Extension of the Asset Pricing Models

1. Capital Market Theory: An overview

Capital market theory followed modern portfolio theory by Markowitz, as researchers explored the implications of introducing a risk-free asset. Sharpe is generally credited with developing the CAPM, but Lintner and Mossin derived similar models independently in the mid 1960s. Assumptions made regarding Capital Market Theory include:

- All investors are Markowitz efficient investors who choose investments on the basis of expected return and risk.
- Investors can borrow or lend any amount at a risk-free rate of interest.
- All investors have homogeneous expectations for returns.
- All investments are infinitely divisible.
- No transactions costs or taxes, no inflation or any change in interest rates and capital markets are in equilibrium.

1.1. Combining A risk-free asset with a risky portfolio

Before discussing this part, note the following two observations:

1) The covariance of a risky asset with the risk-free asset is zero.

2) The correlation coefficient between the risk-free asset and any risky asset is also zero.

The question is: what happens to the average rate of return and the standard deviation of returns when you combine a risk-free asset with a portfolio of risky assets such as those that exist on the Markowitz efficient frontier. The risk free rate is fixed over the investment horizon, so it has some special properties, namely

$$R_f = E(R_f) = R_f$$

$$Var(R_f) = 0$$

$$Cov(R_i, R_f) = 0$$

this is because the risk-free asset has no variability and therefore does not move at all with the return on the market portfolio which is a risky asset. We also know that: $w_f + w_i = 1$ where i represents a risky asset (i.e., we could just include the market portfolio there, M).

So the expected return and risk of this portfolio:

$$E(R_P) = w_f R_f + w_i\, E(R_i)$$

or :

$$E(R_P) = w_f R_f + (1 - w_f) E(R_i)$$

where w_f is the proportion invested in the risk-free asset and $(1-w_f)$ is the weight invested in the risky asset.

What about the risk or standard deviation in this portfolio:

$$Var(R_P) = w_f^2 var(R_f) + w_i^2\, var(R_i) + 2w_i\, w_f std(R_f) std(R_i) corr(R_f, R_i)$$

The portfolio's variance becomes:

$Var(R_P) = (1-w_f\,)^2 var(R_i) = w_i^2 var(R_i)$ that is the portfolio variance is porportional to the variance of asset i. Or the standard deviation is:

$$\sigma_p = \sqrt{(1 - w_f\,)^2 var(R_i)} = (1 - w_f\,)\sigma_i = w_i \sigma_i$$

We can see that the standard deviation of a portfolio that combines the risk-free asset with risky assets is the linear porportion of the standard deviation of the risky assets portolio. In other words the risk of this portfolio is porportional to the risk associated with the risky assets.

We can further say, $\sigma_p = w_i \sigma_i$ so, $w_i = \sigma_p / \sigma_i$

We can use the expected return on the portfolio and find that:

$$E(R_P) = w_f R_f + (1 - w_f) E(R_i) = (1 - w_i) R_f + w_i E(R_i) = R_f + (E(R_i) - R_f) w_i$$

or

$$E(R_P) = R_f + [(E(R_i) - R_f)]\,(\sigma_p / \sigma_i)$$

Which is simply straight line in $(E(R_P), \sigma_p)$ with intercept R_f and slope $(E(R_i) - R_f)/\sigma_i$. The slope of the combination line between the risk free asset and a risky asset is called the Sharpe Ratio or Sharpe's slope and it measures the risk premium on the asset per unit of risk (as measured by the standard deviation of the asset).

If we take two assets, A and B, with $E(R_A) = 0.175, E(R_B) = 0.055, \sigma_A = 0.258, \sigma_B = 0.115$ and $R_f = 0.03$, we can find the Sharpe slopes for both asset A and Asset B. Portfolios of asset A and the risk-free asset are efficient relative to portfolios of asset B and risk-free asset.

Why: $(E(R_A) - R_f)/\sigma_A = 0.562$ and that for B is: 0.217.

1.1.1. The Risk-Return combination

The combination of risk and return for a portfolio of risky assets and the risk-free security is also linear.

1. On a graph with the old Markowitz efficient frontier for risky assets, draw a line from the risk-free rate on the vertical axis to any point on the efficient frontier.

2. Any point on this line is attainable, while points below it are attainable but inefficient in a risk-return sense.

3. Repeat the process until the line is tangent to the efficient frontier at a point M. Points above the line are not possible, all points below are inefficient, and all points on the line are feasible. In particular, any point on the line between the RFR and M represents a portoflio that has some positive amount in the risk-free asset and some positive amount in portfolio M of risky assets.

1.2. Risk-Return possiblities with Leverage

An investor may want to attain a higher expected return than is available at point M in exchange for accepting higher risk. One alternative would be to invest in one of the risky asset portfolios on the efficient frontier beyond point M such as the portfolio at point D. A second alternative is to add leverage to the portfolio by borrowing money at RFR and investing the proceeds in the risky asset portfolio at point M. Points on the line extending above M to the right are feasible and are efficient.

The return on this portfolio is given by:

$$E(R_P) = w_f R_f + (1 - w_f)E(R_M)$$

where $E(R_M)$ is the expected return on the market portfolio. Suppose you borrow an amount equal to 50% of your original wealth at the RFR, w_f will not be a positive fraction in this case, but rather a negative 50% ($w_f = -50\%$). The effect of borrowing on expected return and variance will be as follows:

$$E(R_P) = w_f R_f + (1 - w_f)E(R_M) = -0.5(R_f) + (1.5)(E(R_M))$$

Again the return will increase in a linear fashion along the line RFR - M.

Assume that $E(R_M) = 0.12$ and $R_f = 0.06$ then E(R$_P$) $= 0.15$

What about the effect on standar deviation:

We know that : $\sigma_p = \sqrt{(1 - w_i\ )^2 var(R_i)} = (1 - w_f\ )\sigma_i$, however, instead of a risky asset i, we have the market portfolio, so $\sigma_p = \sqrt{(1 - w_f\ )^2 var(R_M)} = (1 - w_f\ )\sigma_M = [1 - (-0.5)]\sigma_M = 1.50\sigma_M$

The standar deviation also increases in a linear fashion.

We now have a linear relationship between risk and return. This new efficient frontier, the line RFR-M, is called the **Capital Market Line (CML).**

All portfolios along the CML are perfectly positively correlated with each other. Also, each portfolio along the CML has two parts: portfolio M, and the risk-free asset, either a positive amount (lending), or a negative amount (borrowing).

The market portfolio M contains all risky assets and all assets are represented in M in porportion to their market value. In addition, M is completely diversified, so that it has no unique risk attributable to any individual security. This unique risk is called unsystematic, or diversifiable, or firm specific risk. The only risk left in portfolio M is systematic, or nondiversifiable, or market risk.

1.2.1. Few points to note

- All investors who choose to be on the CML will choose the same combination of risky securities, that is, M.

- How much of his initial wealth an investor will put into M depends on his risk preference. A highly risk averse investor will choose to put a larger portion of his or her wealth in the risk-free asset (lend) and therefore less in M than an investor who is less risk averse.

- Tobin's "separation theorem" the decision as to which risky securities to invest in is separate from the decision of how much wealth to put in the risky assets.

- In the case of the Markowitz model, the relevant risk measure for a risky security added to a portfolio is the covariance of that asset with the other securities in the portfolio. However, in the case of the CML, the only combination of risky assets worth considering is the market portfolio M. All risky assets are in the market portfolio, so their returns can be shown according to what is called the **market model: (i.e.,** if the market has gone up then it is likely that the stock has gone up, and if the market has gone down then it is likely that the stock has gone down):

$$R_{it} = a_i + b_i * R_{Mt} + \epsilon$$

where R_{it} is the return for asset i for period t, a_i constant term for asset i, b_i slope coefficient for asset i, R_{Mt} is the return for portfolio M during period t, and ϵ is an error term. The above equation indicates that the higher the return on the market index, the higher the return on the security is likely to be. (Note that the expected value of the random error is zero).

Consider stock A, for example, which has $a_i = 2\%$ and $b_i = 1.2$. This means that the market model for stock A is:

$$R_{At} = 2\% + 1.2 * R_{Mt} + \epsilon_A$$

so that if the market index has a return of 10%, the return on the security is expected to be $2\% + (1.2 \times 10\%) = 14\%$. Similarly, if the market index's return is -5%, then the return on security A is expected to be $2\% + (1.2 \times -5\%) = -4\%$.

$R_{At} = 2\% + 1.2 * R_{Mt} + \epsilon_A$ can be graphed without the random error term. It would be on a space of R_{At} and R_{Mt} where the vertical axis measures the return on the particular security R_{At} whereas the horizontal axis measures the return on the market index R_{Mt}. The line goes through the point on the vertical axis corresponding to the value of a_i, which in this case is 2%. In addition, the line has a slope equal to b_i, or 1.2.

The slope term in the market model is often referred to as beta, and is equal to :

$$\beta_{iM} = Cov(R_{it}, R_{Mt})/\sigma_M^2$$

Where $Cov(R_{it}, R_{Mt})$ denotes the covariance of the returns on stock A and the market index, and σ_M^2 denotes the variance of returns on the market index. A stock that has a return that mirrors the return on the market index will have a beta equal to one. (note: stocks with betas greater than one are more volatile than the market index and are known as **aggressive stocks**. In contrast, stocks with betas less than one are less volatile than the market index and are known as **defensive stocks**).

By taking the variance of both sides and applying statistical rules for variance, we find

$$Var(R_{it}) = 0 + Var(b_i * R_{Mt}) + Var(\epsilon)$$

or

$$\sigma_i^2 = b_i^2 \sigma_M^2 + \sigma_{ei}^2$$

Thus the variance of a security can be stated again as follows:

$$Var(R_{it}) = SystematicVariance + UnsystematicVariance.$$

Proper diversification forces the last term, Var(ϵ) or unsystematic risk, to be zero. The risk of the security is a function of the risk of the market portfolio (market or systematic risk). Because only systematic risk is relevant, it is the only risk that an investor will be rewarded for taking on. The unsystematic risk can be diversified away and therefore an investor should not be compensated for taking on such risk.

Example 1.1. *The single index model*

The single index model related security returns to their betas, thereby measuring how each security varies with the overall market. By observing how two independent securities behave relative to some third value, we learn something about how the two securities are likely to behave relative to each other in the future. Beta is the statistic relating an individual security's returns to those of the market index. Beta can be defined in terms of standard deviations and covariance, as shown in the following

$$\beta_i = \frac{\rho_{im}\sigma_i}{\sigma_m} = \frac{cov(R_i, R_m)}{\sigma_m^2} \tag{1.1}$$

where R_m is the return on the market index, R_i is the return on security i, σ_i is the standard deviation of security $i$returns, σ_m is standard deviation of market returns, and ρ_{im} is the correlation between security i returns and market returns. The relationship between beta and expected return is the essense of the capital asset pricing model as stated by:

$$E(R_i) - R_f = \beta_i E(R_m - R_f) \tag{1.2}$$

Security prices are determined more by the future than the past. Beta is theoretically an estimated of future stock return volatility. The future, of course, is unobservable. Consequently, we must estimate beta using current and historical information. We typically, use the **market model** as follows:

$$R_{stock} = \alpha + \beta R_m \tag{1.3}$$

For example, if we fit the market model to the Royal Bank of Canada stock returns and those of TSE 300, getting the following:

$$R_{stock} = \alpha + \beta R_m = 0.004 + 1.08 R_{TSE300} \tag{1.4}$$

The intercept, according to finance theory, should be zero, actually 0.004 is close to zero. A nonzero value of the intercept indicates a rate of return (positive or negative) that is inexplicable by current finance theory. The intercept is known as **alpha**, and is sometimes used as a measure of peformance. A positive alpha is good; it means an investor earned a return greater than necessary given the level of risk. The analysis also suggests a beta of 1.08, which is greater than 1.00 (market beta) and indicates that RBC stock has a higher systematic risk than that of TSE300. It is important to note that beta can change over time. It will change if the firm's leverage ratios change, if its product sales becomes more volatile, or if it acquires new subsidiaries. Beta is also dependent on the time period used to estimate it. Sixty monthly observations are frequently used to estimate published betas, but this convention may produce a different estimate than if analysts use periods of 100 weeks or 50 days.

To used the market model to forecast tomorrow's returns, we use the beta estimate of 1.08 and observe one-year T-bill rates of 5.5%, and say that the risk premium is $(E(R_m - R_f) = 8.4\%))$. To forecast the expected return on RBC we get:

$$E(R_{RBC}) - 0.055 = 1.08 E(R_{TSE300} - R_f) = 1.08(0.084) \tag{1.5}$$

$$E(R_{RBC}) = 14.6\%$$

1.2.2. Single-Index model and Portfolio theory

Using the single-index model specification, we expressed the expected return of an individual security as $R_i = \alpha_i + \beta_i E(R_m)$ with α_i representing the specific return component and $\beta_i E(R_m)$ related to market-related return component. The residual return disappears from the expression because its average value is zero; that is, it has an expected value of zero. In calculating risk and return of a portfolio, we can use similar formulas and aggregrate across the individual securities to measure those aspects of the portfolio. In particular, the expected return of the portfolio becomes a weighted average of the specific returns (alphas) of the individual securities plus a weighted average of the market-related returns $\beta_i E(R_m)$ of the individual securities. Defining $\alpha_P = \sum_i^N W_i \alpha_i$ as the portfolio alpha and $\beta_P = \sum_i^N W_i \beta_i$ as the portfolio beta, we can directely represent portfolio return as a portfolio alpha plus a portfolio beta times expected market return, as shown: $R_P = \alpha_P + \beta_P E(R_m)$.

Because of the assumption that securities are related only through a common market effect, the risk of a portfolio is also simply a weighted average of the market-related risks of individual securities plus a weighted average of the specific risks of individual securities in the portfolio. Therefore, portfolio risk can be expressed as:

$$Var(R_P) = \beta_P^2 Var(R_m) + \sum_i^N W_i^2 Var(e_i) \tag{1.6}$$

note that the diversifiable risk component W_i^2 will become smaller as securities are added to the portfolio. This is because, according to the single-index model, these risks are uncorrelated. The reduction thus becomes similar to the issue of diversification benefits when adding more uncorrelated securities; the variance is reduced when securities are uncorrelated and have equal weight in portfolio. The effect would be the same in this case, except that we are dealing with one component, diversifiable risk, rather than total risk. The market-related component in this case remain unaffected by the addition of extra securities, because systematic risk is the component that cannot be reduced by diversification.

Example 1.2. *The following table illustrates the use of the single-index model to calculate expected return and variance for a hypothetical portfolio of four securites: (1) Merck, a drug company, to represent a rapid-growth security; (2) Behtlehem Steel, to represent a cyclically oriented security; (3) Kellog, a food company, to represent a stable type of security; and (4) Chevron, to represent an energy-oriented security. The table shows the weighting of these companies in the hypothetical portfolio; the weights sum to 1.00 to represent a portfolio fully invested in stocks. It also shows basic input data: alphas, betas, and residual variances for calculating portfolio return and variance.*

We set the alpha values of the individual stocks at zero, to assume that no special information regarding the relative attractiveness of the individual stocks is considered. In this case, the weighted average alpha of the portfolio is simply zero. The betas of the individual stocks have been estimated with historical data, and the projected market return is 11 percent, a reasonable approximation of the consensus of expectations at the end fo 1993. The weighted average beta of the portfolio is 1.02, and using the projected market return of 11 percent results in a market-related return of 11.2 percent. With an expected alpha of zero, the expected return of the portfolio is entirely market-related and is also projected at 11.2 percent.

We further calculate the portfolio standard deviation. It was observed that over the period 1975-1993 the market standard deviation is about 18% (or, alternatively, its variance was 324 percent), we will assume that this historical figure is appropriate for projection into the future. The specific risk estimates for each of the stocks are also historically derived. The bottom line shows the market-related risk calculated from the weighted portfolio beta and the projected market variance, along with the weighted average of the specific risk of the individual stocks. Note that the specific risk of the portfolio is less than that of any of the individual stocks. This is, of course, consistent with the single index model's assumption of being able to diversify specific risk by adding securities to the portfolio.

Portfolio risk and return, single-index model

Security	Weighting	α_i	β_i	$Var(e)$
Merck	0.25	0	1.20	446
Bethlehem Steel	0.25	0	1.09	653
Kellog	0.25	0	0.89	579
Chevron	0.25	0	0.89	310
Portfolio value	1.00	-	1.02	124
$E(R_P) = \alpha_P + \beta_P E(R_P) = 0 + 1.02(11\%) = 11.2\%$				
$Var(R_P) = \beta_P^2 Var(R_P) + \sum_i^N W_i^2 Var(e_i) = (1.02)^2(324) + 124\% = 337\% + 124\% = 461\%$				