Part I

Portfolio Theory

1. Introduction

Before discussing the portfolio, it is important to make sure the following concepts are understood:

Efficient Portfolios: That is when investors seek to maximize the expected return from their investment given some level of risk they are willing to accept.

Risk Aversion: Individuals according to those theories are assumed to be risk averse: is one who, when faced with two investments with the same expected return but two different risks, will prefer the one with the lower risk.

Risky assets: Those are the ones which the return that will be realized in the future is uncertain. Corporate bonds are riskier than public bonds, because of the possibility of default, inflation and so on.

Assets in which the return that will be realized in the future is known with certainty today are referred to as **risk-free assets** or **riskless assets.**

1.1. Returns And Variances

In order to understand the relationship between risk and return, we must be able to measure both risk and return for an investment.

1.1.1. Dollar returns

The return on an investment has two components: the income component; and the capital gain (or capital loss) component. For a common stock, your return comes in two components: dividends (periodic income) and capital gains.

Suppose you buy 100 shares of ABC common stock for \$25 per share. Over one year, the ABC company pays a cash dividend of \$2 per share and the value of the stock rises to \$30. At the end of the year you have received (\$2 $\times$100) = \$200 in dividends. In addition you own 100 shares of stock worth \$30 per share; The capital gain is [(\$30-\$25)$\times$100] = \$500.

Then:

Total Dollar Return = Dividend Income + Capital Gain (or loss) = \$200 + \$500 = \$700.

The total value of your position is \$3200, total investment is \$2500, and total return is \$700.

1.1.2. Percentage Return

Percentage returns are easy to work with. In the previous example, you received \$700 on a \$2500 investment, so your total percentage return is:

[\$200 + (\$3000 - \$2500)]/\$2500 = .28 = 28%

That can be expressed in terms of per share basis, that is:[\$2 + (\$30 - \$25)]/\$25 = .28 = 28%

The 28% total return is comprised of the dividend yield plus the percentage capital gain (or capital gains yield). The **dividend yield** is given by:

$D_{t+1}/P_t = \$2/\$25 = 0.08 = 8\%$

Where P_t is the price of the stock at the beginning of the year, and D_{t+1} is the dividend paid during the year. The **capital gains yield** is given by:

$(P_{t+1} - P_t)/P_t = (\$30 - \$25)/\$25 = .20 = 20\%$

1.1.3. Average Returns

The average return (or mean) is simply the sum of all returns divided by the number of observations:

Mean $= \overline{R} = E(R) = [R_1 + R_2 + \ldots\ldots\ldots\ldots\ldots + R_T]/T$

Where T is the number of time periods.

1.1.4. The variability of returns

Since riskiness affects return, we must be able to measure the degree of risk associated with an investment in order to understand the relationship between risk and return. The most common measures are the **Variance** and the **Standard Deviation.** In financial context, we use the variance to measure the variability of returns from the average return; the greater the deviation from the average, the more variable the rate of return and the higher the level of risk.

Take an example, say an investment returned 8%, 14%, -4%, and 6% over the last 4 years. The average return is:

$\overline{R} = E(R) = [R_1 + R_2 + R_3 + R_{4T}]/4 = (0.08+0.14-0.04+0.06)/4 = 0.06 = 6\%$

Rate of Return	Average Return	Deviation from Average(R-$\overline{R}$)	(R-$\overline{R}$)^2
0.08	0.06	0.02	0.004
0.14	0.06	0.08	0.0064
-0.04	0.06	-0.10	0.0100
0.06	0.06	0.00	0.000

The calculation of the variance is based on :

Var (R) $= \sigma^2 = (1/(T-1))\,[(R_1-\overline{R})^2+(R_2-\overline{R})^2+(R_3-\overline{R})^2 + \ldots\ldots\ldots\ldots(R_T-\overline{R})^2]$

The standard deviation is the square root of the variance.

2. Expected Portfolio Return

2.1. Expected Returns and Variances

The previous discussion about the calculation of average returns are based on historical data. But usually the focus is on rates of return that might occur in the future. A useful way to quantify the uncertainty about the portfolio return is to specify the probability associated with each of the possible future returns. So each outcome has a probability. The average return in this case will be referred to as **the expected return.** An expected value in an average, or mean, of possible future outcomes. To illustrate consider the following example:

State of the Economy	Probability	Possible Rates of Return
1	0.20	0.10
2	0.20	0.04
3	0.20	-0.09
4	0.20	0.20
5	0.20	0.05

The table provide forecasts of possible return of a stock of ABC during the coming year. The forecasts are based on five equally-likely possible states of the economy. A state of the economy might be a severe recession, or a depression, or moderate growth in the economy. (Note it is by coincidence that $\Pr_i$ the probability of state i is the same, but in general, different states will be attached different probabilities. In case where we have equal weights for Pr's, E (R) is exactly calculated as in the case of the historical data).

Expected return is calculated as a weighted average of the possible returns. Each possible return is multiplied (weighted) by its probability of occurrence. Then the expected return will be:

E(R)= $\overline{R} = (\Pr_1 \times R_1)+(\Pr_2 \times R_2) + (\Pr_3 \times R_3)....................(\Pr_T \times R_T)$

Where T is the number of possible states of the economy, Pr's are the probabilities of the respective states, and R's are the possible rates of return. For the above example, E(R) $= 0.06$

To calculate the variance of the data representing future possible rates of return, we have to determine the squared deviations from the expected return. Each possible squared deviation is multiplied by its probability of occurrence. These are added; the sum is the variance. The standard variance is simply the square root of the variance, or to say:

Var (R) $= \sigma^2 = [\Pr_1 \times (R_1 - \overline{R})^2] + [\Pr_2 \times (R_2 - \overline{R})^2] + ... + [\Pr_T \times (R_T - \overline{R})^2]$

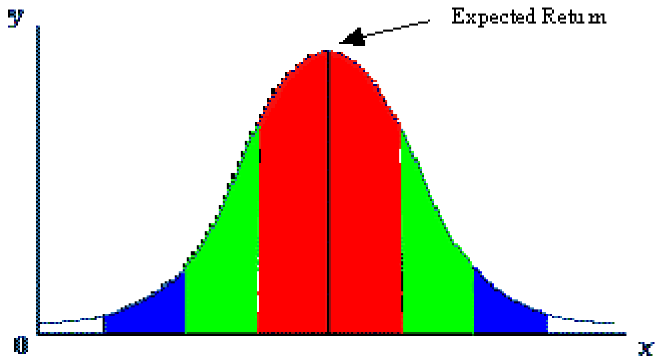


Figure 2.1: Return distribution

State	Probability (Pr)	Rate of Return	$(R-\overline{R})$	$(R-\overline{R})^2$
1	0.20	0.10	0.04	0.016
2	0.20	0.04	-0.02	0.0004
3	0.20	-0.09	-0.15	0.0225
4	0.20	0.20	0.14	0.0196
5	0.20	0.05	-0.01	0.0001

So the Variance can be calculated, for the above example, we get Var(R) = 0.00884, and then the Standard deviation is $\sqrt{0.00884} = 0.09402$.

2.2. Return and Return Volatility

To explain the relationship between expected return and return volatility, and to show why it is important to consider more than just expected returns when selecting assets for inclusion in a portfolio.

Expected Return The return expected on an investment (an asset or a portfolio) based on a probability distribution, taking into account all possible return scenarios.

Return Volatility Represents the variability or uncertainty of an assets return; it is measured by a value called standard deviation.

Essentially, return volatility (or standard deviation) tells us how much an assets actual return is likely to deviate above or below its expected return. Consider the graph above, where each segment represents one standard deviation:

"Red" Zone = The assets actual return has approximately 68% probability of falling within this zone (i.e. within 1 standard deviation of the assets expected return).

"Red+Green" Zone = The assets actual return has approximately 95% probability of falling within this zone (i.e. within 2 standard deviations of the assets expected return).

"Red+Green+Blue" Zone = The assets actual return has approximately 99% probability of falling within this zone (i.e. within 3 standard deviations of the assets expected return).

Asset A has an expected return of 22%, and a return volatility (standard deviation) of 15%. With this information, we can infer the following:

Asset A has a 68% probability of achieving an actual return between 7% and 37% (i.e. one standard deviation below and above expected return On the graph, this range is represented the red area).

Asset A has a 95% probability of achieving an actual return between -8% and 52% (i.e. two standard deviations below and above expected return On the graph, this range is represented the red+green area).

Asset A has a 99% probability of achieving an actual return between -23% and 67% (i.e. three standard deviations below and above expected return On the graph, this range is represented the red+green+blue area).

The important point here is that return volatility (standard deviation) can have a tremendous impact on actual return. The oft-quoted cliché says, High risk, high reward, but thats only half of the story. With an understanding of return volatility, it's clear that the cliché fails to mention that high risk also means the potential for great loss.

When considering an asset for inclusion in a portfolio, it's natural to look to the upside associated with expected return. However, we also need to factor in return volatility and decide whether or not we can live with the potential downside associated it represents.

2.3. Portfolios

An investor's portfolio of assets is the combination of assets the investor owns. A portfolio consists of several assets. Therefore, an investor is more concerned about the characteristics of the portfolio than the characteristics of the individual assets which comprise the portfolio.

Portfolio Weights: the respective percentages of portfolio's total value invested in each of the assets in the portfolio are referred to as the **portfolio weights**.

Example: An investor's portfolio has a total value of $1000 and is comprised of three assets. The values of assets A, B and C are $450, $250 and $300 respectively. What are the portfolio weights for this portfolio.

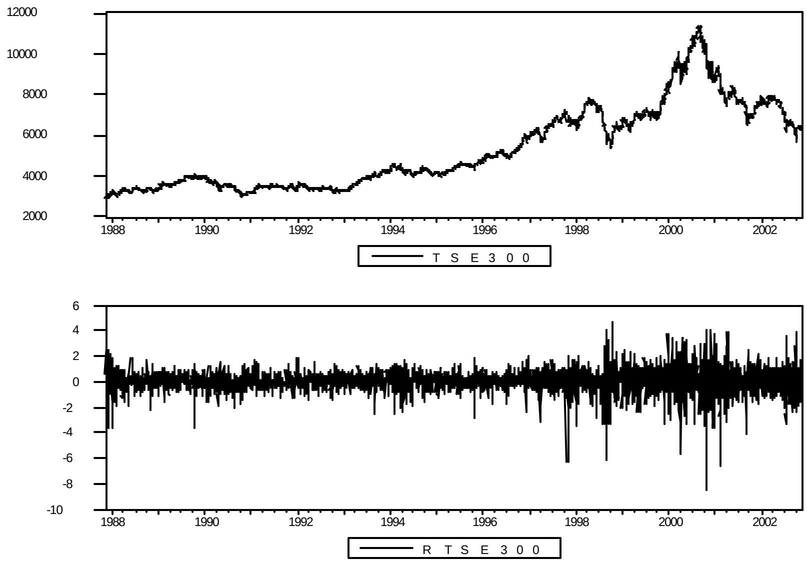


Figure 2.2: TSE 300 in levels and returns. Sample period is from 11/16/1987 to 11/14/2002

The percentage of the total value of the portfolio invested in asset A is ($450/$1000) =.45 = 45%. Similarly, the percentages for assets B and C are ($250/$1000) = 25% and 30%. Therefore, the weights are 45%, 25% and 30%.

2.3.1. Portfolio Expected Returns

Consider again the following example from before, except we have a new asset, let us call the two assets, X and Y.

State	Pr	Rate of Return on X	Rate of Return on Y
1	0.20	0.10	0.03
2	0.20	0.04	-0.08
3	0.20	-0.09	0.07
4	0.20	0.20	0.12
5	0.20	0.05	0.21

Suppose that an investor invests 60% of his money in Asset X and 40% in Asset Y; that is the portfolio weights are 60% and 40%, respectively. For each of the five possible states of he economy indicated in the example, the investor's return on his investment will be equal to a weighted average of the returns in that state for the respective stocks. For example, in state 1, the portfolio return:

$R_P = (0.60 \times 0.10) + (0.40 \times 0.03) = 0.072 = 7.2\%$. That says, if state 1 occurs, then the rate of return for the portfolio will be 7.2%. The table below makes all the calculation to derive at the expected return of the portfolio E(R_P)

State	Probability (Pr)	R_X	R_Y	R_P	$(Pr \times R_P)$
1	0.20	0.10	0.03	0.072	0.0144
2	0.20	0.04	-0.08	-0.008	-0.0016
3	0.20	-0.09	0.07	-0.026	-0.0052
4	0.20	0.20	0.12	0.168	0.0336
5	0.20	0.05	0.21	0.114	0.0228
Total	1.00	0.30	0.35	0.320	0.0640

So the expected return for the portfolio is $0.064 = 6.4\%$.

A simpler approach to compute the expected return for a portfolio is based on the fact that the expected return for the portfolio is a weighted average of the expected returns of the securities which comprise the portfolio; the weights are the weights of portfolio. The general formula would be:

$\mathrm{E}(R_P) = [x_1 \times E(R_1)] + [x_2 \times E(R_2)] + [x_3 \times E(R_3)] + ... + [x_n \times E(R_n)]$

where n is the number of assets in the portfolio, x_1, x_2, x_3, and x_n are the portfolio weights for the assets and $E(R_1), E(R_2), E(R_3)$ and $E(R_n)$ are the expected returns for the assets. For the above example, we have:

$\mathrm{E}(R_P) = [x_1 \times E(R_1)] + [x_2 \times E(R_2)] = (0.60 \times 0.06) + (0.40 \times 0.07) = 6.4\%$

2.3.2. Portfolio Variance

As opposed to the expected return for a portfolio, the variance and standard deviation (of a portfolio) are not equal to weighted averages of the corresponding characteristics of the individual securities. The actual standard deviation for a portfolio is virtually always less than the weighted average of the standard deviations or the securities in the portfolio. Let us find the variance and standard deviation for the portfolio we have above:

State	Pr	R_X	R_Y	R$_P$	$(R_P - E(R_P))^2$	$[\mathbf{Pr} \times (R_P - E(R_P))]^2$
1	0.20	0.10	0.03	0.072	0.000064	0.0000128
2	0.20	0.04	-0.08	-0.008	0.005184	0.0010368
3	0.20	-0.09	0.07	-0.026	0.008100	0.0016200
4	0.20	0.20	0.12	0.168	0.010816	0.0021632
5	0.20	0.05	0.21	0.114	0.002500	0.0005000
Total	1.00	0.30	0.35	0.320	**0.026664**	**0.0053328**

So the variance of the portfolio is **0.0053328** and the standard deviation would be **0.07303.**

2.3.3. Portfolio standard deviation and Diversification

That means, constructing a portfolio in such a way as to reduce portfolio risk without sacrificing return. Observations from the capital markets indicate that deviation of returns for the stock alone can be greater than that of the portfolio, and the stock average return is less than the portfolio return.

The answer to this question lies in the fact that not all risk is relevant. Much of the total risk is diversifiable. That is, if that investment had been combined with other securities, a portion of the variation in its returns could have been smoothed or canceled by complementary variations in the other securities.

It is very important to note here, that in order to reduce risk, diversification of combining securities should involve less than perfectly correlated ones. Diversification will not affect the portfolio return, but it will reduce the variability of return. The less the correlation among security returns, the greater the impact of diversification on reducing variability. Empirically speaking, securities are always positively correlated, they are affected always by business cycles and interest rates: elimination of risk is impossible in this case.

Systematic risk, which is related to market conditions, will always exist and affect the portfolio variability.

In general, combining securities into portfolios reduces the level of risk. The actual standard deviation for a portfolio is virtually always less than the weighted average of the standard deviations for the securities in the portfolio. This is because the correlation between the returns of the securities in the portfolio. Mathematically, it can be shown that the portfolio variance for a two stock portfolio, stock A and stock B is:

$$\sigma_P^2 = X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_BCorr_{A,B}\sigma_A\sigma_B$$

where $Corr_{A,B}$ = Correlation betwee the returns of A and B and $Corr_{A,B}\sigma_A\sigma_B$ = Covariance between the returns of A and B. As long as the correlation **is less than 1.0**, the standard deviation of a portfolio of two securites is less than the weighted average of the standard deviation of the individual securities.

Combining securities or assets whose Corr is less than +1.0 offers the investor the opportunity to reduce portfolio risk as measured by the portfolio standard deviation. This process is commonly known as **diversification.** As investors diversify their portfolios by adding securities that have correlation coefficients less than 1.0, and continue to adjust the respective weights in these portfolios, the many possible combinations can be respresented in a feasible set of the **efficient set of portfolio combinations.**

To illustrate through an example, take a portfolio of two assets A, and B. Suppose that each asset makes up 50% of the total value of the portfolio. The expected return and standard deviation for Asset A are 10% and 0.30, and for Asset B, 25% and 0.60, respectively.

Then, the expected return and variance on the portfolio are:

$E(R_P) = 0.5(R_A) + 0.5(R_B) = 0.5(0.1) + 0.5(0.25) = 0.175.$

The standard deviation is derived from:

$\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_AX_BCorr_{A,B}\sigma_A\sigma_B]^{1/2} = [0.5^2(0.3)^2 + 0.5^2(0.6)^2 + 2(0.5)(0.5)Corr_{A,B}(0.3)(0.6)]^{1/2} =$

$=[0.1125 + 0.09Corr_{A,B}]^{1/2}$

Now taking different values for $Corr_{A,B} = +1, 0, -1$ we get the risk of returns below:

Case	$\rho_{AB,BC}$	Expected return $_{AB}$	Risk of Return $_{AB}$	Risk of returns with replacement asset C
1	+1.0	17.5%	45%	50%
2	0.0	17.5%	33%	36%
3	-1.0	17.5%	15%	10%

The table shows the "magic" of Tobin-Markowitz diversification. The risk of return of the portfolio is shown to decline as the degree of correlation between the assets is reduced, and the expected return on the portfolio is not affected.

There is another lesson we can learn from diversification. One can increase the expected return of a portfolio, and at the same time reduce the risk of the expected return, by substituting a riskier asset for a less risky asset - if the expected return of the replacement asset is correlated much less with the other assets than was the case of the asset it replaced.

Suppose that, in the above example, asset A is replaced by asset C with an expected rate of return of 20% and a standard deviation of 0.40. The expected return of the portfolio is now 22.5% rather than 17.5%, and the risk of returns, (standard deviatioin) becomes $[0.13 + 0.12\ Corr_{C,B}]^{1/2}$. We can see from the table that in case 3, risk of portfolio has been reduced.

When there are only two assets in a portfolio, we could get the variance of this portfolio in the following manner:

$$Var(R_P) = w_i^2 Var(R_i) + w_j^2 Var(R_j) + 2w_i w_j std(R_i) std(R_j) cor(R_i, R_j)$$

w_i and w_j are the percentage of he portfolio's funds invested in asset i, and asset j.

i, and j are the only assets in the portfolio. The above equation states that the variance of the portfolio is the sum of the weighted variances of the two assets plus the weighted correlation between the two assets. So the correlation between the two assets affect the variance of the portfolio. The lower the correlation between the two assets, the lower the portfolio's variance, and vice versa. (perfect positive correlation, when it is 1, and -1 means negative perfect correlation). To expand the above equation to n assets in portfolio, say we have three assets, i, j and k, we get:

Var $(\mathrm{R}_P) = w_i^2$ Var $(\mathrm{R}_i) + w_j^2$ Var $(\mathrm{R}_j) + w_k^2 Var(R_k) + 2w_i w_j$ std(R_i) std(R_j) cor$(\mathrm{R}_i,\mathrm{R}_j)$+$2w_i w_kstd(\mathrm{R}_i)$ std(R_k) cor$(\mathrm{R}_i,\mathrm{R}_k)$+$2w_j w_k$ std(R_j) std(R_k) cor$(\mathrm{R}_j,\mathrm{R}_k)$

again, w_k is the percentage of the portfolio's funds invested in asset k.

To explain the concept of asset correlation, and to show why it is important to understand the implications of correlation when considering assets for possible inclusion in your portfolio.

2.3.4. Adding Securities to eliminate risk

In this part, we further illustrate the power of diversification. First note that the general formula for the variance of a portfolio is as follows:

$$Var(R_P) = \sum_{i=1}^{N} w_i^2 var(R_i) + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j Cov(R_i, R_j) \text{ for } i \neq j \tag{2.1}$$

The formula says that the variance of a portfolio is a weighted average of the variances of the individual securities plus the covariance between each security and every other security in the portfolio. If securities have zero correlation, and hence zero covariance, the second term goes to zero, and the expression reduces to:

$$Var(R_P) = \sum_{i=1}^{N} w_i^2 var(R_i) \tag{2.2}$$

Assume for purposes of illustration that only securities with zero covariance are available, that each security has the same variance, and that equal amounts are invested in each security,

$$Var(R_P) = \sum_{i=1}^{N} (1/N)^2 var(R_i) = N(1/N)^2 var(R_i) = 1/N var(R_i) \tag{2.3}$$

and

$$\text{s tan dard deviation} R_P = \frac{\sqrt{var(R_i)}}{\sqrt{N}} = \frac{\text{S tan dard deviation} R_i}{\sqrt{N}} \tag{2.4}$$

Using this formula and assuming that the individual securities have standard deviations of (0.4 percent), the data in the following table show how the risk (standard deviation) of the portfolio declines as zero-correlated securities with identical standard deviations are added to the portfolio. Risk is reduced to less than 10 percent of that of a single-stock portfolio when 128 securities are in the portfolio and to less than 2 percent of the original risk when 510 are included. As the number of securities added becomes larger and larger (approaches infinity, in technical terms), the standard deviation of the portfolio approaches zero.

Portfolio risk and number of securities (zero correlation)

Number of securities	1	2	8	16	32	128	510
Standard deviation of portfolio return	40.0	28.3	14.1	10.0	7.1	3.5	1.8

The principle here is that if there are sufficient numbers of securities with zero correlation (zero covariance), the portfolio analyst can make the portfolio risk arbitrarily small. This is the basis for insurance, which explains why insurance companies attempt to write many individual policies and spread their coverage so as to minimize overall risk. It also has direct relevance in providing a benchmark for assessing the extent to which diversification can be effective in reducing risk for equity investors.

Example 2.1. *Correlation*

Asset correlation measures the extent to which the returns on two assets move together (i.e the extent to which those returns behave similarly in response to market events or stimuli).How it works

Asset correlation ranges from a maximum of +1.00 to a minimum of -1.00. If two assets have a perfect positive correlation (+1.00), their returns will tend to move simultaneously in the same direction. With a perfect negative correlation (-1.00), their returns will tend to move simultaneously in opposite directions. A correlation of 0 indicates that there is no relationship at all between the price movements of two assets.

Since few asset pairs will come anywhere close to perfect positive or negative correlation, the following rules of thumb can be helpful:

High Correlation: Asset correlation greater than 0.75; implies that the two assets respond very similarly to the market and that their prices will very often move in the same direction.

Moderate Correlation: Asset correlation between 0.25 and 0.75; implies that the two assets respond in somewhat similar ways to the market and that their prices will move more or less in the same direction, depending on how strong the correlation.

Low Correlation: Asset correlation between 0.00 and 0.25; implies that the two assets respond fairly independently to the market and that their prices also tend to move independently of one another.

Negative Correlation: Asset correlation below 0.00; implies that the two assets respond fairly differently to the market and that their prices will tend to move in opposite directions.

An individual investor maintains a portfolio that includes shares of the following assets: Intel (INTC), AT&T (T), Walmart (WMT), Vanguard Wellington Income Fund (VWELX), Janus Fund (JANSX), and Vanguard Total International Stock Index Fund (VGTSX). The matrix below shows how those assets correlate to one another:

Correlation	INTC	T	WMT	VWELX	JANSX	VGTSX
INTC	1.00	0.53	0.21	0.35	0.42	0.25
T	0.53	1.00	0.29	0.27	0.42	0.12
WMT	0.21	0.29	1.00	0.24	0.31	0.12
VWELX	0.35	0.27	0.24	1.00	-0.08	0.84
JANSX	0.42	0.42	0.31	-0.08	1.00	0.72
VGTSX	0.25	0.12	0.12	0.84	0.72	1.00

In viewing this matrix, we can make the following observations:

To illustrate the extent to which one asset can behave differently from other assets, the Vanguard Wellington Income Fund serves as a good example: This fund has moderate or low/moderate correlation with the Intel, AT&T and Walmart stocks, a negative correlation with the Janus Fund, and a high correlation with the Vanguard Total International Stock Index Fund. While not precise, it is generally true that the closer your median or average correlation is to +1.00, the less diversified your portfolio is likely to be. In this portfolio, VGTSX's high correlations with VWELX and JANSX suggest that this fund may be contributing less to portfolio diversification than the investor thinks.

Correlation is important because it serves as a checking mechanism on portfolio diversification. This is true because portfolio diversification depends upon both the number and weightings of portfolio assets, as well as their relative correlations to one another. By including low-correlation asset pairs in your portfolio, you can "hedge" the risk of otherwise volatile assets, diversify, and possibly lower your portfolio's overall volatility.

2.4. Constructing Markowitz Efficient Portfolios

To illustrate how the risk-return character of a portfolio changes as we vary the weighting of securities in the portfolio, we use the data shown in the following table for two hypothetical stocks. Panel shows the expected returns and standard deviations for each of the stocks. Note that stock A has both a higher expected return and standard deviation than stock B to illustrate a trade-off between risk and return. Also, for purposes of illustration, we assume that the correlation between the two stocks is (1), +1.0 to represent perfect correlation; (2) 0 to represent independence; or (3) +0.5 for a correlation that is approximately in line with the average correlation between stocks in the United States. Using this date in Panel B, we calculate the expected return and risk for a portfolio of these two stocks over a range of portfolio weights for each of the three correlation assumptions. In making these calculations, we assume that the portfolio is fully invested and that the weight of stock A is W_A, so the weight of stock B is $(1-W_A)$

	Panel A		
	Expected Return	Standard Deviation	Correlation
Stock A	15%	24%	+1.0, 0.0, +0.5
Stock B	125	18%	

	Panel B					
W_A	0	20	40	60	80	100
$W_B = (1 - W_A)$	100	80	60	40	20	0
Expected return	12.0	12.6	13.2	13.8	14.4	15.0
Standard deviation						
Corr=1.0	18.0	19.2	20.4	21.6	22.8	24.0
Corr=0.5	18.0	17.3	17.7	19.0	22.0	24.0
Corr=0.0	18.0	15.2	14.4	16.1	19.5	24.0

Panel B shows the calculated risk and return data for these portfolios where the weight of stock A ranges in increments of 0.20 from 0 percent to 100 percent with the corresponding weight of stock B ranging down from 100% to 0 percent. The third row in the panel shows the expected return for the portfolio and indicates that the portfolio consisting of 100 percent of stock A has the highest return and the portfolio consisting of 100 percent of stock B has the lowest expected return. The bottom three rows show the calculated portfolio standard deviation for each of the three correlations. This illustrates how portfolio variance changes as the security weighting vary.

For ease of comparison, we show a plot of the varying risk-return trade-off of the two stock portfolios at each of the three assumed correlation levels. Note that there is a direct linear trade-off of risk and return when the assumed correlation is perfectly positive. On the other hand, there is a curvilinear relationship and a lesser level of risk for the portfolios when the correlation is zero and at 0.5. This illustrates the general principle that there is productive diversification and risk reduction when correlation is less than perfectly positive. Correspondingly, there is a more favorable risk-return trade-off at a correlation of zero than at a 0.5 correlation level, again illustrating the potential gains from diversification as correlation is lower.

In general, investors will calculate the risk and expected return for each portfolio. For the portfolios with the same level of risk, there will be a large number of portfolios, each with a different expected return. The investor will choose the portfolio with the greatest expected return for a given level of risk. This portfolio is the **Markowitz efficient portfolio.** Any portfolio that can be created is called a *feasible portfolio*. The collection of all the feasible portfolios is called the *feasible set of portfolios*. The Markowitz efficient portfolio is that one that gives the highest expected return of all feasible portfolios with the same risk. (it is called also the **mean-variance efficient portfolio**).

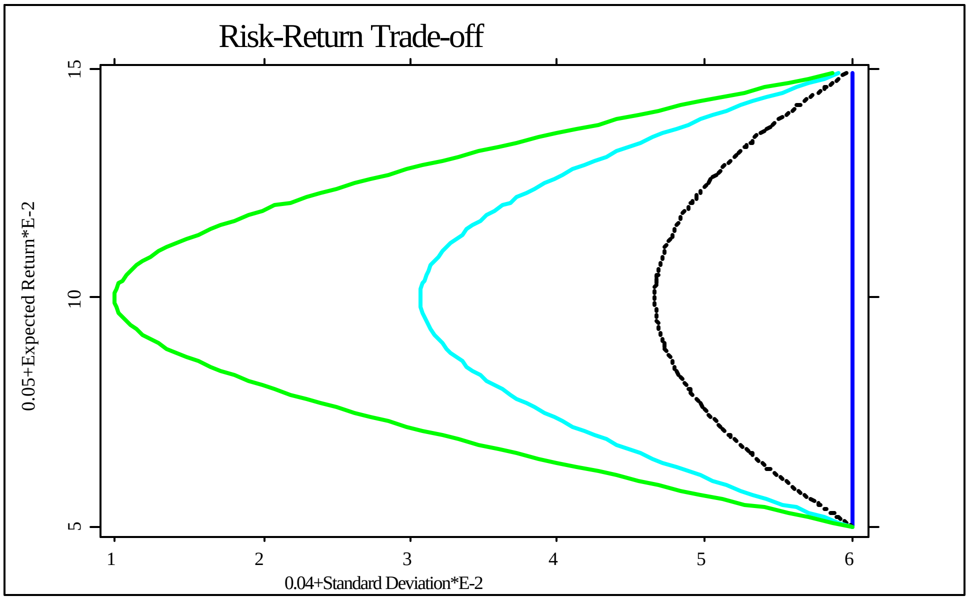


Figure 2.3:

2.5. Announcements, surprises and expected returns

The actual return on a security **(R)** consists of two parts: the **expected return**, which is predicted by participants in the **financial markets**; and, a surprise or **unexpected part**. The expected return is based on a large number of factors that may influence a given company. The risk for a particular asset comes from the unexpected part of the return. For a given stock, we can write:

$$R = E(R) + U \tag{2.5}$$

The difference between R and E(R) is attributed to surprises reflected in the unexpected return, U.

2.5.1. Risk: Systematic and Unsystematic

The risk of owning an asset, such a common stock, emanates from the unexpected events; if events always occurred exactly as predicted, there would be no risk. We identify two types of unexpected events, which give rise to two categories of risk: **systematic and unsystematic risk**.

A **systematic risk** tends to affect a large number of assets to a greater or lesser degree; systematic risks are also called **market risks**. An **unsystematic risk** affects only a single asset or a small group of assets. Systematic risks arise from uncertainty about economy-wide factors, such as inflation and interest rates, whereas the possibility of a labor strike or lawsuit involving a single company is an unsystematic or **"idiosyncratic" risk.**

We know that :

$$R = E(R) + U \tag{2.6}$$

where U is the surprise component. The surprise component is influenced by two kinds of risk. So we write:

$$U = \text{Systematic portion} + \text{Unsystematic portion} \tag{2.7}$$

or

$$U = m + \epsilon \tag{2.8}$$

Then:

$$R = E(R) + m + \epsilon \tag{2.9}$$

The principle of diversification indicates that spreading an investment across many assets eliminates some of the risk. The risk which is eliminated by diversification is called **diversifiable risk**. However, there is a minimum level of risk in a portfolio which can not be eliminated by diversification. This minimum risk is called **nondiversifiable risk**.

Why is not all risk diversifiable:

The answer is found in the distinction between systematic and unsystematic risk. In a large portfolio, the unsystematic risk associated with one stock typically has no impact on the unsystematic risk associated with any other stock. Therefore, in a large portfolio, we would not expect to observe any particular relationship between the unsystematic risks for the individual stocks in the portfolio.

Systematic risk has an impact on all the stocks in a portfolio. For example, a recession will have a negative impact on virtually all the securities in the portfolio, and consequently on the investor's return from the portfolio. **Systematic risk** cannot be diversified away; hence, the terms **systematic risk** and **nondiversifiable risk** are synonymous.

In this respect, investors will not accept risk without compensation in the form of a higher expected return. Investors can eliminate unsystematic risk by diversifying, and the market will not reward investors for taking unnecessary risks. So, investors are not paid for bearing unsystematic risks. In other words, the reward for bearing risk depends only on the systematic risk of an investment.

2.5.2. Measuring Systematic Risk

The systematic risk of an asset is measured by its beta coefficient, represented by β. Beta measures the systematic risk for a particular asset, relative to the systematic risk for the average asset. For example, the average TSE common stock has a beta value of 1.0, so a stock with $\beta = 0.5$ has half the systematic risk of the average TSE stock. Or, a stock of $\beta = 1.25$ has 25% more systematic risk than the average stock.

It is important to remember that, since systematic risk is the relevant risk for an investor, the expected return for an asset is dependent on β, a stock with a high value of β has a high expected return, and a stock with a lower value of β has a lower expected return.

2.5.3. Portfolios Betas

The value of beta for a portfolio (β_P) is a weighted average of the betas of the securities which comprise the portfolio, as indicated in the following:

$$\beta_P = (x_1\beta_1) + (x_2\beta_2) + (x_3\beta_3) + \ldots\ldots\ldots + (x_n\beta_n) \tag{2.10}$$

Where, x's are the weights, n is the number of assets in portfolios, and $\beta's$ are the beta values for the assets in the portfolio.

Example 2.2. *Suppose, two assets, A and B, with :* $\beta_A = 1.10$, $\beta_B = 0.70$, *and* $x_A = 0.40$ *and* $x_B = 0.60$

then: $\beta_P = 1.10 \times 0.40 + 0.70 \times 0.60 = 0.86$.

2.6. The Security Market Line

Now we develop the relationship between systematic risk, measured by Beta, and the expected return. The graphic representation is called the security market line (SML) and the algebraic representation is the Capital Asset Pricing Model (CAPM).

2.6.1. Beta and the Risk Premium

We take an example: Let asset A be an individual security with β_A =1.2 and E(R_A) = 16%. Assume that the risk-free rate (R_f) = 5%. Note that beta for the risk-free asset is zero. Since the risk-free asset does not have any risk, it clearly does not have any systematic risk. If x is the percentage of the portfolio invested in asset A, then (1-x) is invested in the risk-free asset. The expected return for a portfolio is:

$\mathrm{E}(\mathrm{R}_P) = [x \times E(R_A)] + [(1-x) \times E(\mathrm{R}_f)] = [x \times 16\%] + [(1-x) \times 5\%]$

Beta for this portfolio is:

$\beta_P = [x \times \beta_A] + [(1-x) \times 0] = [x \times 1.2] + 0 = 1.2x$

By selecting various values of x, and then computing the corresponding value of E(R_P) and β_P we can plot expected returns and beta. This relationship between E(R_P) and β_P is a straight line through the point R_f and the point represented by $\beta_A = 1.2$ and E(R_A)

x	0%	20	50	90	100	150	200
E(R_P)	5.0%	7.2	10.5	14.9	16.0	21.5	27.0
β_P	0.00	0.24	0.60	1.08	.120	1.80	2.40

The slope of the line, is called the reward-to-risk ratio. The slope of a straight line can be computed between any two points on the line. Take β_A, E(R_A),R_f to find the slope:

Slope: $= (E(R_A) - R_f) \,/\, (\beta_A - 0) = 0.09167 = 9.167\%$

This can be regarded in the following manner: for each unit increase in systematic risk (an increase from beta =0 to beta = 1), an investor will increase her expected return by 9.16%.

This reward-to-risk ratio can be explained as follow: assume an investor is considering placing her entire investment portfolio in the risk-free asset. Prior to making her decision, the investor might want to know what her compensation would be for taking on risk. That is represented by the reward-to-risk ratio, for each unit increase in the systematic risk, (Beta), she will increase her expected rate of return by 9.16%.

Another example, say we have another asset B, with $\beta_B = 0.6$ and $E(R_B) =$ 10%. Should an investor consider purchasing this asset?. In this case, we compare asset B to a portfolio (P), comprised of asset A and the risk-free asset, with $\beta_P = 0.6$.

We know earlier that, $\beta_P = 1.2x$, so x must be 0.5, then:

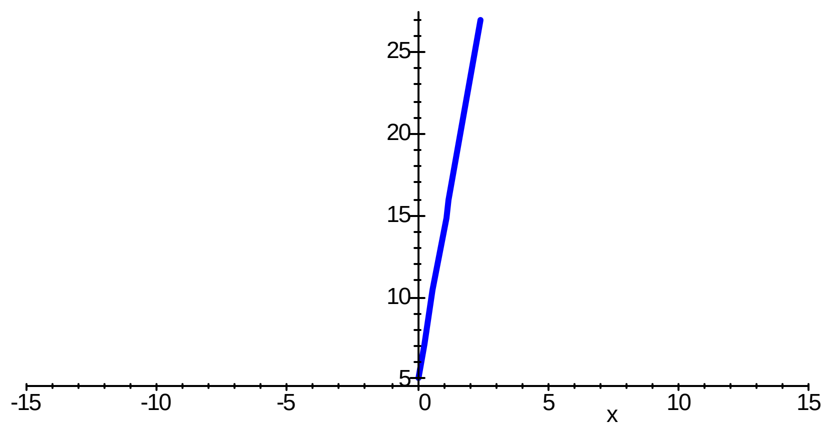
$E(R_P) = (0.5 \times 16\%) + (0.5 \times 5\%) = 10.5\%$

For the same risk, portfolio P has a higher expected return than does asset B, so that P is preferable to B.

B is inferior to A, and that can be demonstrated through the reward-to-risk ration, that is:

Slope $= (E(R_B) - R_f)/(\beta_B - 0) = 0.0833 = 8.33\%$.

8.33% is less than 9.167%, then, an investor would not consider buying asset B.



Security Market Line

2.6.2. The Security Market Line

The above calculation indicates that investors would prefer to hold asset A not asset B. This would lead to a decrease in the price of asset B and thus an increase in its expected return; similarly, the price of asset A would increase and its expected return would decrease. Consequently, in equilibrium, asset B, and all assets, must fall directly on a straight line, this line is called the **Security Market Line.**

A portfolio comprised of all assets in the market is called the market portfolio. The market portfolio itself is an asset, so its expected return $E(R_M)$ and β_M must be on the security market line.

what then is the value of β_M

The market portfolio is comprised of assets in the market and a portfolio beta is a weighted average of the betas of the securities in the portfolio. Therefore, the beta for a market portfolio must be equal to beta for the average security; that is, $\beta_M = 1$.

The slope of SML would be:

$\left(E(R_M) - \beta_f\right)/(\beta_M - 0) = \left(E(R_M) - \beta_f\right)/(1-0) = E(R_M) - \beta_f$

Since this slope is the excess return on the market over the risk-free rate, it is often called **market risk premium**, and the equation for the SML becomes:

$E(R_i) = R_f + [E(R_M) - \beta_f] \times \beta_i$

where, $E(R_i)$ and β_i are the expected return and beta for any asset. This equation is referred to as the **Capital Asset Pricing Model (CAPM)**, and it indicates the expected return for any asset for a given level of systematic risk.

2.7. The Capital Asset Pricing Model (CAPM)

The **Capital Asset Pricing Model (CAPM)** shows how the rate of return, or alternatively the price of an asset, is determined in efficient markets.

Assumptions underlying the CAPM

1. All investors prefer the highest expected return for any given level of risk

2. Capital markets are efficient in that asset prices and yields reflect all information.

3. All investors have the same expectations about each asset's and return. This condition is called **homogeneous expectations.**

4. Investors can both borrow and lend at a risk-free rate of interest.

Investors will calculate the risk and expected return for each portfolio. For the portfolios with the same level of risk, there will be a large number of portfolios, each with a different expected return. The investor will choose the portfolio with the greatest expected return for a given level of risk (or alternatively, the lowest risk for a given expected return.) This portfolio is the **Markowitz efficient portfolio.**

Any portfolio that can be created is called a **feasible portfolio**. The collection of all the feasible portfolios is called the **feasible set of portfolios**. The Markowitz efficient portfolio is that one that gives the highest expected return of all feasible portfolios with the same risk. (it is called also the **mean-variance efficient portfolio**).

2.7.1. Combining portfolios of Risk-Free and Risky Assets

Suppose investors want to hold a portfolio of assets that combines a risk-free asset with an efficient portfolio of risky assets found on an efficient frontier.

The question would be: Which portfolio among those along the efficient frontier should be included in the combined portfolio.

The question can be answered with the figures. The curve AD is an efficient frontier of portfolios of risky assets. We draw a straight line from R^S the risk-free rate of return for the risk-free asset, to intersect the efficient frontier at V. The line $R^S V$ shows combinations of the risk-free asset with the V portfolio of risky assets on the efficient frontier. At R^S on the line only the risk-free asset is held, while at V only the V portfolio of risky assets is held. In between, as we move along the ling from R^S to V, increasingly less of the investor's given funds are held in the risk-free asset and more are held in the V portfolio of risky assets. An investor can go behind the broken portion of the line by borrowing at the risk-free rate.

Combined portfolios along the straight line running from R^S to V are inefficient portfolios relative to those lying on a similar line drawn through R^S but intersecting the efficient frontier above V. This is because higher expected return can be obtained at the same risk. Rotating the $R^S V$ line upward until it is just tangent to the efficient frontier produces the **optimum portfolio** of risky assets to be combined with the riskless asset. This optimum portfolio of risky asset is shown at point M. The optimum portfolio M is called the market portfolio of risky assets, or simply the **market portfolio**. The straight line $R^S MC$ is called **the capital market line (CML)**, which is the efficient frontier of combined portfolios.

If all investors have the same expectations about asset returns and risk, each will want to hold a combined portfolio lying along the capital market line. No investor will want to hold any portfolio or risky assets other than the market portfolio.

2.8. Arbitrage Pricing Theory (APT)

The CAPM is a normative theory, it does not necessarily describes what investors do, but rather what they should do if their behavior reflects the underlying assumptions of the market model. It has been subject to extensive empirical testing, but there has been little evidence to support it. This is not surprising given its assumption of investor behavior. It is unrealistic that all investors have the same expectations concerning the future rate of return and risk of every asset in the market, and even less likely that the lending rate equals the borrowing rate (which equals the "risk-free" rate of interest).

An alternative to the CAPM, which attempts to avoid its shortcoming, is the **Arbitrage Pricing Theory (APT).** It is not based on unrealistic assumptions, however, like the CAPM, the APT states that the expected return, or alternatively the price of an asset, depends on its systematic risk. Whereas the CAPM defines only one source of systematic risk - market conditions - the APT assumes multiple sources referred t **as factors .** That APT can be expressed by the following:

$$E(R^A) = R^F + \beta_1 F_1 + \beta_2 F_2 + ... + \beta_K F_K \tag{2.11}$$

where $E(R^A)$ is the expected return on an individual asset, R^F is the risk-free rate of return, $F_1, F_2, ...F_K$ are systematic risk factors (inflation, interest rates, industrial production, ...), and $\beta_1, \beta_2, ...\beta_K$ indicate the sensitivity of the asset's expected return to the respective factors.

Preliminary findings suggest that APT is better at explaining the historical data than the CAPM, but that may be the result of the empirical methods and tests employed by different researchers.

2.8.1. Derivation of the Capital Market Line

It was demonstrated where individuals would like to be regarding risky assets. We constructed the Markowitz efficient frontier. Also we derived the security market line, where individuals are able to borrow at the risk free rate. Individuals facing the possibility of borrowing and being on the efficient frontier, would prefer to have the highest expected return for a given level of risk. It was demonstrated by Sharpe, Treynor and Mossin that the opportunity to borrow or lend at the risk-free rate implies a capital market where risk-averse investors will prefer to hold portfolios consisting of combinations of the risk-free and some portfolios M on the Markowitz efficient frontier. Sharpe called this line (up to the tangent point on the efficient frontier) the **capital market line (CML).**

How does an investor constructs portfolio M. Eugene Fama answered this question by demonstrating that M must consist of all assets available to investors, and each asset must be held in proportion to its market value relative to the total market value of all assets. So, for example, if the total market value of some asset is $500 million and the total market value of the all assets is $X, then the percentage of the portfolio that should be allocated to that asset is $500 million divided by $X.

2.8.2. How to derive the Capital Market Line

Suppose an investor creates two-fund portfolio: a portfolio consisting of w_f invested in the risk-free asset and w_M in the market portfolio, where w represents the corresponding percentage (weight) of the portfolio allocated to each asset. Thus, $w_f + w_M = 1$

What is the expected return and risk of this portfolio: $E(R_P) = w_f R_f + w_M E(R_M)$
since we know that $w_f = 1 - w_M$
then $E(R_P) = (1 - w_M) R_f + w_M E(R_M) = R_f + w_M (E(R_M) - R_f)$

What about the risk

Var(R$_P$) =w$_f^2 var(R_f)$+w$_M^2$ $var(R_M)$+2w$_M$ w$_f std(R_f)std(R_M)corr(R_f, R_M)$

we know that $var(R_f) = 0$, also $corr(R_f, R_M) = 0$ this is because the risk-free asset has no variability and therefore does not move at all with the return on the market portfolio which is a risky asset. So the portfolio variance becomes:

Var(R$_P$) =w$_M^2$ $var(R_M(R_M)$

or to say:

Std (R$_P$) =w$_M$ $Std(R_M)$

and therefore: w$_M$ =Std (R$_P$)$/Std(R_M)$

Now let us use the equation for the expected return:

E(R$_P$) $= R_f$+w$_M$ $(E(R_M) - R_f) = R_f +$ [Std (R$_P$)$/Std(R_M)](E(R_M) - R_f)$

This is the equation for the Capital Market Line. The numerator is the expected return on the market beyond the risk-free return. It is a measure of the risk premium or the reward for holding the risky market portfolio rather than the risk-free asset. The denominator is the risk of the market portfolio. Thus, the slope measures the reward per unit of market risk. Since the CML represents the return offered to compensate for a perceived level of risk, each point on the line is a balanced market condition, or equilibrium. The slope is also referred to as the **market price of risk.**

According to capital market theory, the risk premium is equal to the market price of risk times the quantity of risk for the portfolio (as measured by the standard deviation of the portfolio), that is:

E(R$_P$) $= R_f$+ Market price of risk $\times$ Amount of portfolio risk

2.8.3. The Minimum Variance Portfolio

In a two-security portfolio, some particular combination of the two securities will result in the least possible variance. Not surprisingly, this basked of securities is called the **minimum variance portfolio**. To find the security proportions that will produce the minimum variance portfolio, it is necessary to understand basic calculas. To find the minimum risk portfolio, take the first derivative of the risk requation, set is equal to zero, and solve for the value of the variable needed.

The variance of a two-security portfolio is as follows:

$$\sigma_P^2 = w_a^2\sigma_a^2 + w_b^2\sigma_b^2 + 2w_a w_b \rho_{ab}\sigma_a\sigma_b \tag{2.12}$$

We also know that $w_a + w_b = 1$, so $w_b = 1 - w_a$. Substituting this expression for w_b gives the following version of the above equation:

$$\sigma_P^2 = w_a^2\sigma_a^2 + (1 - w_a)^2\sigma_b^2 + 2w_a(1 - w_a)\rho_{ab}\sigma_a\sigma_b \tag{2.13}$$

Taking the derivative of the equation for σ_P^2 with respect to w_a and setting it equal to zero, and solving for w_a

we get:

$$w_a = \frac{\sigma_b^2 - \rho_{ab}\sigma_a\sigma_b}{\sigma_a^2 + \sigma_b^2 - 2\rho_{ab}\sigma_a\sigma_b}$$