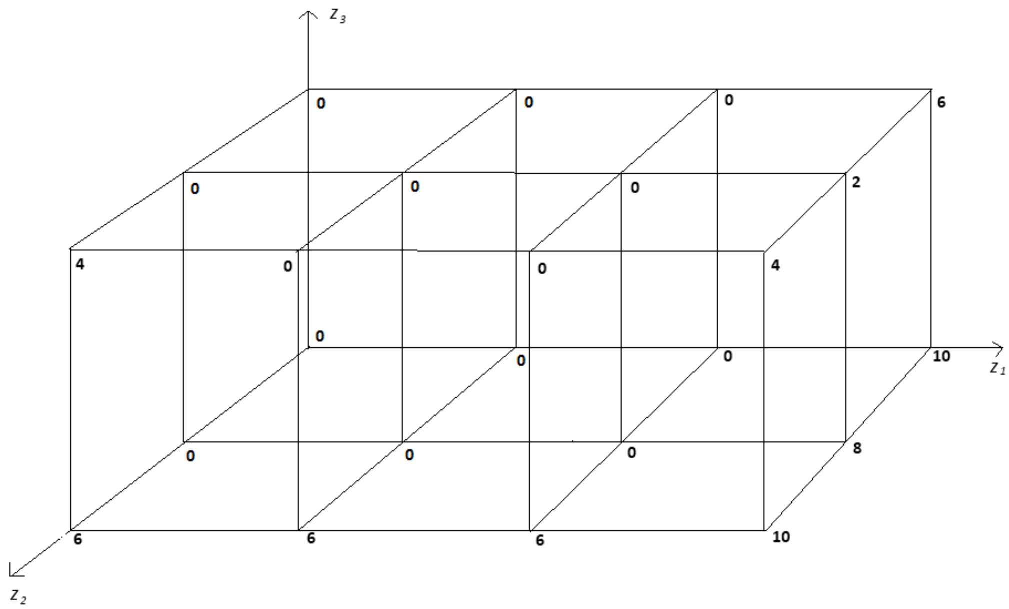

Figure 2: Flows for the public Shapley-Shubik (left) and private Shapley-Shubik (right) methods in Example 1

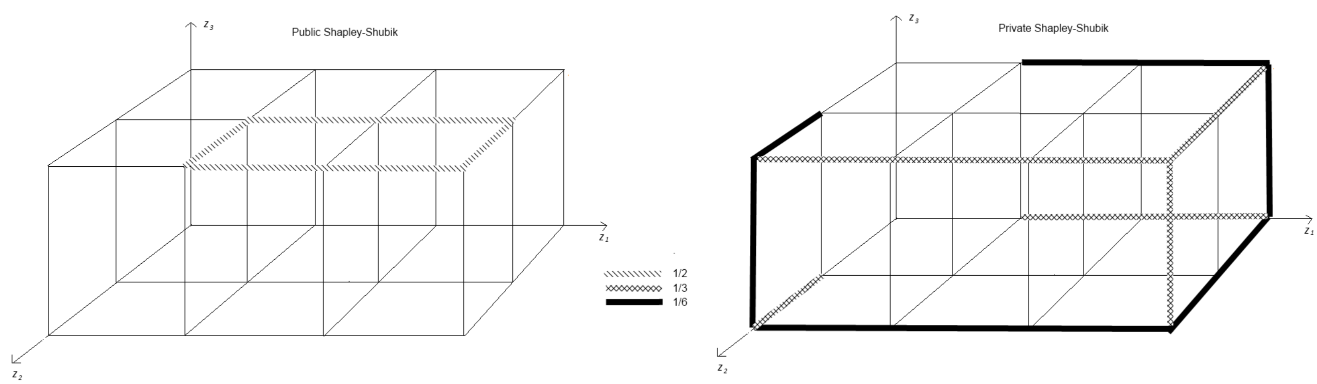

Figure 3: Cost function C'^{*} from Example 2

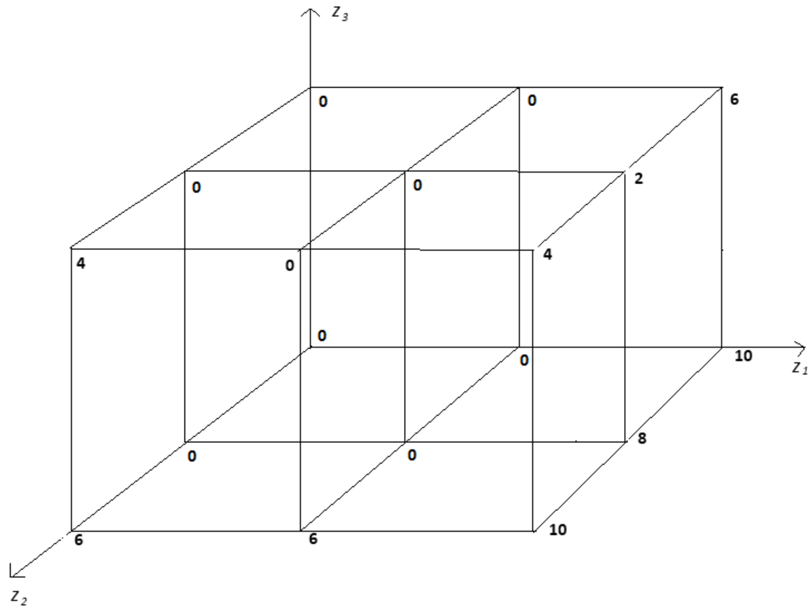

Figure 4: Cost function C''^{*} from Example 2

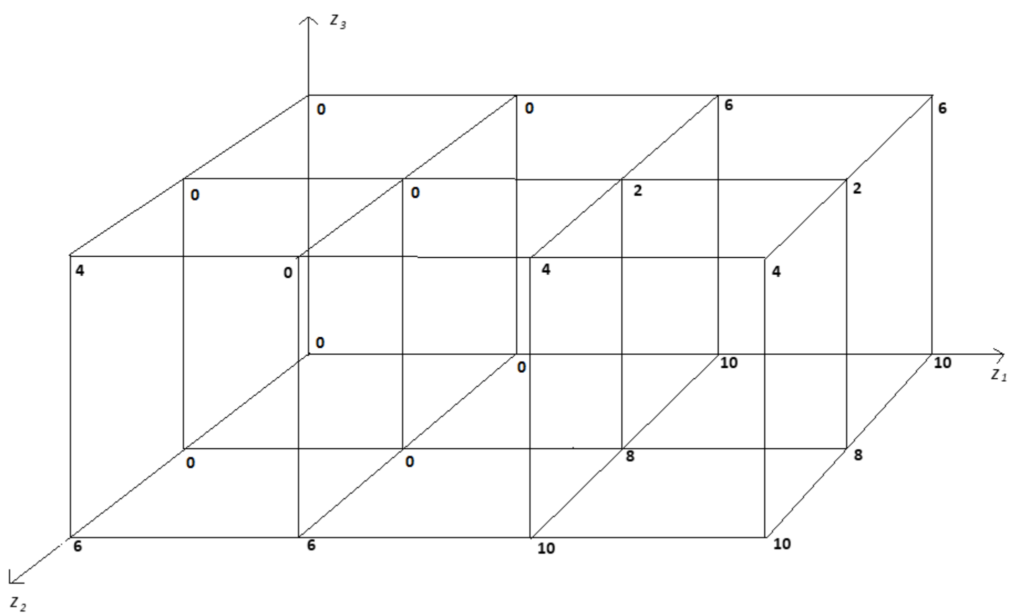

Figure 5: Illustration of Independence of Dummy Units: Cost $C(S \cup \{i\}, (t, t_i))$ as a function of t_i, for a fixed coalition S and a fixed level of production t.

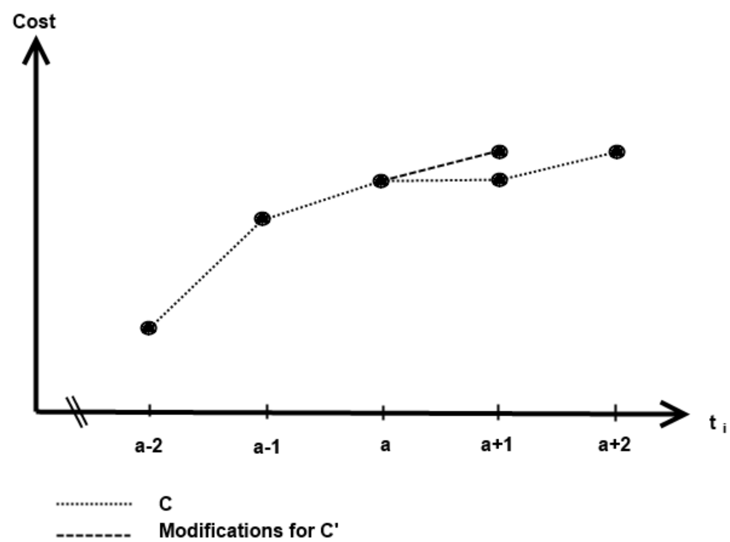

Figure 6: Restrictions on flows coming from independence of dummy units, with $x' = (1,1,0)$ and $x = (2,1,0)$. Edges with the same pattern must have equal flows.

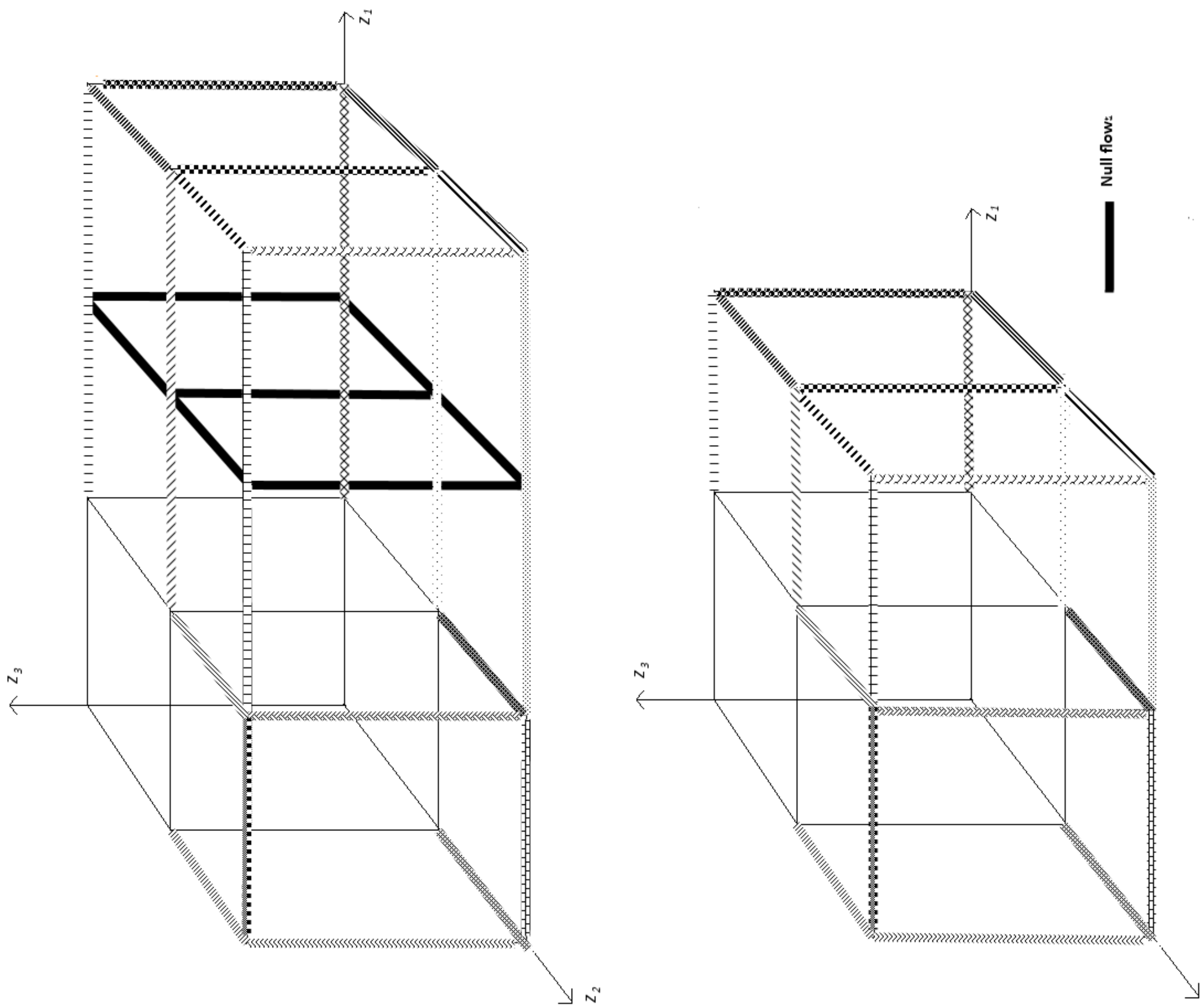

Figure 7: Flows for the Average Single Rent method in Example 4

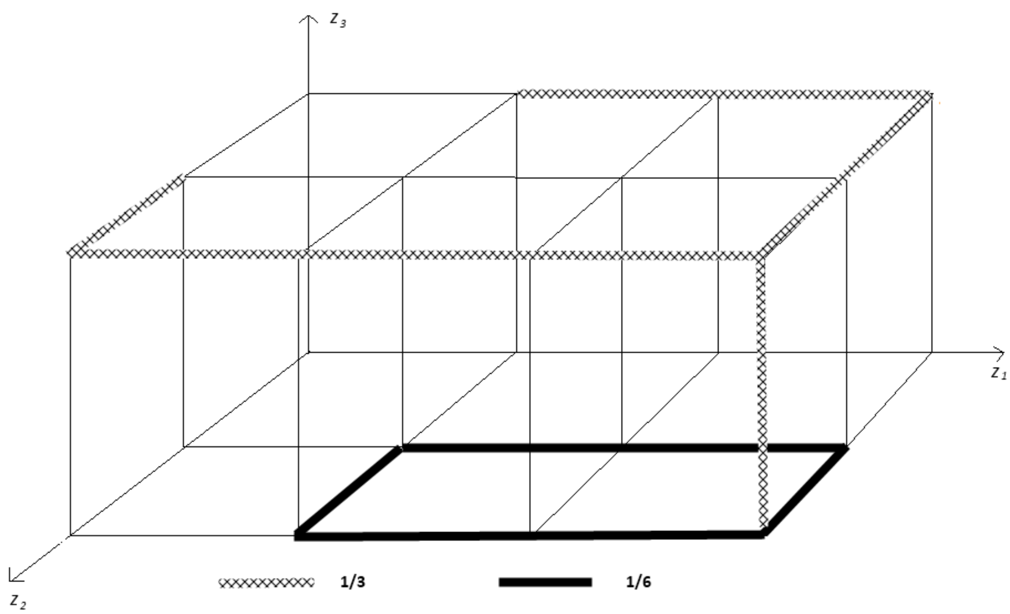

Figure 8: Flows to (2, 2, 2) for the three methods